An averaged model for switched systems with state jumps applicable for PWM descriptor systems

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Abstract— **Switched descriptor systems with pulse width modulation are characterized by modes whose dynamics are described by differential algebraic equations; this type of models belongs to the more general class of switched impulsive systems, i.e. switched systems with ordinary differential equations as modes dynamics and state jumps at the switching time instants. The presence of possible jumps in the state makes the application of the classical averaging technique nontrivial. In this paper we propose an averaged model for switched impulsive systems. The state trajectory of the proposed averaged model is shown to approximate the one of the original system with an error of order of the switching period. The model reduces to the classical averaged model when there are no jumps in the state. The practical interest of the theoretical averaging results is demonstrated through the analysis of the dynamics of a switched electrical circuit.**

I. INTRODUCTION

Switched descriptor systems represent the dynamic behavior of many physical apparatus, e.g. mechanical systems [18] and electronic circuits [13], [7]. The dynamics of switched descriptor systems is determined by switchings among different modes, where each mode is characterized by a set of linear differential equations and algebraic constraints. A mathematical representation of this class of systems can be obtained in terms of switched linear differential algebraic equations (DAE). Switched DAE belong to the more general class of so called switched impulsive systems where each mode is represented by ordinary differential equations (ODE) together with rules for state jumps at the switching time instants.

In this paper we propose an averaged model for switched impulsive systems with pulse width modulation (PWM). Averaging theory for switched systems has a big interest in the control literature considering different approaches and points of view related to the switched system characteristics: non-periodic switching functions [1], pulse modulations [20], [15], [17], dithering [3], hybrid systems framework [21], [22]. On the practical point of view, the averaging approach is a widely used technique in the power electronics community since 1970s [19] and has been also applied to other switched systems of practical interest, see [14], [16] and the references therein.

The presence of state discontinuities at the switching time instants introduces several difficulties for the averaging analysis. An averaged model for homogeneous switched DAE systems is presented in [4], [10] while an averaging result for the non homogeneous case is proposed in [9]. In [11] an averaged model whose dynamic matrices depend on the switching period has been presented and further analyzed in [8]. Unfortunately, the averaged models proposed in the papers above require some technical assumptions on the dynamic modes that are not always satisfied by practical systems [12], [13].

In this paper we propose a discrete-time averaged model for switched impulsive systems under milder assumptions with respect to the averaged models used in the previous literature. We show that the difference between the solution of this model and the moving average of the solution of the original system is of the same order as the switching period. We also conjecture a possible structure for a corresponding continuous-time version of the averaged model. In the work we focus on an homogeneous switched system with two modes and constant duty cycles. Nevertheless the case of multiple modes is a straightforward extension and non-homogeneous systems can be represented in the considered framework for inputs which can be determined as outputs of switched impulsive systems. A ladder electrical circuit is considered as a practical PWM descriptor system example and the numerical simulations validate the effectiveness of the proposed model.

The paper is organized in several sections. In Section II some preliminary definitions and properties of switched impulsive systems are recalled. Section III presents the proposed averaged model and Section IV the main approximation result. In Section V a numerical verification of the theoretical results is proposed. In Section VI the results are summarized.

II. SWITCHED IMPULSIVE SYSTEMS

The switched impulsive system considered in our analysis is characterized by a pulse width modulation between modes with a switching period $p \in \mathbb{R}_+$. For the sake of readability we consider only two modes; the extension to the presence of more modes can be easily obtained by applying similar arguments.

The system behavior is characterized by a pulse width modulation with the commutations from the mode 2 to the mode 1 at the time instants $t_k \in \mathbb{R}_+$, $k \in \mathbb{N}$, being the multiple of the period p, and from the mode 1 to the mode 2 at the switching time instants s_k within the k-th period. Then we have $t_k = kp$ and $s_k = t_k + dp$, where $d \in (0, 1)$ is the duty cycle. In the sequel we consider a finite time interval

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$t \in [0, T]$, we assume that T is a multiple of the switching period p and we indicate $\ell(p) = T/p$. A generalization when this is not the case is straightforward.

A model of the switched impulsive system of interest can be represented as follows

$$\dot{x}(t) = F_1 x(t), \quad t \in (t_k, s_k) \tag{1a}$$
$$x(t_k^+) = \Pi_1 x(t_k^-) \tag{1b}$$
$$\dot{x}(t) = F_2 x(t), \quad t \in (s_k, t_{k+1}) \tag{1c}$$
$$x(s_k^+) = \Pi_2 x(s_k^-) \tag{1d}$$

with $t \in [0, T]$ and $x(0^-) = x_0 \in \mathbb{R}^n$ the initial condition of the state variable. The flow matrices $F_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, characterize the dynamics of the two modes and the matrices $\Pi_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, determine the possible jumps of the state variable at the switching time instants.

The solution of the switched system (1) can be obtained by an iterative process. We consider the switching system at the switching time instant then we have that

$$x_k^+ = \Pi_1 x_k^- \tag{2a}$$
$$x(s_k^-) = G_1(dp) x_k^+ \tag{2b}$$
$$x(s_k^+) = \Pi_2 x(s_k^-) \tag{2c}$$
$$x_{k+1}^- = G_2((1-d)p) x(s_k^+) \tag{2d}$$

where $G_i(s) = e^{F_i s}$, $s \in [0, p]$, $i \in \{1, 2\}$. By using (2), the left solution of (1) at the time instants multiple of the switching periods can be written in the following iterative form

$$x_{k+1}^- = \Theta(d, p) x_k^- \tag{3}$$

for $k \in \{0, \dots, \ell(p)\}$, where

$$\Theta(d, p) = G_2((1-d)p) \Pi_2 G_1(dp) \Pi_1. \tag{4}$$

By repetitive application of (3), the solution of (6) can be written as

$$x_k^- = \Theta(d, p)^k x_0^- \tag{5}$$

for all $k \in \{0, \dots, \ell(p)\}$.

The switched impulsive system (1) include several practical systems and, among them, switched descriptor systems which can be represented in the following homogeneous switched DAE form

$$E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x \tag{6}$$

for $t \in [0, T]$, where $\sigma : [0, T] \to \{1, 2\}$ is a piece-wise constant right-continuous function, that selects at each time instant the index of the active mode from the index set $\{1, 2\}$, i.e.,

$$\sigma(t) = \begin{cases} 1, & t \in [t_k, s_k), \\ 2, & t \in [s_k, t_{k+1}), \end{cases} \tag{7}$$

with $k \in \{0, \dots, \ell(p)\}$, $i = 1, 2$.

The model (6) can be represented in the form (1) if the pairs (E_i, A_i) with $i = 1, 2$ are regular matrix pairs, i.e. the polynomials $\det(sE_i - A_i)$, $i = 1, 2$, are not identically zero. Indeed, under these conditions, there exist transformation matrices S_i and T_i which put (E_i, A_i) into the quasi Weierstrass form

$$(S_i E_i T_i, S_i A_i T_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N_i \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix} \right), \tag{8}$$

$i = 1, 2$, with $T_i = [V_i,\ W_i]$, $\quad S_i = [E_i V_i,\ A_i W_i]^{-1}$ where N_i is a nilpotent matrix, I is the identity matrix, J_i, V_i and W_i are matrices of appropriate size. Then, for any regular matrix pair (E_i, A_i) it is possible to define the flow matrix F_i and the consistency projector Π_i as follow:

$$F_i = T_i \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1} \tag{9a}$$
$$\Pi_i = T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1} \tag{9b}$$

$i = 1, 2$. The consistency projector is an idempotent matrix, i.e., $\Pi_i^2 = \Pi_i$ and commutes with the corresponding flow matrix, i.e., $F_i \Pi_i = F_i = \Pi_i F_i$, $i = 1, 2$. In [11, Theorem 12] it is shown that $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the impulse free part of any (distributional) solution of (6) if and only if it is a solution of (1).

It is important to highlight that our averaging result do not require that the matrices Π_i, $i = 1, 2$, in (1) are projectors.

III. Discrete-time averaged model

In this section we present the proposed averaged model and we prove a corresponding approximation result with respect to the solution of (1) at the multiple of the switching period.

Let us consider the following discrete-time model

$$z_{k+1} = \Phi(d, p) z_k \tag{10a}$$
$$\mu_k = \Gamma(d) z_k \tag{10b}$$

with $k \in \{0, \dots, \ell(p) - 1\}$, with

$$\Gamma(d) = \Pi_1 d + \Pi_2 \Pi_1 (1 - d). \tag{11}$$

and

$$\Phi(d, p) = \Pi_\cap + \Lambda(d) p \tag{12}$$

where

$$\Pi_\cap = \Pi_2 \Pi_1, \tag{13a}$$
$$\Lambda(d) = \Pi_2 F_1 \Pi_1 d + F_2 \Pi_2 \Pi_1 (1 - d), \tag{13b}$$

It should be noticed that in the case of a switched ODE, the projectors are equal to the identity matrix and the matrix $\Lambda(d)$ reduces to the dynamic matrix of the classical continuous-time averaged model of pulse width modulated systems with two modes, i.e. $F_1 d + F_2 (1 - d)$. In general, the matrix $\Pi_\cap$ is not a projector, i.e. $\Pi_\cap^2 \neq \Pi_\cap$; in this case the hypothesis used for the averaged model proposed in [11], [12] do not hold.

The solution of (10b) can be written as

$$z_k = \Phi(d, p)^k z_0 \tag{14}$$

for all $k \in \{0, \dots, \ell(p) - 1\}$.

We are interested to compare (5) and (14). To this aim we need to recall the Big O notation and some of its properties.

Definition 1 ($\mathrm{O}(p^r)$): For any finite integer $r \in \mathbb{N}$, a matrix function $G : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is said to be an $\mathrm{O}(p^r)$ function as $p \to 0$ ($G(p) = \mathrm{O}(p^r)$ for short), if there exist positive constants α and $\bar{p}$ such that

$$\|G(p)\| \le \alpha p^r, \quad \forall p \in (0, \bar{p}).$$

Any linear combination of functions which are $\mathrm{O}(p^r)$ is an $\mathrm{O}(p^r)$ function itself.

By considering the Taylor approximation we can write, for any matrix $F \in \mathbb{R}^{n\times n}$ and any $s \in [0, p]$

$$G(s) = e^{Fs} = I + Fs + \mathrm{O}(p^2) = I + \mathrm{O}(p) \tag{15}$$

where I is the identity matrix.

In (5) the power of the matrix $\Theta(d, p)$ appears. Since k represents the number of switching periods within a given time interval, when p goes to zero k tends to infinity. For the averaging result it is crucial to analyze what happens to the matrix $\Theta(d, p)^k$ when p goes to zero. In particular, the following Lemma holds.

Lemma 2: Consider a finite $T \in \mathbb{R}_+$ multiple of $p \in \mathbb{R}_+$, $\ell(p) = T/p$, a duty cycle $d \in (0, 1)$ and generic Lipschitz continuous matrix functions $\Phi : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $M : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$. Assume that there exists a $\gamma_1 \ge 0$ such that

$$\|\Phi(d, p)\| \le 1 + \gamma_1 p \tag{16a}$$

$$M(d, p) = \mathrm{O}(p^2), \tag{16b}$$

for all $k \in \{0, \ldots, \ell(p)\}$. Then

$$\Phi(d, p)^k = \mathrm{O}(1) \tag{17a}$$

$$(\Phi(d, p) + M(d, p))^k = \Phi(d, p)^k + \mathrm{O}(p). \tag{17b}$$

for all $k \in \{0, \ldots, \ell(p)\}$.

Proof: The proof of this lemma follows as a direct application of Lemma 3 in [8]. ■

We are now ready to prove that the difference between (5) and (14) is of order of the switching period.

Lemma 3: Consider the systems (1) and (10) with the corresponding solutions (5) and (14), respectively. Assume that there exists $\gamma_1 \ge 0$ such that (16a) holds. Then, for any $z_0 = x_0^- + \mathrm{O}(p)$ the following condition

$$x_k^- = z_k + \mathrm{O}(p) \tag{18}$$

holds for all $k \in \{0, \ldots, \ell(p) - 1\}$.

Proof: By using (15) in (4) we can write

$$\begin{aligned}\Theta(d, p) &= G_2((1-d)p)\Pi_2 G_1(dp)\Pi_1\\ &= (I + F_2(1-d)p)\Pi_2(I + F_1 dp)\Pi_1 + \mathrm{O}(p^2)\\ &= \Phi(d, p) + \mathrm{O}(p^2)\end{aligned} \tag{19}$$

By applying Lemma 2 with (16a), one can write

$$\Theta(d, p)^k = \Phi(d, p)^k + \mathrm{O}(p) \tag{20}$$

for all $k \in \{0, \ldots, \ell(p)\}$. By subtracting (14) to (5) one obtains

$$\begin{aligned}x_k^- &= z_k + \Theta(d, p)^k x_0^- - \Phi(d, p)^k z_0\\ &\overset{a}{=} z_k + \Phi(d, p)^k (x_0^- - z_0) + \mathrm{O}(p)\\ &\overset{b}{=} z_k + \mathrm{O}(p)\end{aligned} \tag{21}$$

where in $\overset{a}{=}$ we used (20) and in $\overset{b}{=}$ we used (17a) together with the hypothesis on the initial conditions. ■

The verification of (16a) is not easy to be checked in a formal way. Therefore it is useful to consider the following operative sufficient conditions which allow us to verify the assumption.

Lemma 4: Consider a finite $T \in \mathbb{R}_+$ multiple of $p \in \mathbb{R}_+$, $\ell(p) = T/p$, a duty cycle $d \in (0, 1)$, and the matrices $\Phi(d, p)$ and $\Pi_\cap$ given by (12)–(13). If there exists a symmetric matrix P such that

$$P \succ 0 \tag{22a}$$

$$\Pi_\cap^\top P \Pi_\cap - P \preceq 0 \tag{22b}$$

then (16a) holds for any $k \in \{0, \ldots, \ell(p)\}$ and for all p, with $\|\cdot\|$ being the norm induced by P.

Proof: By using (12) one can write

$$\Phi(d, p) = \Pi_\cap + \mathrm{O}(p) \tag{23}$$

Let us consider the difference equation $\xi_{k+1} = \mathcal{P}_k \xi_k$ with $\xi_k \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $\mathcal{P}_k \in \mathcal{F}$ and $\mathcal{F} = \{\Pi_1, \Pi_2\}$. By using the piecewise quadratic stability based on Lyapunov theory [5] it follows that the existence of a matrix P which solves the LMIs (22) imply that the system is absolutely stable for any sequence of matrices in $\mathcal{F}$, see Sec. 5 in [2]. Then, by using Theorem 3 in [6] it follows that

$$\|\Pi_\cap\| \le 1 \tag{24}$$

with $\|\cdot\|$ being the norm induced by the matrix P. Therefore, by applying such norm to (23) and by using (24), the condition (16a) follows. ■

IV. Approximation of the moving average

In this section we show that solution of the proposed averaged model approximates the moving average of the solution of (1) with a state error which goes to zero when p tends to zero. The section is concluded with a candidate continuous-time averaged model of (1) derived by exploiting the approximation result.

A. Sampled data moving average

It is interesting to compare the variable μ_k given by (10b) with the averaged of the state $x(t)$ computed over the switching period p. The discrete-time average of the state at the multiple kp of the switching period can be written as

$$m_k = \frac{1}{p}\int_{kp}^{(k+1)p} x(t)\mathrm{d}t \tag{25}$$

for all $k \in \{0, \ldots, \ell(p) - 1\}$. The following lemma shows that the difference between (10b) and (25) is of order of the switching period.

Lemma 5: Consider the systems (1) and (10) with the corresponding solutions (5) and (14), respectively and the discrete-time average (25). Assume that there exists $\gamma_1 \ge 0$ such that (16a) holds. Then, for any $z_0 = x_0^- + \mathrm{O}(p)$ the following condition

$$m_k = \mu_k + \mathrm{O}(p) \tag{26}$$

holds for all $k \in \{0, \dots, \ell(p)-1\}$.

Proof: By using (2) in (25) one can write

$$\begin{aligned} pm_k &= \int_0^{dp} G_1(t)\Pi_1 x_k^- \mathrm{d}t + \int_0^{(1-d)p} G_2(t)\Pi_2 x(s_k^-)\mathrm{d}t \\ &= \left[\int_0^{dp} G_1(t)\mathrm{d}t + \int_0^{(1-d)p} G_2(t)\Pi_2 G_1(dp)\mathrm{d}t\right]\Pi_1 x_k^- \end{aligned} \tag{27}$$

for all $k \in \{0, \dots, \ell(p)-1\}$. Then, from (27) by using (15) one has:

$$\begin{aligned} pm_k &= [Id + \Pi_2(1-d)]\, p\Pi_1 x_k^- + \mathrm{O}(p^2) \\ &= \Gamma(d) p x_k^- + \mathrm{O}(p^2). \end{aligned} \tag{28}$$

Then by using Lemma 3 and (10b) it follows that (26) holds. ■

B. Continuous-time moving average

Under some extra conditions on the projectors it can be proved that the continuous-time moving average of the state variable defined as

$$m(t) = \frac{1}{p}\int_t^{t+p} x(\tau)\mathrm{d}\tau \tag{29}$$

for any $t \in [0, T-p]$, remains close to (25). This is proved by the following lemma.

Lemma 6: Consider the systems (1) and (10) with the corresponding solutions (5) and (14), respectively, the discrete-time average (25) and the continuous-time moving average (29). Consider (13a) and assume that there exists an $\alpha > 0$ such that

$$\|\Pi_\cap^k - \Pi_\cap^{k-1}\| \le \frac{\alpha}{k^2} \tag{30}$$

for all $k \in \mathbb{N}$. Then the following condition

$$m(t) = m_k + \mathrm{O}(p) \tag{31}$$

holds for any $t \in (0, T-p]$ where $k = \left\lfloor \frac{t}{p} \right\rfloor$.

Proof: Since $m(t) = m_k$ for any $t = t_k = kp$, $k \in \{0, \dots, \ell(p)-1\}$, in the time instants multiple of the switching period the condition (31) is trivially satisfied.

Let us consider the moving average over a time interval of length p which starts in the first mode. For any $t \in [t_k, s_k]$, $k \in \{0, \dots, \ell(p)-1\}$, $\tau_1 = t - t_k$, i.e. $\tau_1 \in [0, dp]$, by using (2) in (29) one can write

$$\begin{aligned} pm(t) = &\int_\tau^{dp} G_1(\xi)\Pi_1 x_k^- \mathrm{d}\xi \\ &+ \int_0^{(1-d)p} G_2(\xi)\Pi_2 G_1(dp)\Pi_1 x_k^- \mathrm{d}\xi \\ &+ \int_0^{\tau_1} G_1(\xi)\Pi_1 x_{k+1}^- \mathrm{d}\xi \end{aligned} \tag{32}$$

By taking the difference between (27) and (32) one obtains

$$\begin{aligned} p(m(t) - m_k) &= \int_0^{\tau} G_1(\xi)\Pi_1(x_{k+1}^- - x_k^-)\mathrm{d}\xi \\ &= \Pi_1(\Pi_\cap - I)\tau_1 x_k^- + \mathrm{O}(p^2) \end{aligned} \tag{33}$$

for any $t \in [t_k, s_k]$, $\tau_1 = t - t_k$, for all $k \in \{0, \dots, \ell(p)-1\}$.

Let us consider the moving average over a time interval of length p which starts in the second mode. For any $t \in [s_k, t_{k+1}]$, $k \in \{0, \dots, \ell(p)-1\}$, $\tau_2 = t - s_k$, i.e. $\tau_2 \in [0, (1-d)p]$, one can write

$$\begin{aligned} pm(t) = &\int_{\tau_2}^{(1-d)p} G_2(\xi)\Pi_2 G_1(dp)\Pi_1 x_k^- \mathrm{d}\xi \\ &+ \int_0^{\tau_2} G_2(\xi)\Pi_2 G_1(dp)\Pi_1 x_{k+1}^- \mathrm{d}\xi \\ &+ \int_0^{dp} G_1(\xi)\Pi_1 x_{k+1}^- \mathrm{d}\xi \\ =&[\Pi_\cap((1-d)p - \tau_2) + \Pi_\cap \tau_2 \\ &+ \Pi_1\Pi_\cap dp]x_k^- + \mathrm{O}(p^2) \end{aligned} \tag{34}$$

By taking the difference between (27) and (34) one obtains

$$\begin{aligned} p(m(t) - m_k) = &\int_0^{dp} G_1(\xi)\Pi_1(x_{k+1}^- - x_k^-)\mathrm{d}\xi \\ &+ \int_0^{\tau_2} G_2(\xi)\Pi_2 G_1(dp)\Pi_1(x_{k+1}^- - x_k^-)\mathrm{d}\xi \\ =&(\Pi_2\tau_2 + Idp)\Pi_1(\Pi_\cap - I)x_k^- + \mathrm{O}(p^2) \end{aligned} \tag{35}$$

for any $t \in [s_k, t_{k+1}]$, $\tau_2 = t - s_k$, i.e. $\tau_2 \in [0, (1-d)p]$, for all $k \in \{0, \dots, \ell(p)-1\}$.

By using (5) and (20), the expression (33) can be rewritten as

$$p(m(t) - m_k) = \Pi_1\tau_1(\Pi_\cap - I)\Phi(d,p)^k x_0^- + \mathrm{O}(p^2) \tag{36}$$

for any $t = \tau_1 + t_k$, $k \in \{0, \dots, \ell(p)-1\}$, $\tau_1 \in [0, dp]$, and (34) can be rewritten as

$$\begin{aligned} p(m(t) - m_k) = &(\Pi_2\tau_2 + Id_1 p)\Pi_1(\Pi_\cap - I)\Phi(d,p)^k x_0^- \\ &+ \mathrm{O}(p^2) \end{aligned} \tag{37}$$

for any $t = \tau_2 + s_k$, $k \in \{0, \dots, \ell(p)-1\}$, $\tau_2 \in [0, (1-d)p]$.

By using (12) one can write

$$\Phi(d,p)^k = \Pi_\cap^k + \sum_{i=1}^k \Pi_\cap^{i-1}\Lambda(d)\Pi_\cap^{k-i} p + k\mathrm{O}(p^2) \tag{38}$$

By considering the first term in the right hand side of (38), in (36) and (37) it will appear the term $(\Pi_\cap - I)\Pi_\cap^k = \Pi_\cap^{k+1} - \Pi_\cap^k$. By taking the norms and by using (30) one obtains:

$$\|(\Pi_\cap - I)\Pi_\cap^k\| \le \frac{\alpha}{(k+1)^2} \tag{39}$$

for all $k \in \{0, \dots, \ell(p)-1\}$. Moreover, by considering the second term in the right hand side of (38), it is

$$(\Pi_\cap - I)\sum_{i=1}^k \Pi_\cap^{i-1}\Lambda(d)\Pi_\cap^{k-i} = \sum_{i=1}^k (\Pi_\cap^i - \Pi_\cap^{i-1})\Lambda(d)\Pi_\cap^{k-i}. \tag{40}$$

By taking the norms on both sides of (40) one obtains:

$$\begin{aligned}
&\|(\Pi_\cap - I)\sum_{i=1}^{k}\Pi_\cap^{i-1}\Lambda(d)\Pi_\cap^{k-i}\| \\
&\quad\le \|\Lambda(d)\|\sum_{i=1}^{k}\|\Pi_\cap^{i}-\Pi_\cap^{i-1}\|\,\|\Pi_\cap^{k-i}\| \\
&\quad\le \|\Lambda(d)\|\sum_{i=1}^{k}\frac{\alpha}{i^2}\,\|\Pi_\cap^{k-i}\| \le \frac{1}{6}\alpha\beta\pi^2\|\Lambda(d)\| \qquad (41)
\end{aligned}$$

where we used that $\|\Pi_\cap^k\| \le \beta := 1+\sum_{i=1}^{\infty}\frac{\alpha}{i^2} < \infty$ for all $k \in \{0,\dots,\ell(p)-1\}$, which follows from $\|\Pi_\cap^k - I\| = \|\sum_{i=1}^{k}(\Pi_\cap^i - \Pi_\cap^{i-1})\|$ and (30).

We have by (38), (39) (for $k = \lfloor \frac{t}{p} \rfloor$) and (41) that

$$(\Pi_\cap - I)\Phi(d,p)^k = O(p^2) + O(p) = O(p)$$

Dividing (36) and (37) by p and taking $\tau_1 = \mathrm{O}(p)$ and $\tau_2 = \mathrm{O}(p)$ into account we arrive at (31). ■

We can now prove that the error between the continuous moving average of the state behaviour related to the system of interest and the output of the discrete model is of order of the switching period.

Theorem 7: Consider the switched system (1) and the discrete average (10), by assuming that there exist the constants $\gamma_1 \ge 0$ and $\alpha > 0$ such that (16a) and (30) holds, then for all $z_0 = x_0^- + \mathrm{O}(p)$ the following condition

$$m(t) = \mu_k + \mathrm{O}(p) \qquad (42)$$

for any $t \in (0, T-p]$ and any $k \in \{0,\dots,\ell(p)-1\}$ with $k = \left\lfloor \frac{t}{p} \right\rfloor$ holds.

Proof: The proof directly follows by applying in sequence Lemma 6 and Lemma 5. ■

C. Continuous-time model

Theorem 7 shows that the discrete-time model (10) provides and approximation of the moving average of the solution of the switched impulsive system (1). One may be interested to a continuous-time averaged model. In order to conjecture a structure for this model one can consider the following system

$$\dot{\tilde{z}}(t) = A(d,p)\tilde{z}(t), \quad t \in \mathbb{R}_+ \qquad (43a)$$

$$\mu(t) = \Gamma(d)\tilde{z}(t) \qquad (43b)$$

with $\tilde{z}(0) = z_0$ initial condition, the dynamic matrix

$$A(d,p) = \frac{1}{p}\left(\Phi(d,p) - I\right) \qquad (44)$$

where $\Phi(d,p)$ is given by (12)–(13), and $\Gamma(d)$ is given by (11).

A motivation for the choice of the matrices in (43) can be obtained by discretizing the model (43) with the forward Euler method. The expression (44) clearly shows that this discretization procedure leads to the discrete-time averaged model (10). A further motivation is the fact that the models (10) and (43) have the same steady state subspaces.

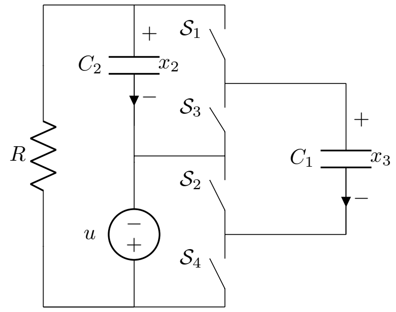

Fig. 1. Elementary cell of a ladder step-up switched capacitor converter.

The model (43) has an interesting interpretation. Indeed, in the particular case $\Pi_i = I$ for $i = 1, 2$, it is $\Phi(d,p) = I + [F_1 d + F_2(1-d)]p$ which implies $A(d) = F_1 d + F_2(1-d)$ and $\Gamma(d) = I$, i.e. the model (43) reduces to the classical continuous-time averaged model for switched ODE. Unfortunately, in the case of switched impulsive ODE to found a bound on the error $\mu(t) - \mu_k$ is a nontrivial issue because the dynamic matrix of $A(d,p)$ of the continuous-time system (43) depends on p which is the sampling period used for the discretization. The solution of this problem is left for future investigations.

V. Example

In this section we consider a practical switched capacitor electrical circuit, shown in Fig. 1.

The circuit represents the typical elementary cell of a ladder step-up switched capacitor and it consists of two capacitors and four switches that are controlled in a complementary way. Then the modes of the system are two, corresponding to the pair $\{\mathcal{S}_1, \mathcal{S}_2\}$ turned on together with the pair $\{\mathcal{S}_3, \mathcal{S}_4\}$ turned off and viceversa.

By considering as input a constant voltage source u, say x_1, the circuit can be modeled with x_2 and x_3 being the state variables corresponding to the voltages on the capacitors C_1 and C_2, respectively. Then the matrices pairs of the two modes together with the input matrices are:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_2R & C_1R \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 0 & C_2\rho & C_1\rho \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_2R & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \quad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where $\rho = \frac{1}{C_1 + C_2}$. It is easy to verify that $\Pi_1\Pi_2 \ne \Pi_2\Pi_1$, but there exists a common matrix that satisfies (22).

The simulation has been carried out by selecting the following parameters: $C_1 = C_2 = 120\,\mu\mathrm{F}$, $R = 10\,\mathrm{k}\Omega$ and $\Delta_p = 0.9p$.

In Fig. 2 and Fig. 3 is shown the behavior of the state variables x_2 and x_3 for different switching periods, i.e. $p = 0.05\,\mathrm{s}$ and $p = 0.1\,\mathrm{s}$ respectively, over a time interval of $1\,\mathrm{s}$. The discrete model well approximates the switching behavior (blue lines); indeed the error between the moving average and the discrete switching evolution decreases by reducing the switching period. It can be noted that during the first periods we have jumps whose amplitude is not related

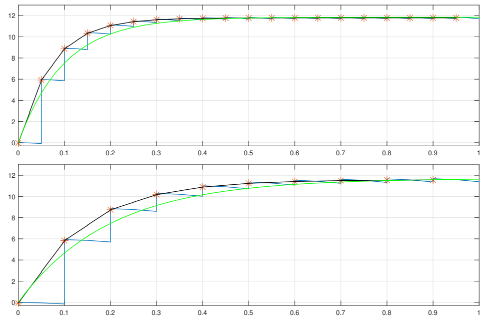

Fig. 2. Time evolution of the first state variables of the switched capacitor circuit with $p = 0.05\,\text{s}$ (top) and $p = 0.1\,\text{s}$ (bottom): switched DAE system (blue lines), proposed averaged model (green lines), discrete model (red stars), moving average (dark lines).

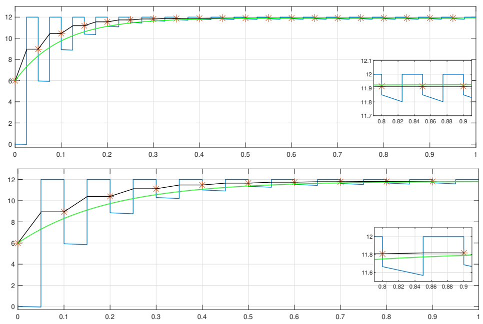

Fig. 3. Time evolution of the second state variables of the switched capacitor circuit with $p = 0.05\,\text{s}$ (top) and $p = 0.1\,\text{s}$ (bottom): switched DAE system (blue lines), proposed averaged model (green lines), discrete model (red stars), moving average (dark lines).

to the decreasing of the switching period. Nevertheless the discrete model and especially the continuous model are still able to catch the averaged behavior of the switched system. Furthermore the validity of the proposed continuous model is evident; indeed a good approximation between the continuous model and the moving average by increasing the simulation time, i.e. the number of switching period analyzed, is shown.

VI. Conclusion

In this paper a new averaged model for a class of switched system that present state jumps at the switching time instants has been presented. This general result do not require strictly condition as the one in the previous literature and it can be applied to practical electronics circuits, among the others power converters, that can be model by switched descriptor systems; indeed switched DAEs belongs to the class of switched impulsive system. In this paper, we have considered a class of homogeneous switched system with two modes and constant duty cycle. A circuit with two parallel capacitors has been used to validate the results.

Future work will be the formal proof of the continuous averaged model whose idea has been presented and numerical validated in this paper. Moreover this result will be extended to the case of more than two modes and by considering duty cycle different for each mode. A further step will be the analysis with state-dependent duty cycles.

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