Edge-wise funnel output synchronization of heterogeneous agents with relative degree one[⋆]

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Abstract

In a recent work by three of the authors, in order to enforce synchronization for scalar heterogeneous multi-agent systems with some useful characteristics, a *node-wise* funnel coupling law was proposed. The emergent dynamics, to which each of the agents synchronizes, was characterized and it was studied how networks can be synthesized which exhibit these emergent dynamics. The advantage of this synthesis is its suitability for plug-and-play operation. However, the aforementioned emergent dynamics under node-wise funnel coupling are determined by an algebraic equation which does not admit an explicit solution in general, and even its pointwise solution proves rather difficult. Furthermore, the contractivity assumption on the emergent dynamics, required to establish the synchronization, is hard to be checked without solving the algebraic equation. To resolve these drawbacks, in the present paper we present a new funnel coupling law that uses *edge-wise* output differences. Under this novel coupling the benign properties of node-wise funnel coupling are retained, but the emergent dynamics are given explicitly by the blended dynamics of the multi-agent system, which already proved an advantageous tool in the analysis and design of such networks. Additionally, our results are not restricted to scalar systems and treat the case that neighboring agents only communicate their output information, and not their complete state.

Keywords: Synchronization, heterogeneous multi-agents, blended dynamics, funnel control

1. Introduction

In the recent work [1], arbitrary precision approximate synchronization for scalar heterogeneous multi-agent systems

$$\dot{x}_i(t) = f_i(t, x_i(t)) + u_i(t), \quad i \in \mathcal{N}, \tag{1}$$

under a so-called *node-wise funnel coupling law*

$$u_i(t) = \mu_i\left(\frac{\nu_i(t)}{\psi_i(t)}\right), \quad \nu_i(t) = \sum_{j\in\mathcal{N}_i}(x_j(t) - x_i(t)), \tag{2}$$

was studied. Here, $\mathcal{N} := \{1, \ldots, N\}$ is the set of agent indices, the number of agents is N, $\mathcal{N}_i \subseteq \mathcal{N}$ is the set of agents that send information to agent i, and $f_i : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is sufficiently smooth. As design parameters in the coupling law, the coupling function $\mu_i : (-1, 1) \to \mathbb{R}$, which satisfies $\lim_{s\to\pm1} \mu_i(s) = \pm\infty$, and the performance function $\psi_i : [t_0, \infty) \to \mathbb{R}_{>0}$ can be used to achieve the following prescribed behavior of the diffusive coupling terms:

$$\forall\, i \in \mathcal{N}\ \forall\, t \geq t_0 :\ |\nu_i(t)| < \psi_i(t). \tag{3}$$

The present brief paper is devoted to an extension of the results of [1] and to overcome some of its limits. In this spirit, we skip an extensive literature review on synchronization of multi-agent systems and simply refer to the precursor [1]. Similarly, some of the proof techniques used here will resemble those in [1] in some parts, which we will skip likewise, but refer the readers to the unabridged version of the paper on arXiv.

1.1. Benefits and limits of node-wise funnel coupling

The coupling law (2) exhibits the following benign characteristics.

- It can be used in a fully decentralized manner.
- It does not require additional stability or synchronizability conditions for every *individual agents*.
- It only utilizes the information of the diffusive coupling term $\nu_i(t)$.
- The synchronization performance can be prescribed both for the transient and the steady-state behavior.

Under a set of mild assumptions, it was shown in [1] that (3) holds and that this implies the desired synchronization performance

$$\forall\, i, j \in \mathcal{N}\ \forall\, t \geq t_0 :\ |x_i(t) - x_j(t)| \leq \frac{2\sqrt{N}}{\lambda_2} \max_{l\in\mathcal{N}} \psi_l(t),$$

where λ_2 is the algebraic connectivity of the interconnection graph of the multi-agent system. Furthermore, we may observe the following additional characteristics.

- Asymptotic synchronization can be achieved even without the common internal model assumption by choosing ψ_i such that $\lim_{t\to\infty} \psi_i(t) = 0$, $i \in \mathcal{N}$.

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- Finite-time synchronization can be achieved by choosing ψ_i such that $\psi_i(t) > 0$ for $t \in [t_0, t_0 + T)$ and $\psi_i(t_0 + T) = 0$ with some $T > 0$, for all $i \in \mathcal{N}$.
- Pseudo-global convergence can be achieved by choosing ψ_i such that $\lim_{t \to t_0} \psi_i(t) = \infty$, for instance with $\psi_i(t) = 1/(t - t_0)$.

The emergent dynamics was shown to be determined by an algebraic equation in [1], and under a contractivity assumption each of the agents synchronizes to its solution. It was further studied how networks can be synthesized which exhibit these emergent dynamics. Characterization of emergent behavior can be used for the synthesis of a heterogeneous network with some specific purposes. For instance, one can design the emergent behavior with the desired characteristics, and then provide a design guideline for each agent so that the designed vector field and/or the coupling function yield the desired emergent behavior. This method of designing a heterogeneous network with the desired collective behavior was introduced in [2]. The networks designed in this way have beneficial properties such as that

- the collective behavior does not depend on the initial conditions, hence the network is amenable to plug-and-play operation, i.e., agents can join or leave online,
- each individual agent may be unstable, for instance, malfunctioning or even malicious, as long as their combination, i.e., the emergent dynamics, is stable,
- and the collective behavior is robust against disturbances, noise, and uncertainty in the vector field.

Although the above properties are quite convincing, the difficulty lies in determining and investigating the emergent dynamics. As mentioned above, they are determined by an algebraic equation, and this equation does not admit an explicit solution in general; even its pointwise solution proves rather difficult. Furthermore, the contractivity assumption on the emergent dynamics is hard to be checked a priori, i.e., without solving the algebraic equation. Additionally, to make the network synchronously behaves as the designed emergent behavior it is required that the performance functions are sufficiently narrow, which may depend on some global information such as the algebraic connectivity or the number of agents, thus preventing the use of the node-wise funnel coupling law in a fully decentralized manner.

1.2. Contribution of the present paper

The main purpose of the present paper is to present a novel funnel coupling law which uses edge-wise output differences instead of the node-wise coupling terms in (2), and it achieves that

- all the benign properties of the node-wise funnel coupling law are retained,
- the emergent dynamics are given explicitly by the blended dynamics, which is a generalization of the blended dynamics for the scalar system (1) given by

$$\dot{\hat{s}}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, \hat{s}(t)), \quad \hat{s}(t_0) = \frac{1}{N}\sum_{i=1}^{N} x_i(t_0),$$

- and the designed emergent behavior can be realized without additional constraints on the performance function.

We emphasize that, as shown in [2], the blended dynamics characterize the emergent behavior under linear diffusive coupling, and already proved advantageous in various control design problems and for the analysis of properties such as the robustness against uncertainties or malicious agents. Compared to the emergent dynamics under node-wise funnel coupling, the blended dynamics are given directly as the average of the individual agent dynamics and stability assumptions are thus easier to check a priori. Furthermore, the task of designing a heterogeneous network with a desired collective behavior is much simpler, since it is not necessary to solve a complicated algebraic equation.

Additional extensions to the results of [1], which are also applicable to, but not studied in [1], are as follows:

- We consider multi-input, multi-output (MIMO) systems.
- We consider non-trivial internal dynamics.
- We investigate the feasibility of plug-and-play operation.

The idea of an edge-wise funnel coupling law was first proposed in [3] and the specific use of this design to solve distributed consensus optimization can be found in [4]. Both this novel coupling law and the node-wise funnel coupling law (2) are inspired by the funnel control introduced in [5]; see also the recent works [6, 7] and the literature review therein. We note that the problem of dynamic average consensus, where its goal is for each agent to follow the average of the given time-varying signals, which is known only to each agent, has been solved in a similar manner in [8] using the prescribed performance control methodology (which is related to the funnel control).

1.3. Organization of the present paper

In Section 2, we present the problem statement in an extended setting, introduce the novel edge-wise funnel coupling law, and show that it achieves synchronization with a prescribed performance of the edge-wise output differences. In Section 3 we show that, under a mild input-to-state stability assumption, each agent synchronizes to the blended dynamics of the system, which thus represents the emergent behavior. Section 4 provides an illustration of the plug-and-play operation by a simulation, and some conclusions are given in Section 5. The Appendix contains the proofs of the main results and some preparatory lemmas.

2. Edge-wise funnel coupling law

In the present paper, we consider a heterogeneous multi-agent system given by

$$\begin{aligned} \dot{\mathbf{x}}_i(t) &= \mathbf{F}_i(t, \mathbf{x}_i(t)) + \mathbf{G}_i(t, \mathbf{x}_i(t)) \cdot \mathbf{u}_i(t), \quad i \in \mathcal{N}, \\ \mathbf{y}_i(t) &= \mathbf{H}_i(\mathbf{x}_i(t)). \end{aligned} \tag{4}$$

Here, the internal state at time $t \in \mathbb{R}$ is represented by $\mathbf{x}_i(t) \in \mathbb{R}^{n_i}$ with (agent-dependent) state-dimension $n_i \in \mathbb{N}$, $\mathbf{u}_i(t) \in \mathbb{R}^m$ is the control input, and $\mathbf{y}_i(t) \in \mathbb{R}^m$ is the output of agent i with (agent-independent) dimension $m \leq n_i$, $\forall i \in \mathcal{N}$.

Assumption 1 (open loop dynamics). *For each $i \in \mathcal{N}$, the functions $\mathbf{F}_i : [t_0,\infty) \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $\mathbf{G}_i : [t_0,\infty) \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m}$ are measurable in t, locally Lipschitz continuous with respect to $\mathbf{x}_i$, and bounded on each compact subset of $\mathbb{R}^{n_i}$ uniformly in $t \in [t_0,\infty)$. The function $\mathbf{H}_i : \mathbb{R}^{n_i} \to \mathbb{R}^m$ is continuously differentiable.*

Note that in this paper, solutions of differential equations such as (4) are considered in the sense of Carathéodory. Furthermore, under Assumption 1 and for a coupling $\mathbf{u}_i(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ that preserves the properties of $\mathbf{F}_i$ and $\mathbf{G}_i$, the corresponding closed-loop system of (4) (and similar equations) will have unique (local) solutions. Throughout the paper, when speaking of solutions, we will always mean the unique Carathéodory solution.

In order to utilize the funnel control methodology, we require that the dynamics of each agent are globally equivalent to a system with strict relative degree one, as stated in the following.

Assumption 2 (normal form). *For each $i \in \mathcal{N}$, there exists a diffeomorphism $\Phi_i : \mathbf{x}_i \mapsto (\mathbf{y}_i,\mathbf{z}_i)$ with $\mathbf{y}_i = \mathbf{H}_i(\mathbf{x}_i) \in \mathbb{R}^m$ and $\mathbf{z}_i \in \mathbb{R}^{n_i-m}$ such that, by $(\mathbf{y}_i,\mathbf{z}_i) = \Phi_i(\mathbf{x}_i)$, (4) is transformed into*

$$\begin{aligned}\dot{\mathbf{y}}_i(t) &= \widetilde{\mathbf{F}}_i(t,\mathbf{y}_i(t),\mathbf{z}_i(t)) + \mathbf{\Gamma}_i(t,\mathbf{y}_i(t),\mathbf{z}_i(t)) \cdot \mathbf{u}_i(t),\\ \dot{\mathbf{z}}_i(t) &= \mathbf{Z}_i(t,\mathbf{z}_i(t),\mathbf{y}_i(t)),\end{aligned}$$

for suitable $\widetilde{\mathbf{F}}_i$, $\mathbf{\Gamma}_i$, and $\mathbf{Z}_i$.

For later use, we define

$$\overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i) := \nabla\mathbf{H}_i(\mathbf{x}_i)\mathbf{G}_i(t,\mathbf{x}_i), \quad \mathbf{x}_i \in \mathbb{R}^{n_i}, \tag{5}$$

then, under Assumption 2, $\mathbf{\Gamma}_i$ has the form

$$\mathbf{\Gamma}_i(t,\mathbf{y}_i,\mathbf{z}_i) = \overline{\mathbf{\Gamma}}_i(t,\Phi_i^{-1}(\mathbf{y}_i,\mathbf{z}_i)).$$

Remark 1. *To be precise, Assumption 2 differs from the typical property of relative degree one as given in [9]. In the case that $\mathbf{G}_i$ is time-invariant, system (4) has uniform (strict) relative degree one in the virtue of [9], if $\overline{\mathbf{\Gamma}}_i(\mathbf{x}_i)$ from (5) is invertible for all $\mathbf{x}_i \in \mathbb{R}^{n_i}$. If additionally $\mathbf{H}_i^{-1}(0)$ is diffeomorphic to $\mathbb{R}^{n_i-m}$ and for $\tilde{\mathbf{G}}_i := \mathbf{G}_i\,\overline{\mathbf{\Gamma}}_i^{-1}$ its columns $\tilde{g}_i^j$ are complete vector fields so that their pairwise Lie bracket satisfies $[\tilde{g}_i^j(\mathbf{x}_i),\tilde{g}_i^k(\mathbf{x}_i)] = \nabla\tilde{g}_i^j(\mathbf{x}_i)\tilde{g}_i^k(\mathbf{x}_i) - \nabla\tilde{g}_i^k(\mathbf{x}_i)\,\tilde{g}_i^j(\mathbf{x}_i) = 0$ for all $1 \le j,k \le m$ and all $\mathbf{x}_i \in \mathbb{R}^{n_i}$, then [9, Cor. 5.7] yields the existence of a global diffeomorphism Φ_i as in Assumption 2.*

Remark 2. *Note that Assumption 2 inherently excludes the case where the images of the output maps $\mathbf{H}_i$, $i \in \mathcal{N}$, have no common element, i.e., $\cap_{i\in\mathcal{N}}\mathbf{H}_i(\mathbb{R}^{n_i}) = \emptyset$, which would prevent output synchronization.*

The required invertibility of $\overline{\mathbf{\Gamma}}_i$ is captured in the following assumption, together with boundedness of its inverse, which is necessary to infer boundedness of the control inputs of each agent under edge-wise funnel coupling.

Assumption 3 (gain matrix). *For each $i \in \mathcal{N}$, the gain matrix $\overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i)$ in (5) is known and available for the design of the coupling law, it is invertible for all $t \ge t_0$ and all $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and its inverse is uniformly bounded, i.e., there exists $M_{\mathbf{\Gamma}} > 0$ such that $\|\overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i)^{-1}\|_\infty \le M_{\mathbf{\Gamma}}$ for all $t \ge t_0$ and all $\mathbf{x}_i \in \mathbb{R}^{n_i}$.*

Under the above assumptions, we propose for each $i \in \mathcal{N}$ the *edge-wise* funnel coupling law

$$\begin{aligned}\mathbf{u}_i(t) &= \mathbf{u}_i\big(t,\overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i),\{\boldsymbol{\nu}_{ij}\}\big) = \overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i)^{-1}\sum_{j\in\mathcal{N}_i}\mathbf{u}_{ij}(t,\boldsymbol{\nu}_{ij}),\\ \mathbf{u}_{ij}(t,\boldsymbol{\nu}_{ij}) &= \mathrm{col}\left(\mu_{ij}^1\left(\frac{\nu_{ij}^1(t)}{\psi_{ij}^1(t)}\right),\ldots,\mu_{ij}^m\left(\frac{\nu_{ij}^m(t)}{\psi_{ij}^m(t)}\right)\right),\end{aligned} \tag{6}$$

where $\boldsymbol{\nu}_{ij} := \mathbf{y}_j - \mathbf{y}_i = \mathrm{col}(\nu_{ij}^1,\ldots,\nu_{ij}^m)$ and the functions ψ_{ij}^p and μ_{ij}^p satisfy assumptions given below.

Assumption 4 (communication graph). *The communication graph $\mathcal{G} = (\mathcal{N},\mathcal{E})$ induced by the neighborhoods $\mathcal{N}_i$ for $i \in \mathcal{N}$ (i.e., $\mathcal{N}$ is the set of nodes and $(j,i) \in \mathcal{E}$ if, and only if, $j \in \mathcal{N}_i$) is undirected and connected.*[1]

For the basics of graph theory we refer to [10]; some specific lemmas required for the proofs of the main results can also be found in Appendix A.

Assumption 5 (design parameters for coupling). *For each edge $(j,i) \in \mathcal{E}$ and index $p \in \mathcal{M} := \{1,\ldots,m\}$, the performance function $\psi_{ij}^p : [t_0,\infty) \to \mathbb{R}_{>0}$ is bounded and differentiable with bounded derivative; there are $\overline{\psi} > 0$ and $\theta_\psi > 0$ such that*

$$\forall\, t \in [t_0,\infty): \quad 0 < \psi_{ij}^p(t) \le \overline{\psi} \quad \textit{and} \quad \left|\dot{\psi}_{ij}^p(t)\right| \le \theta_\psi.$$

The coupling function $\mu_{ij}^p : (-1,1) \to \mathbb{R}$ satisfies $\lim_{s\to\pm1}\mu_{ij}^p(s) = \pm\infty$. Finally, we have symmetry, i.e., $\psi_{ij}^p(t) = \psi_{ji}^p(t)$ for all $t \ge t_0$ and $\mu_{ij}^p(-s) = -\mu_{ji}^p(s)$ for all $s \in (-1,1)$.

The performance functions ψ_{ij}^p in the coupling law (6) reflect the two objectives of ν_{ij}^p approaching zero with prescribed transient behavior and asymptotic accuracy, that is

$$\forall\, t \ge t_0\ \forall\,(j,i) \in \mathcal{E}\ \forall\, p \in \mathcal{M}: \ \left|\nu_{ij}^p(t)\right| < \psi_{ij}^p(t). \tag{7}$$

We particularly allow for $\lim_{t\to\infty}\psi_{ij}^p(t) = 0$, which means that asymptotic synchronization can be achieved.[2] Furthermore, we stress that the choice of the functions ψ_{ij}^p is completely up to the designer. While it is often convenient to adopt a monotonically shrinking funnel (through the choice of monotonically decreasing functions ψ_{ij}^p), it might be beneficial to widen the funnel over some later time intervals to accommodate, e.g., periodic disturbances or joining agents during plug-and-play operation.

Note that the control input $\mathbf{u}_i$ in (6) only requires the information of the output difference terms $\boldsymbol{\nu}_{ij}(t)$ with the neighboring agents, $j \in \mathcal{N}_i$, and the gain matrix $\overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i) = \nabla\mathbf{H}_i(\mathbf{x}_i)\mathbf{G}_i(t,\mathbf{x}_i)$, but does not directly use the information of the outputs $\mathbf{y}_j$, $j \in \mathcal{N}_i$ and $\mathbf{y}_i$, nor the

[1] Different from the literature, in the present paper edges $(j,i) \in \mathcal{E}$ always have a direction (from node j to node i), and a graph is undirected, if for any $(j,i) \in \mathcal{E}$ we also have $(i,j) \in \mathcal{E}$.

[2] However, in this case we have $\lim_{t\to\infty}\nu_{ij}^p(t) \to 0$ as well and the coupling law (6) thus contains the quotient of these two "infinitesimally small" terms. Therefore, the case of asymptotic synchronization seems to be of limited practical utility; similar to asymptotic tracking by funnel control, cf. [7, Rem. 1.7].

state $\mathbf{x}_i$. Beyond that, knowledge of the diffeomorphisms $\Phi_i(\cdot)$, the vector fields $\widetilde{\mathbf{F}}_i$ and $\mathbf{Z}_i$ (or $\mathbf{F}_i$) is not required.

Due to the symmetry requirement in Assumption 5, new agents may have to communicate with their neighbors once when they join the network. We emphasize that, even with this possibility of local communication, edge-wise funnel coupling can be still used in a fully decentralized manner.

Assumptions 4 and 5 are crucial in the recovery of the blended dynamics as the emergent collective behavior of the network under edge-wise funnel coupling. In particular, the strong nonlinearity introduced in $\mathbf{u}_i$ by μ_{ij}^p is not present in the time derivative of the averaged variable $\mathbf{s}(t) := (1/N)\sum_{i=1}^N \mathbf{y}_i(t)$, as

$$\begin{aligned}\dot{\mathbf{s}}(t) &= \frac{1}{N}\sum_{i=1}^N \Big[\widetilde{\mathbf{F}}_i(t,\mathbf{y}_i(t),\mathbf{z}_i(t)) + \overline{\mathbf{\Gamma}}_i(t,\mathbf{x}_i(t))\cdot \mathbf{u}_i(t)\Big] \\ &= \frac{1}{N}\sum_{i=1}^N \widetilde{\mathbf{F}}_i(t,\mathbf{y}_i(t),\mathbf{z}_i(t))\end{aligned}$$

where the second term cancels out because of the symmetry. Now, denoting the synchronization error by $\mathbf{e}_i(t) := \mathbf{y}_i(t) - \mathbf{s}(t)$, we have

$$\begin{aligned}\dot{\mathbf{s}}(t) &= \frac{1}{N}\sum_{i=1}^N \widetilde{\mathbf{F}}_i(t,\mathbf{s}(t)+\mathbf{e}_i(t),\mathbf{z}_i(t)), \\ \dot{\mathbf{z}}_i(t) &= \mathbf{Z}_i(t,\mathbf{z}_i(t),\mathbf{s}(t)+\mathbf{e}_i(t)), \qquad i\in\mathcal{N},\end{aligned} \tag{8}$$

which becomes exactly the blended dynamics introduced in [2] when $\mathbf{e}_i \equiv 0$ for all $i \in \mathcal{N}$.

Therefore, if synchronization with prescribed performance as in (7) is achieved (similar to (3) in the case of node-wise funnel coupling), then $\mathbf{e}_i$ also evolves within a prespecified error margin, as we have

$$\forall\, t \ge t_0\ \forall\, i,j \in \mathcal{N}:\ \|\mathbf{y}_i(t) - \mathbf{y}_j(t)\|_\infty \le d_{\mathcal{G}}\Psi(t), \tag{9}$$

where $d_{\mathcal{G}}$ is the diameter of the communication graph $\mathcal{G} = (\mathcal{N},\mathcal{E})$ and $\Psi(t) := \max_{p\in\mathcal{M}} \max_{(j,i)\in\mathcal{E}} \psi_{ij}^p(t) \le \overline{\psi}$.[3]

The synchronization objective (7) is obtained under the additional assumption that solutions of (8) do not exhibit a finite escape time.

Assumption 6 (no finite escape time). *For any initial time t_0, the perturbed blended dynamics (8) with any absolutely continuous inputs $\mathbf{e}_i : [t_0,\infty) \to \mathbb{R}^m$, $i \in \mathcal{N}$, such that $\|\mathbf{e}_i(t)\|_\infty \le d_{\mathcal{G}}\overline{\psi}$ for all $t \ge t_0$, has a global solution for any initial values $\mathbf{s}(t_0) \in \mathbb{R}^m$, $\mathbf{z}_i(t_0) \in \mathbb{R}^{n_i-m}$, $i \in \mathcal{N}$.*

We stress that if the functions $\widetilde{\mathbf{F}}_i$ and $\mathbf{Z}_i$ are globally Lipschitz in their arguments, then Assumption 6 holds. Note that Assumption 6 is a stability condition on the emergent behavior, in a very relaxed sense.

Lemma 1. *Under Assumptions 1–6, assume that the solution of system (4) with (6) exists on $[t_0,\omega)$ for some $\omega > t_0$ and satisfies $|\nu_{ij}^p(t)| < \psi_{ij}^p(t)$ for all $t \in [t_0,\omega)$, $(j,i) \in \mathcal{E}$, and $p \in \mathcal{M}$. Then $\mathbf{x}_i$ is bounded on $[t_0,\omega)$ for all $i \in \mathcal{N}$.*

[3] The diameter of a graph $\mathcal{G}$ is the maximum length among the shortest paths between any two nodes.

The proof of Lemma 1 is a direct consequence of the representation (8) and Assumption 6, where $\|\mathbf{e}_i(t)\|_\infty \le d_{\mathcal{G}}\overline{\psi}$, $t \in [t_0,\omega)$, is guaranteed by (9); the details are omitted.

Theorem 1. *Consider the system (4) with the edge-wise funnel coupling law (6). Under Assumptions 1–6, if the initial values $\mathbf{x}_i(t_0)$ of (4) and the performance functions ψ_{ij}^p satisfy $|\nu_{ij}^p(t_0)| < \psi_{ij}^p(t_0)$ for all $(j,i) \in \mathcal{E}$ and $p \in \mathcal{M}$, then there exists a global solution $(\mathbf{x}_1,\ldots,\mathbf{x}_N) : [t_0,\infty) \to \mathbb{R}^{n_1+\ldots+n_N}$ of (4) and (6) which satisfies the synchronization objective (7).*

The proof is relegated to Appendix B. Note that, under the assumptions of Theorem 1, the inequality (9) holds, and thus, approximate (when $\limsup_{t\to\infty}\Psi(t) > 0$ is small) or asymptotic (when $\lim_{t\to\infty}\Psi(t) = 0$) output synchronization is achieved.

In virtue of Theorem 1, the multi-agent system (4) under edge-wise funnel coupling (6) is amenable to plug-and-play operation. Agents can always leave the network (which, however, may decompose the network into several connected components) and can always join the network by a simple handshake with the agents to be connected; by local communication the functions μ_{ij}^p and ψ_{ij}^p are set so that the output differences ν_{ij}^p, at the moment of joining, reside inside the funnel. We stress that for each connection of a new agent with one of its neighbors a separate performance function ψ_{ij}^p can be chosen, which allows for a straightforward inclusion. On the other hand, although the node-wise funnel coupling law (2) also exhibits a similar property, the inclusion of a new agent may cause all of its neighbors to change their individual performance function, since (2) requires the diffusive term ν_i to reside inside the funnel. If the neighbors are unable to adapt their performance functions, then the inclusion of the new agent may even be infeasible when its (output) difference with one of the neighbors is too large.

We note that, similar to [1, Rems. 1 & 2], the edge-wise funnel coupling law (6) is also able to achieve finite-time synchronization and/or pseudo-global convergence. Moreover, the control action is guaranteed to remain bounded (even when the performance functions ψ_{ij}^p converge to zero), under mild additional assumptions.

Theorem 2. *In addition to the assumptions of Theorem 1, assume that one of the following conditions holds.*

(a) $\widetilde{\mathbf{F}}_i(t,\mathbf{y},\mathbf{z}) \equiv \widehat{\mathbf{F}}(t,\mathbf{y}) + \mathbf{g}_i(t,\mathbf{y},\mathbf{z})$, where $\widehat{\mathbf{F}}(t,\mathbf{y})$ is globally Lipschitz with respect to $\mathbf{y}$ uniformly in t and there exists $M_{\mathbf{g}}$ such that $\|\mathbf{g}_i(t,\mathbf{y},\mathbf{z})\|_\infty \le M_{\mathbf{g}}$ for all $i \in \mathcal{N}$, $t \ge t_0$, $\mathbf{y} \in \mathbb{R}^m$, and $\mathbf{z} \in \mathbb{R}^{n_i-m}$.

(b) There exists $M_{\mathbf{x}}$ such that $\|\mathbf{x}_i(t)\|_\infty \le M_{\mathbf{x}}$ for all $i \in \mathcal{N}$ and $t \ge t_0$.

Then the input $\mathbf{u}_i$ of (6) for (4) is bounded on $[t_0,\infty)$, i.e., there exists $M_{\mathbf{u}} > 0$ such that for all $t \ge t_0$ and $i \in \mathcal{N}$, we have $\|\mathbf{u}_i(t)\|_\infty \le M_{\mathbf{u}}$.

The proof is similar to that of [1, Thm. 3], when we additionally invoke the boundedness of $\overline{\mathbf{\Gamma}}_i^{-1}$ from Assumption 3; hence it is omitted.

It is important to note that Theorem 2 includes the case $\lim_{t\to\infty}\psi_{ij}^p(t) = 0$, i.e., asymptotic synchronization can be achieved while the input remains bounded.

Remark 3. *We present an example of a network that utilizes the coupling (6), but the output difference fails to reside inside the funnel when the graph symmetry in Assumption 4 is not satisfied, even though the individual dynamics satisfy condition (a) of Theorem 2. To be precise, consider a network of four agents interconnected via a strongly connected directed graph*[4] *induced by* $\mathcal{N}_1 = \{4\} = \mathcal{N}_2$, $\mathcal{N}_3 = \{1,2\}$, *and* $\mathcal{N}_4 = \{3\}$. *The dynamics (4) are given by* $\dot{\mathbf{x}}_i(t) = \mathbf{u}_i(t)$, $\mathbf{y}_i(t) = \mathbf{x}_i(t)$ *for* $i \in \mathcal{N}$, *with initial conditions* $\mathbf{x}_1(t_0) = -1$, $\mathbf{x}_2(t_0) = 1$, $\mathbf{x}_3(t_0) = 0$, $\mathbf{x}_4(t_0) = 0$, *and the performance and coupling functions are chosen such that* $\psi^1_{31} = \psi^1_{32} = \tilde{\psi}$, $\psi^1_{14} = \psi^1_{24} = \hat{\psi}$ *for some* $\tilde{\psi}, \hat{\psi}$, *and* $\mu^1_{31}(-s) = -\mu^1_{32}(s)$, $\mu^1_{14}(-s) = -\mu^1_{24}(s)$, $\mu^1_{43}(0) = 0$. *Under this setting, it is straightforward to see that* $\dot{\mathbf{x}}_1(t) = -\dot{\mathbf{x}}_2(t)$ *and* $\dot{\mathbf{x}}_3(t) = \dot{\mathbf{x}}_4(t) = 0$ *for all* $t \ge t_0$, *and hence the network resides in the manifold* $\{\mathbf{x}_1 = -\mathbf{x}_2, \mathbf{x}_3 = \mathbf{x}_4 = 0\}$. *Then, the trajectories of agents 1 and 2 only depend on the performance function* $\hat{\psi}$. *Therefore, for appropriate performance function* $\tilde{\psi}$ *(which converges to zero relatively faster than* $\hat{\psi}$*), the output difference* $\nu^1_{31} = -\nu^1_{14}$ *and* $\nu^1_{32} = -\nu^1_{24}$ *fails to reside inside the funnel. Such pathological cases are avoided when the communication graph is undirected.*

3. Blended dynamics as the emergent behavior

In this section, we show that, under a stability condition on the emergent behavior (8), the behavior of the blended dynamics appear as the emergent collective behavior of the network (4) coupled via (6). We consider stability in the following sense.

Definition 1 ([11]). *A system* $\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{u}(t))$ *is* incremental input-to-state stable (δ-ISS)*, if there exists a class-*$\mathcal{KL}$ *function* $\hat{\beta}$ *and a class-*$\mathcal{K}_\infty$ *function* $\hat{\gamma}$,[5] *so that for any initial conditions* $\hat{\mathbf{x}}(t_0)$, $\mathbf{x}(t_0)$ *and locally essentially bounded, measurable inputs* $\hat{\mathbf{u}}$, $\mathbf{u}$, *the solutions* $\hat{\mathbf{x}}$, $\mathbf{x}$ *exist globally on* $[t_0, \infty)$ *and satisfy*

$$\|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\|_\infty \le \hat{\beta}\left(\|\hat{\mathbf{x}}(t_0) - \mathbf{x}(t_0)\|_\infty, t - t_0\right) + \hat{\gamma}\left(\sup_{s \in [t_0, t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty\right)$$

for all $t \ge t_0$.

Note that the δ-ISS property of a system implies a semi-global fading memory property [12] of the input difference in the state difference as follows.

Lemma 2. *If the system* $\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t), \mathbf{u}(t))$ *is δ-ISS, then for any given* $M_{\mathbf{u}}, M_{\mathbf{x}_0} > 0$ *and a decreasing function* $w : \mathbb{R}_{\ge 0} \to (0, 1]$ *that converges to zero, there exists a class-*$\mathcal{KL}$ *function* β *and a class-*$\mathcal{K}_\infty$ *function* γ, *so that for any initial conditions* $\hat{\mathbf{x}}(t_0)$, $\mathbf{x}(t_0)$ *satisfying* $\|\hat{\mathbf{x}}(t_0) - \mathbf{x}(t_0)\|_\infty \le M_{\mathbf{x}_0}$ *and for any locally essentially bounded, measurable inputs* $\hat{\mathbf{u}}$, $\mathbf{u}$ *satisfying* $\sup_{t \ge t_0} \|\hat{\mathbf{u}}(t) - \mathbf{u}(t)\|_\infty \le M_{\mathbf{u}}$, *the solutions* $\hat{\mathbf{x}}$, $\mathbf{x}$ *exists globally on* $[t_0, \infty)$ *and satisfy*

$$\|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\|_\infty \le \beta\left(\|\hat{\mathbf{x}}(t_0) - \mathbf{x}(t_0)\|_\infty, t - t_0\right) + \gamma\left(\sup_{s \in [t_0, t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty w(t - s)\right) \tag{10}$$

for all $t \ge t_0$.

The proof of Lemma 2 is inspired by [13].

Proof. First of all, note that by δ-ISS, there are $\hat{\beta} \in \mathcal{KL}$ and $\hat{\gamma} \in \mathcal{K}_\infty$ such that

$$\forall\, t \ge t_0 : \ \|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\|_\infty \le \hat{\beta}(M_{\mathbf{x}_0}, 0) + \hat{\gamma}(M_{\mathbf{u}}) =: M_{\mathbf{x}}.$$

Then, define $\hat{\beta}_{M_{\mathbf{x}}}(\cdot) := \hat{\beta}(M_{\mathbf{x}}, \cdot)$ and

$$\delta : \left(0, 2\hat{\beta}_{M_{\mathbf{x}}}(0)\right] \to \mathbb{R}, \ \epsilon \mapsto \ w\left(\hat{\beta}^{-1}_{M_{\mathbf{x}}}\left(\frac{\epsilon}{2}\right)\right) \hat{\gamma}^{-1}\left(\frac{\epsilon}{2}\right).$$

Note that $\hat{\beta}^{-1}_{M_{\mathbf{x}}}(\cdot)$ is strictly decreasing, hence $\delta(\cdot)$ is strictly increasing with $\operatorname{im}\delta = (0, w(0)\hat{\gamma}^{-1}(\hat{\beta}_{M_{\mathbf{x}}}(0))]$ and $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$. Therefore, $\delta^{-1}(\cdot)$ can be extended to a class-$\mathcal{K}_\infty$ function, which we call $\gamma(\cdot)$. We claim that (10) holds with $\beta(\cdot,\cdot) = \hat{\beta}(\cdot,\cdot)$ and $\gamma(\cdot)$. It is easy to see that

$$\forall\, \epsilon \in \left(0, 2\hat{\beta}_{M_{\mathbf{x}}}(0)\right] : \ \hat{\gamma}\left(\frac{\delta(\epsilon)}{w\left(\hat{\beta}^{-1}_{M_{\mathbf{x}}}(\epsilon/2)\right)}\right) = \frac{\epsilon}{2},$$

hence, with $\Delta := \delta(\epsilon)$, we find that

$$\forall\, \Delta \in \left(0, w(0)\hat{\gamma}^{-1}\left(\hat{\beta}_{M_{\mathbf{x}}}(0)\right)\right] : \ \hat{\gamma}\left(\frac{\Delta}{w(T^*(\Delta))}\right) = \frac{\gamma(\Delta)}{2},$$

where $T^*(\Delta) := \hat{\beta}^{-1}_{M_{\mathbf{x}}}(\gamma(\Delta)/2)$.

To prove the claim, let

$$\delta_t := \sup_{s \in [t_0, t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty w(t - s), \quad t \ge t_0,$$

and first suppose that $t - t_0 \ge T^*(\delta_t)$. Then, we have $w(T^*(\delta_t)) \le w(t - s)$ for all $t - T^*(\delta_t) \le s \le t$, which implies that

$$\begin{aligned}
&\hat{\gamma}\left(\sup_{s \in [t - T^*(\delta_t), t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty\right) \\
&\le \hat{\gamma}\left(\sup_{s \in [t - T^*(\delta_t), t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty \frac{w(t - s)}{w(T^*(\delta_t))}\right) \\
&\le \hat{\gamma}\left(\frac{1}{w(T^*(\delta_t))} \sup_{s \in [t_0, t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty w(t - s)\right) \\
&= \hat{\gamma}\left(\frac{\delta_t}{w(T^*(\delta_t))}\right) = \frac{1}{2}\gamma(\delta_t).
\end{aligned}$$

Note that in the last equality we have used that

$$\delta_t \le w(0) M_{\mathbf{u}} \le w(0)\hat{\gamma}^{-1}(M_{\mathbf{x}}) \le w(0)\hat{\gamma}^{-1}\left(\hat{\beta}_{M_{\mathbf{x}}}(0)\right),$$

where the fact that $M_{\mathbf{x}} \le \hat{\beta}_{M_{\mathbf{x}}}(0) = \hat{\beta}(M_{\mathbf{x}}, 0)$ implicitly follows from the δ-ISS property. Therefore, we get

$$\begin{aligned}
&\|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\|_\infty \\
&\le \hat{\beta}\left(\|\hat{\mathbf{x}}(t - T^*(\delta_t)) - \mathbf{x}(t - T^*(\delta_t))\|_\infty, t - (t - T^*(\delta_t))\right) \\
&\quad + \hat{\gamma}\left(\sup_{s \in [t - T^*(\delta_t), t)} \|\hat{\mathbf{u}}(s) - \mathbf{u}(s)\|_\infty\right) \\
&\le \hat{\beta}\left(M_{\mathbf{x}}, T^*(\delta_t)\right) + \frac{1}{2}\gamma(\delta_t) = \gamma(\delta_t)
\end{aligned}$$

[4] A directed graph is strongly connected, if there is a directed path between any two nodes.

[5] Recall that a function $\hat{\beta}$ is of class-$\mathcal{KL}$, if $\hat{\beta}(\cdot, t)$ is strictly increasing with $\hat{\beta}(0, t) = 0$ for all $t \ge 0$ and $\hat{\beta}(s, \cdot)$ is strictly decreasing with $\lim_{t \to \infty} \hat{\beta}(s, t) = 0$ for all $s \ge 0$. A function $\hat{\gamma}$ is of class-$\mathcal{K}_\infty$, if it is strictly increasing, vanishes at zero and converges to infinity.

which gives (10). Next, suppose that $t-t_0 \le T^*(\delta_t)$. Then, we have $w(T^*(\delta_t)) \le w(t-s)$ for all $t_0 \le s \le t$. Thus, we get

$$
\begin{aligned}
&\|\hat{\mathbf{x}}(t)-\mathbf{x}(t)\|_\infty \\
&\le \hat\beta\left(\|\hat{\mathbf{x}}(t_0)-\mathbf{x}(t_0)\|_\infty, t-t_0\right)+\hat\gamma\left(\sup_{s\in[t_0,t)}\|\hat{\mathbf{u}}(s)-\mathbf{u}(s)\|_\infty\right)\\
&\le \hat\beta\left(\|\hat{\mathbf{x}}(t_0)-\mathbf{x}(t_0)\|_\infty, t-t_0\right)\\
&\quad+\hat\gamma\left(\sup_{s\in[t_0,t)}\|\hat{\mathbf{u}}(s)-\mathbf{u}(s)\|_\infty\frac{w(t-s)}{w(T^*(\delta_t))}\right)\\
&= \hat\beta\left(\|\hat{\mathbf{x}}(t_0)-\mathbf{x}(t_0)\|_\infty, t-t_0\right)+\hat\gamma\left(\frac{\delta_t}{w(T^*(\delta_t))}\right)\\
&= \beta\left(\|\hat{\mathbf{x}}(t_0)-\mathbf{x}(t_0)\|_\infty, t-t_0\right)+\frac{1}{2}\gamma\left(\delta_t\right)
\end{aligned}
$$

which gives (10). □

Theorem 3. *Let the assumptions of Theorem 1 hold, let $\mathbf{x} = \mathrm{col}(\mathbf{x}_1,\dots,\mathbf{x}_N)$ be a global solution of (4) coupled via (6), and assume that the system (8) with input $\mathrm{col}(\mathbf{e}_1,\dots,\mathbf{e}_N)$ is δ-ISS.*[6] *Then the blended dynamics*

$$
\begin{aligned}
\dot{\hat{\mathbf{s}}}(t) &= \frac{1}{N}\sum_{i=1}^{N}\widetilde{\mathbf{F}}_i(t,\hat{\mathbf{s}}(t),\hat{\mathbf{z}}_i(t)),\\
\dot{\hat{\mathbf{z}}}_i(t) &= \mathbf{Z}_i(t,\hat{\mathbf{z}}_i(t),\hat{\mathbf{s}}(t)), \qquad i\in\mathcal{N},
\end{aligned}
\tag{11}
$$

have a global solution under the initial condition $\hat{\mathbf{s}}(t_0) = (1/N)\sum_{i=1}^N \mathbf{y}_i(t_0)$ and $\hat{\mathbf{z}}_i(t_0) = \mathbf{z}_i(t_0)$, $i\in\mathcal{N}$, which satisfies for all $i\in\mathcal{N}$ and $t\ge t_0$:

$$
\begin{aligned}
&\left\|\begin{bmatrix}\hat{\mathbf{s}}(t)\\ \hat{\mathbf{z}}_i(t)\end{bmatrix}-\begin{bmatrix}\mathbf{y}_i(t)\\ \mathbf{z}_i(t)\end{bmatrix}\right\|_\infty\\
&\qquad\le d_{\mathcal{G}}\Psi(t)+\gamma\left(\sup\nolimits_{s\in[t_0,t)} d_{\mathcal{G}}\Psi(s)w(t-s)\right)
\end{aligned}
\tag{12}
$$

where $d_{\mathcal{G}}$ and $\Psi(\cdot)$ are as in (9), $w:\mathbb{R}_{\ge0}\to(0,1]$ is an arbitrarily given decreasing function that converges to zero, and $\gamma(\cdot)$ is given by Lemma 2 for $M_{\mathbf{u}} = d_{\mathcal{G}}\overline{\psi}$ and any $M_{\mathbf{x}_0}>0$. If additionally for system (8) there exists a bounded input and a corresponding global solution that is bounded, then $\mathbf{x}$ and the solution $\hat{\mathbf{s}}$, $\hat{\mathbf{z}}_i$, $i\in\mathcal{N}$, of (11) are bounded.

Proof. The existence of a global solution $\hat{\mathbf{s}}(\cdot)$ and $\hat{\mathbf{z}}_i(\cdot)$, $i\in\mathcal{N}$, of (11) follows from the fact that (8) is δ-ISS. Since the assumptions of Theorem 1 are satisfied, it follows from (9) that for $\mathbf{e}_i = \mathbf{y}_i-\mathbf{s}$ with $\mathbf{s} = (1/N)\sum_{i=1}^N\mathbf{y}_i$ we have that $\|\mathbf{e}_i(t)\|_\infty < d_{\mathcal{G}}\Psi(t)\le d_{\mathcal{G}}\overline{\psi}$ for all $t\ge t_0$. Then (12) can be deduced from Lemma 2 as follows:

$$
\begin{aligned}
&\left\|\begin{bmatrix}\hat{\mathbf{s}}(t)\\ \hat{\mathbf{z}}_i(t)\end{bmatrix}-\begin{bmatrix}\mathbf{y}_i(t)\\ \mathbf{z}_i(t)\end{bmatrix}\right\|_\infty \le \left\|\begin{bmatrix}\hat{\mathbf{s}}(t)\\ \hat{\mathbf{z}}_i(t)\end{bmatrix}-\begin{bmatrix}\mathbf{s}(t)\\ \mathbf{z}_i(t)\end{bmatrix}\right\|_\infty+\|\mathbf{e}_i(t)\|_\infty\\
&\le\gamma\left(\sup\nolimits_{s\in[t_0,t)}\max\nolimits_{j\in\mathcal{N}}\|\mathbf{e}_j(s)\|_\infty w(t-s)\right)+\|\mathbf{e}_i(t)\|_\infty.
\end{aligned}
$$

Clearly, if (8) has a bounded global solution for some bounded input, then $\hat{\mathbf{s}}$ and $\hat{\mathbf{z}}_i$, $i\in\mathcal{N}$, are bounded and hence $\mathbf{x}$ is bounded. □

Remark 4. *Let the assumptions of Theorem 3 hold.*

(i) *Observe that, if the solution $\mathbf{x}$ of (4) and (6) is bounded under the additional assumption of the theorem, then, by Theorem 2, the input signals $\mathbf{u}_i$, $i\in\mathcal{N}$, are bounded.*

(ii) *If all the performance functions ψ_{ij}^p converge to zero, then $\lim_{t\to\infty}\Psi(t)=0$, hence the right-hand side of (12) also converges to zero as $t\to\infty$.*

(iii) *Although the function w which determines the rate of the fading memory in (12) can be arbitrarily chosen, we like to highlight that γ, determined by Lemma 2, depends on w.*

(iv) *The additional assumption that for system (8) there exists a bounded input with a corresponding bounded global solution can be satisfied if, for instance, the blended dynamics (11), represented as $\dot{\mathbf{x}} = \mathbf{F}(t,\mathbf{x})$, which are incrementally stable and have a Lyapunov function $U(\hat{\mathbf{x}},\mathbf{x})$ as in [14] such that*

$$
\frac{\partial U}{\partial\hat{\mathbf{x}}}(\hat{\mathbf{x}},\mathbf{x})\mathbf{F}(t,\hat{\mathbf{x}})+\frac{\partial U}{\partial\mathbf{x}}(\hat{\mathbf{x}},\mathbf{x})\mathbf{F}(t,\mathbf{x})\le-\alpha(\|\hat{\mathbf{x}}-\mathbf{x}\|)
$$

with some class-$\mathcal{K}_\infty$ function α, which also satisfy $\alpha(\|\mathbf{x}\|)\ge\|(\partial U/\partial\mathbf{x})(\hat{\mathbf{x}},0)\|\|\mathbf{F}(t,0)\|$. Then $U(\mathbf{x},0)$ may serve as a Lyapunov function to infer boundedness of a solution.

(v) *One might tighten the bound (12) by finding for each $i,j\in\mathcal{N}$ the minimal length of a path d_{ij} between them. In particular, we instead get $\|\mathbf{e}_i(t)\|_\infty\le d_i\Psi(t)$, where $d_i=\sum_{j\ne i}d_{ij}/N < d_{\mathcal{G}}$. Then the right-hand side of (12) can be replaced by $d_i\Psi(t)+\gamma(\sup_{s\in[t_0,t)}\max_{j\in\mathcal{N}}d_j\Psi(s)w(t-s))$.*

(vi) *The stability condition on the emergent behavior can be modified to other concepts in a straightforward way. For instance, motivated by [15], we may consider input-to-state stability to a compact set A:*[7]

$$
\|\mathbf{x}(t)\|_A\le\hat\beta\left(\|\mathbf{x}(t_0)\|_A,t-t_0\right)+\hat\gamma\left(\sup_{s\in[t_0,t)}\|\mathbf{u}(s)\|_\infty\right).
$$

Then, a similar conclusion can be made:

$$
\begin{aligned}
&\left\|\begin{bmatrix}\mathbf{y}_i(t)\\ \mathbf{z}_1(t)\\ \vdots\\ \mathbf{z}_N(t)\end{bmatrix}\right\|_A\le\beta\left(\left\|\begin{bmatrix}(1/N)\sum_{j=1}^N\mathbf{y}_j(t_0)\\ \mathbf{z}_1(t_0)\\ \vdots\\ \mathbf{z}_N(t_0)\end{bmatrix}\right\|_A, t-t_0\right)\\
&\qquad+d_{\mathcal{G}}\Psi(t)+\gamma\left(\sup_{s\in[t_0,t)}d_{\mathcal{G}}\Psi(s)w(t-s)\right);
\end{aligned}
$$

the compact attractor A of the blended dynamics approximates the behavior of the network.

In the remainder of this section we consider the special case when all the internal dynamics (the differential equation for $\mathbf{z}_i$, $i\in\mathcal{N}$) share the same vector field, i.e.,

[6] Note that Assumption 6 is already a consequence of (8) being δ-ISS.

[7] For a set $\Xi\subseteq\mathbb{R}^n$, $\|\mathbf{x}\|_\Xi$ denotes the distance between the point $\mathbf{x}\in\mathbb{R}^n$ and Ξ, i.e., $\|\mathbf{x}\|_\Xi := \inf_{\mathbf{y}\in\Xi}\|\mathbf{x}-\mathbf{y}\|_\infty$.

$\mathbf{Z}_i = \mathbf{Z}$ for all $i \in \mathcal{N}$, but not necessarily the same initial condition. Then we can reduce the dimension of the blended dynamics (11) and consider

$$\dot{\hat{\mathbf{s}}}(t) = \frac{1}{N}\sum_{i=1}^{N} \widetilde{\mathbf{F}}_i(t, \hat{\mathbf{s}}(t), \hat{\mathbf{z}}(t)), \tag{13a}$$

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{Z}(t, \hat{\mathbf{z}}(t), \hat{\mathbf{s}}(t)). \tag{13b}$$

This is motivated by the observation that, under the assumption $\mathbf{Z}_i = \mathbf{Z}$, if system (8) is δ-ISS, then the blended dynamics (11) are globally asymptotically stable with respect to the closed set

$$\mathcal{S}_{\mathbf{z}} := \{ \operatorname{col}(\hat{\mathbf{s}}, \hat{\mathbf{z}}_1, \ldots, \hat{\mathbf{z}}_N) \mid \forall i,j \in \mathcal{N}: \ \hat{\mathbf{z}}_i = \hat{\mathbf{z}}_j \} .$$

To see this, let $\operatorname{col}(\mathbf{s}, \mathbf{z}_1, \ldots, \mathbf{z}_N)$ be a solution of (11) with $\mathbf{s}(t_0) = \mathbf{s}^0$ and $\mathbf{z}_i(t_0) = \mathbf{z}_i^0$, $i \in \mathcal{N}$; then it also solves (8) with $\mathbf{e}_i = 0$, $i \in \mathcal{N}$. Further let $(\hat{\mathbf{s}}, \hat{\mathbf{z}})$ be a solution of (13) with $\hat{\mathbf{s}}(t_0) = \mathbf{s}^0$ and $\hat{\mathbf{z}}(t_0) = \hat{\mathbf{z}}^0$; then $\operatorname{col}(\hat{\mathbf{s}}, \hat{\mathbf{z}}, \ldots, \hat{\mathbf{z}})$ solves (8) with $\mathbf{e}_i = 0$, $i \in \mathcal{N}$. Since (8) is δ-ISS we get

$$\left\| \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{z}}(t) \\ \vdots \\ \hat{\mathbf{z}}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{s}(t) \\ \mathbf{z}_1(t) \\ \vdots \\ \mathbf{z}_N(t) \end{bmatrix} \right\|_\infty \leq \hat{\beta}\left(\max_{j\in\mathcal{N}} \left\|\hat{\mathbf{z}}^0 - \mathbf{z}_j^0\right\|_\infty, t - t_0 \right), \tag{14}$$

thus $\|\mathbf{z}_j(t) - \mathbf{z}_i(t)\|_\infty \to 0$ for $t \to \infty$ for all $i,j \in \mathcal{N}$. The approximation result of Theorem 3 can then be extended as follows.

Corollary 4. *Let the assumptions of Theorem 3 hold and assume that $n_i = n$ and $\mathbf{Z}_i = \mathbf{Z}$ for all $i \in \mathcal{N}$. Let $\mathbf{x} = \operatorname{col}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ be a global solution of* (4) *coupled via* (6)*. Then the blended dynamics* (13) *have a global solution under the initial condition $\hat{\mathbf{s}}(t_0) = (1/N)\sum_{i=1}^N \mathbf{y}_i(t_0)$ and $\hat{\mathbf{z}}(t_0) = \hat{\mathbf{z}}^0 \in \mathbb{R}^{n-m}$, which satisfies for all $i \in \mathcal{N}$ and $t \geq t_0$:*

$$\left\| \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{z}}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{y}_i(t) \\ \mathbf{z}_i(t) \end{bmatrix} \right\|_\infty \leq d_{\mathcal{G}}\Psi(t) + \hat{\beta}\left(\max_{j\in\mathcal{N}} \left\|\hat{\mathbf{z}}^0 - \mathbf{z}_j^0\right\|_\infty, t - t_0 \right) + \gamma\left(\sup\nolimits_{s\in[t_0,t)} d_{\mathcal{G}}\Psi(s) w(t-s)\right).$$

The proof of Corollary 4 is a direct consequence of Theorem 3 and the inequality (14); the details are omitted.

In terms of the reduced blended dynamics (13), the assumption of Corollary 4 can be guaranteed by the sufficient condition that the system

$$\begin{aligned} \dot{\mathbf{s}}(t) &= \frac{1}{N}\sum_{i=1}^{N} \widetilde{\mathbf{F}}_i(t, \mathbf{s}(t) + \mathbf{e}_i(t), \mathbf{z}(t) + \mathbf{d}_i(t)), \\ \dot{\mathbf{z}}(t) &= \mathbf{Z}(t, \mathbf{z}(t), \mathbf{s}(t) + \mathbf{e}_0(t)), \end{aligned} \tag{15}$$

with input $\operatorname{col}(\mathbf{e}_0, \ldots, \mathbf{e}_N, \mathbf{d}_1, \ldots, \mathbf{d}_N)$ is δ-ISS with corresponding functions $\bar{\beta}$ and $\bar{\gamma}$, which satisfies the small gain condition $2\bar{\gamma}'(s) < 1$ for all $s > 0$.

Lemma 3. *If the system* (15)*, under the assumption that $n_i = n$ and $\mathbf{Z}_i = \mathbf{Z}$ for all $i \in \mathcal{N}$, is δ-ISS with corresponding functions $\bar{\beta}$ and $\bar{\gamma}$, such that $\bar{\gamma}$ is differentiable and satisfies the small gain condition $2\bar{\gamma}'(s) < 1$ for all $s > 0$, then there exists a class-$\mathcal{KL}$ function $\hat{\beta}$ and a class-$\mathcal{K}_\infty$ function $\hat{\gamma}$, so that the system* (8) *is δ-ISS with corresponding functions $\hat{\beta}$ and $\hat{\gamma}$.*

Proof. Consider any two solutions of (8) denoted as $(\hat{\mathbf{s}}, \hat{\mathbf{z}}_1, \ldots, \hat{\mathbf{z}}_N)$ and $(\mathbf{s}, \mathbf{z}_1, \ldots, \mathbf{z}_N)$ with inputs $(\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_N)$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_N)$, respectively. Fix $i \in \mathcal{N}$. Then $(\hat{\mathbf{s}}, \hat{\mathbf{z}}_i)$ is a solution of (15) with corresponding input $\operatorname{col}(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_N, \hat{\mathbf{z}}_1 - \hat{\mathbf{z}}_i, \ldots, \hat{\mathbf{z}}_N - \hat{\mathbf{z}}_i)$, and $(\mathbf{s}, \mathbf{z}_i)$ is a solution of (15) with corresponding input $\operatorname{col}(\mathbf{e}_i, \mathbf{e}_1, \ldots, \mathbf{e}_N, \mathbf{z}_1 - \mathbf{z}_i, \ldots, \mathbf{z}_N - \mathbf{z}_i)$. Since (15) is δ-ISS, we conclude that

$$\left\| \begin{bmatrix} \hat{\mathbf{s}}(t) - \mathbf{s}(t) \\ \hat{\mathbf{z}}_i(t) - \mathbf{z}_i(t) \end{bmatrix} \right\|_\infty \leq \bar{\beta}\left(\left\| \begin{bmatrix} \hat{\mathbf{s}}(t_0) - \mathbf{s}(t_0) \\ \hat{\mathbf{z}}_i(t_0) - \mathbf{z}_i(t_0) \end{bmatrix} \right\|_\infty, t - t_0 \right) + \bar{\gamma}\left(\sup_{s\in[t_0,t)} \left\| \begin{bmatrix} \tilde{e}(s) \\ 2\tilde{z}(s) \end{bmatrix} \right\|_\infty \right) \tag{16}$$

where $\tilde{e}(t) := \max_{j\in\mathcal{N}} \|\hat{\mathbf{e}}_j(t) - \mathbf{e}_j(t)\|_\infty$ and $\tilde{z}(t) := \max_{j\in\mathcal{N}} \|\hat{\mathbf{z}}_j(t) - \mathbf{z}_j(t)\|_\infty$. Thus, we can conclude that

$$\sup_{s\in[t_0,t)} \left\| \begin{bmatrix} \tilde{s}(s) \\ \tilde{z}(s) \end{bmatrix} \right\|_\infty \leq \bar{\beta}\left(\left\| \begin{bmatrix} \tilde{s}(t_0) \\ \tilde{z}(t_0) \end{bmatrix} \right\|_\infty, 0 \right) + \bar{\gamma}\left(2 \sup_{s\in[t_0,t)} \tilde{z}(s) \right) + \bar{\gamma}\left(\sup_{s\in[t_0,t)} \tilde{e}(s) \right)$$

where $\tilde{s}(t) := \|\hat{\mathbf{s}}(t) - \mathbf{s}(t)\|_\infty$. Now observe that $\alpha(s) = s - \bar{\gamma}(2s)$ satisfies $\alpha'(s) > 0$ for all $s > 0$ and is hence strictly monotonically increasing. Define

$$\hat{\gamma}(s) := \alpha^{-1}\big(2\max\{\bar{\beta}(s,0), \bar{\gamma}(s)\}\big),$$

which is clearly a class-$\mathcal{K}_\infty$ function. Then, with $\zeta := \max\left\{ \left\| \begin{bmatrix} \tilde{s}(t_0) \\ \tilde{z}(t_0) \end{bmatrix} \right\|_\infty, \ \sup_{s\in[t_0,t)} \tilde{e}(s) \right\}$ we have that

$$\begin{aligned} \sup_{s\in[t_0,t)} \tilde{z}(s) &\leq 2\max\{\bar{\beta}(\zeta,0), \bar{\gamma}(\zeta)\} + \bar{\gamma}\left(2 \sup_{s\in[t_0,t)} \tilde{z}(s) \right) \\ &= \alpha(\hat{\gamma}(\zeta)) + \sup_{s\in[t_0,t)} \tilde{z}(s) - \alpha\left(\sup_{s\in[t_0,t)} \tilde{z}(s) \right), \end{aligned}$$

and from the monotonicity of α it follows that $\sup_{s\in[t_0,t)} \tilde{z}(s) \leq \hat{\gamma}(\zeta)$. We further have

$$\begin{aligned} &\sup_{s\in[t_0,t)} \left\| \begin{bmatrix} \tilde{s}(s) \\ \tilde{z}(s) \end{bmatrix} \right\|_\infty \\ &\leq \bar{\beta}\left(\left\| \begin{bmatrix} \tilde{s}(t_0) \\ \tilde{z}(t_0) \end{bmatrix} \right\|_\infty, 0 \right) + \bar{\gamma}\left(2\hat{\gamma}\left(\left\| \begin{bmatrix} \tilde{s}(t_0) \\ \tilde{z}(t_0) \end{bmatrix} \right\|_\infty \right) \right) \\ &\quad + \bar{\gamma}\left(2\hat{\gamma}\left(\sup_{s\in[t_0,t)} \tilde{e}(s) \right) \right) + \bar{\gamma}\left(\sup_{s\in[t_0,t)} \tilde{e}(s) \right) \\ &\leq \hat{\gamma}\left(\left\| \begin{bmatrix} \tilde{s}(t_0) \\ \tilde{z}(t_0) \end{bmatrix} \right\|_\infty \right) + \hat{\gamma}\left(\sup_{s\in[t_0,t)} \tilde{e}(s) \right) \end{aligned} \tag{17}$$

where for the last inequality we utilized the fact that $\bar{\beta}(s,0) + \bar{\gamma}(2\hat{\gamma}(s)) \leq \hat{\gamma}(s)$ and $\bar{\gamma}(s) + \bar{\gamma}(2\hat{\gamma}(s)) \leq \hat{\gamma}(s)$ for all $s \geq 0$. On the other hand, we can conclude from (16) that

$$\limsup_{t\to\infty} \left\| \begin{bmatrix} \tilde{s}(t) \\ \tilde{z}(t) \end{bmatrix} \right\|_\infty \leq \bar{\gamma}\left(\sup_{s\in[t_0,\infty)} \left\| \begin{bmatrix} \tilde{e}(s) \\ 2\tilde{z}(s) \end{bmatrix} \right\|_\infty \right).$$

Since the initial time t_0 was arbitrary, we further get

$$\begin{aligned} \limsup_{t\to\infty} \tilde{z}(t) &\leq \lim_{t_0\to\infty} \bar{\gamma}\left(2 \sup_{s\in[t_0,\infty)} \tilde{z}(s) \right) + \bar{\gamma}\left(\sup_{s\in[t_0,\infty)} \tilde{e}(s) \right) \\ &= \bar{\gamma}\left(2 \limsup_{t\to\infty} \tilde{z}(t) \right) + \bar{\gamma}\left(\sup_{s\in[t_0,\infty)} \tilde{e}(s) \right), \end{aligned}$$

hence $\limsup_{t\to\infty} \tilde{z}(t) \leq \tilde{\gamma}(\sup_{s\in[t_0,\infty)} \tilde{e}(s))$, where $\tilde{\gamma}(s) := \alpha^{-1}\big(\bar{\gamma}(s)\big)$ is a class-$\mathcal{K}_\infty$ function. Now, this further implies that

$$\limsup_{t\to\infty} \left\| \begin{bmatrix} \tilde{s}(t) \\ \tilde{z}(t) \end{bmatrix} \right\|_\infty \leq \bar{\gamma}\left(2\limsup_{t\to\infty} \tilde{z}(t)\right) + \bar{\gamma}\left(\sup_{s\in[t_0,\infty)} \tilde{e}(s)\right) \leq \tilde{\gamma}\left(\sup_{s\in[t_0,\infty)} \tilde{e}(s)\right) \tag{18}$$

where we utilized the fact that $\tilde{\gamma}(s) = \bar{\gamma}(2\tilde{\gamma}(s)) + \bar{\gamma}(s)$ for all $s \geq 0$. Then, from (17) and (18), as in Theorem 1 ((j) $\Rightarrow$ (a)) of [15], we can conclude δ-ISS of (8). □

On the other hand, instead of the small gain condition, if we have that the internal dynamics (13b) are δ-ISS, then we get the following approximation result.

Theorem 5. *Let the assumptions of Theorem 1 hold, let $\mathbf{x} = \mathrm{col}(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ be a global solution of (4) coupled via (6), and assume that the system (15) with input $\mathrm{col}(\mathbf{e}_0,\ldots,\mathbf{e}_N,\mathbf{d}_1,\ldots,\mathbf{d}_N)$ is δ-ISS (with corresponding functions $\hat{\beta}$ and $\hat{\gamma}$). Further assume that the internal dynamics (13b) with input $\hat{\mathbf{s}}$ are δ-ISS (with corresponding functions $\bar{\beta}$ and $\bar{\gamma}$). Then the blended dynamics (13) have a global solution under the initial condition $\hat{\mathbf{s}}(t_0) = (1/N)\sum_{i=1}^N \mathbf{y}_i(t_0)$ and $\hat{\mathbf{z}}(t_0) = (1/N)\sum_{i=1}^N \mathbf{z}_i(t_0)$, which satisfies for all $i \in \mathcal{N}$ and $t \geq t_0$:*

$$\begin{aligned} \left\| \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{z}}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{y}_i(t) \\ \mathbf{z}_i(t) \end{bmatrix} \right\|_\infty &\leq d_{\mathcal{G}}\Psi(t) + \beta(\|\hat{\mathbf{z}}(t_0) - \mathbf{z}_i(t_0)\|_\infty, t-t_0) \\ &\quad + \gamma\left(\sup_{s\in[t_0,t)} \max\{d_{\mathcal{G}}\Psi(s), \tilde{Z}(s)\} w(t-s)\right), \end{aligned} \tag{19}$$

where $d_{\mathcal{G}}$ and $\Psi(\cdot)$ are as in (9), $\bar{w}, w : \mathbb{R}_{\geq 0} \to (0,1]$ are arbitrarily given decreasing functions that converge to zero, $\beta(\cdot)$, $\gamma(\cdot)$ are given by Lemma 2 for $\hat{\beta}(\cdot)$, $\hat{\gamma}(\cdot)$, $\hat{M}_{\mathbf{x}_0} = \max_{j\in\mathcal{N}} \|\hat{\mathbf{z}}(t_0) - \mathbf{z}_j(t_0)\|_\infty$ and $\hat{M}_{\mathbf{u}} = \max\{d_{\mathcal{G}}\overline{\psi}, \tilde{\beta}(2\hat{M}_{\mathbf{x}_0}, 0) + \tilde{\gamma}(d_{\mathcal{G}}\overline{\psi})\}$, and

$$\begin{aligned} \tilde{Z}(t) := \max_{j,i\in\mathcal{N}} &\tilde{\beta}(\|\mathbf{z}_j(t_0) - \mathbf{z}_i(t_0)\|_\infty, t-t_0) \\ &+ \tilde{\gamma}\left(\sup_{s\in[t_0,t)} d_{\mathcal{G}}\Psi(s)\bar{w}(t-s)\right) \end{aligned}$$

where $\tilde{\beta}(\cdot)$,$\tilde{\gamma}(\cdot)$ are given by Lemma 2 for $\bar{\beta}(\cdot)$, $\bar{\gamma}(\cdot)$, $\bar{M}_{\mathbf{x}_0} = 2\hat{M}_{\mathbf{x}_0}$ and $\bar{M}_{\mathbf{u}} = d_{\mathcal{G}}\overline{\psi}$. If additionally for system (15) there exists a bounded input and a corresponding global solution that is bounded, then $\mathbf{x}$ and the solution $(\hat{\mathbf{s}}, \hat{\mathbf{z}})$ of (13) are bounded.

Proof. The existence of a global solution $\hat{\mathbf{s}}(\cdot)$ and $\hat{\mathbf{z}}(\cdot)$ of (13) follows from the fact that (15) is δ-ISS. Fix $i \in \mathcal{N}$. Since the assumptions of Theorem 1 are satisfied, it follows from (9) that for $\mathbf{e}_i = \mathbf{y}_i - \mathbf{s}$ with $\mathbf{s} = (1/N)\sum_{i=1}^N \mathbf{y}_i$ we have that $\|\mathbf{e}_i(t)\|_\infty < d_{\mathcal{G}}\Psi(t)$ for all $t \geq t_0$. Furthermore, we set $\mathbf{e}_0 := \mathbf{e}_i$ and $\mathbf{d}_j := \mathbf{z}_j - \mathbf{z}_i$ for $j \in \mathcal{N}$. Then (19) follows from the fact that (15) is δ-ISS and from the result of Lemma 2 as

$$\begin{aligned} \left\| \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{z}}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{y}_i(t) \\ \mathbf{z}_i(t) \end{bmatrix} \right\|_\infty &\leq \left\| \begin{bmatrix} \hat{\mathbf{s}}(t) \\ \hat{\mathbf{z}}(t) \end{bmatrix} - \begin{bmatrix} \mathbf{s}(t) \\ \mathbf{z}_i(t) \end{bmatrix} \right\|_\infty + \|\mathbf{e}_i(t)\|_\infty \\ &\leq \|\mathbf{e}_i(t)\|_\infty + \beta\left(\|\hat{\mathbf{z}}(t_0) - \mathbf{z}_i(t_0)\|_\infty, t-t_0\right) \\ &\quad + \gamma\left(\sup_{s\in[t_0,t)} \left\| \begin{bmatrix} \|\mathbf{e}_0(s)\|_\infty \\ \max_{j\in\mathcal{N}} \|\mathbf{e}_j(s)\|_\infty \\ \max_{j\in\mathcal{N}} \|\mathbf{d}_j(s)\|_\infty \end{bmatrix} \right\|_\infty w(t-s)\right). \end{aligned}$$

Furthermore, since (13b) is δ-ISS, by Lemma 2, we find that

$$\begin{aligned} \|\mathbf{d}_j(t)\|_\infty &\leq \tilde{\beta}(\|\mathbf{z}_j(t_0) - \mathbf{z}_i(t_0)\|_\infty, t-t_0) \\ &\quad + \tilde{\gamma}\left(\sup_{s\in[t_0,t)} \|\mathbf{y}_j(s) - \mathbf{y}_i(s)\|_\infty \bar{w}(t-s)\right) \\ &\leq \tilde{Z}(t). \end{aligned}$$

Clearly, if (15) has a bounded global solution for some bounded input, then $\hat{\mathbf{s}}$ and $\hat{\mathbf{z}}$ are bounded and hence $\mathbf{x}$ is bounded. □

4. Simulation

We illustrate our results by a simulation for four heterogeneous agents:

$$\begin{aligned} \varepsilon\dot{y}_i(t) &= \tilde{F}_i(t, y_i(t), \mathbf{z}_i(t)) + u_i(t), \\ \dot{\mathbf{z}}_i(t) &= \mathbf{Z}(t, \mathbf{z}_i(t), y_i(t)), \qquad i \in \mathcal{N} = \{1,\ldots,4\}, \end{aligned}$$

where $\varepsilon = 0.01$, $\mathbf{z}_i(t) := \mathrm{col}(z_{i,1}(t), z_{i,2}(t), z_{i,3}(t))$, $\mathbf{Z}(t,\mathbf{z}_i,y_i) := \mathrm{col}(Z_1(t,z_{i,1},y_i), Z_2(t,z_{i,2},y_i), Z_3(t,z_{i,3},y_i))$, and

$$\begin{aligned} Z_1(t,z,y) &:= -100z + 100y, \\ Z_2(t,z,y) &:= \begin{cases} -z + 0.4(y+0.5), & \text{if } y + 0.5 < 0, \\ -z + 7(y+0.5), & \text{if } y + 0.5 \geq 0, \end{cases} \\ Z_3(t,z,y) &:= \frac{1}{20}\begin{cases} -z, & \text{if } y + 1 < 0, \\ -z + 50(y+1), & \text{if } y + 1 \geq 0. \end{cases} \end{aligned}$$

The heterogeneous vector fields $\tilde{F}_i$ are given by

$$\begin{aligned} \tilde{F}_1(t,y_1,\mathbf{z}_1) &:= -\frac{1}{3}y_1^3 + 4z_{1,1} + 11, \\ \tilde{F}_2(t,y_2,\mathbf{z}_2) &:= -\frac{1}{3}y_2^3 - 16z_{2,2} - \frac{55}{3}, \\ \tilde{F}_3(t,y_3,\mathbf{z}_3) &:= -\frac{1}{3}y_3^3 + 16z_{3,2} - 4(z_{3,2} - 1.1)^2 + \frac{11}{3}, \\ \tilde{F}_4(t,y_4,\mathbf{z}_4) &:= -\frac{1}{3}y_4^3 - 4z_{4,3} + \frac{55}{3}. \end{aligned}$$

Each agent represents a neuromorphic circuit with one positive/negative feedback inspired by [16]. We consider a situation of plug-and-play operation. The associated switching graph $\mathcal{G}(t) = (\mathcal{N}(t), \mathcal{E}(t))$ is illustrated in Figure 1.

The initial performance functions are all chosen as $\psi_{ij}(t) = (\pi/2)(0.9\exp(-t) + 0.1)$ for $(j,i) \in \mathcal{E}$. Then, upon the joining of agent 1 at $t = 100$, we set $\psi_{14}(t) = (\pi/2)(8.9\exp(-(t-100)) + 0.1)$. After that, when agent 2 joins at $t = 170$, we set $\psi_{24}(t) = (\pi/2)(0.9\exp(-(t-170)) + 0.1)$. Finally, when agent 3 joins at $t = 220$, we set $\psi_{32}(t) = (\pi/2)(4.9\exp(-(t-220)) + 0.1)$. The coupling

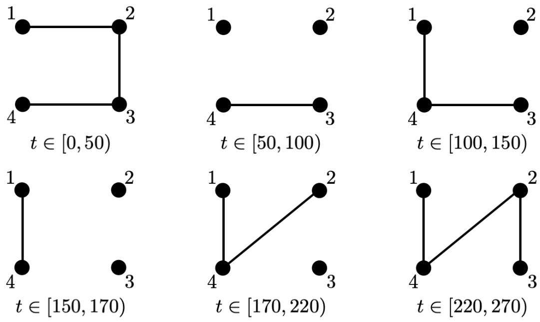

Figure 1: Illustration of the switching graph $\mathcal{G}(t)$ that represents the plug-and-play operation in Section 4.

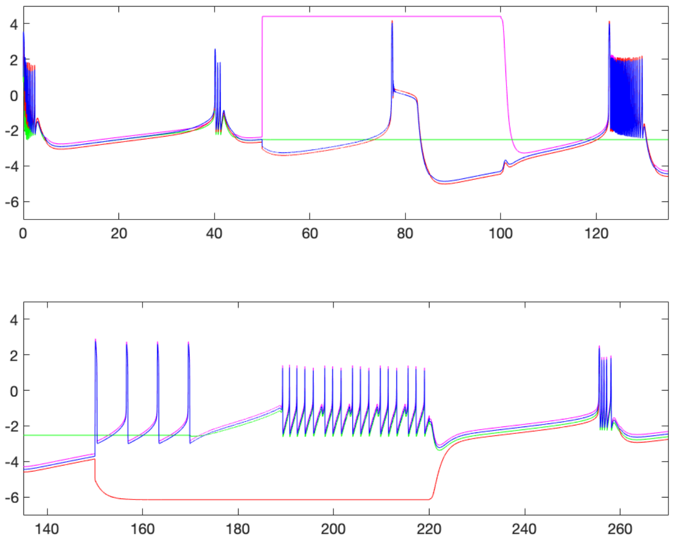

Figure 2: Illustration of the plug-and-play operation among the four heterogeneous agents illustrated in Section 4. (i) Agents 1 and 2 leave the network at $t = 50$. (ii) Agent 1 joins the network at $t = 100$. (iii) Agent 3 leaves the network at $t = 150$. (iv) Agent 2 joins the network at $t = 170$. (v) Agent 3 joins the network at $t = 220$.

functions are all chosen as $\mu_{ij}(s) = \tan((\pi/2)s)$. The simulation is performed in Matlab/Simulink software package with initial conditions $y_i(0) = 1$ and $\mathbf{z}_i(0) = \mathrm{col}(0, 0, 0)$, $i \in \mathcal{N}$. See Figure 2 for the simulation result.

During the plug-and-play operation, one can observe that the emergent collective behavior represents different pulses, e.g., spiking, as the blended dynamics (13) differ by the set $\mathcal{N}(t)$ of connected agents as

$$\varepsilon\dot{\hat{y}}(t) = \frac{1}{|\mathcal{N}(t)|}\sum_{i\in\mathcal{N}(t)} \tilde{F}_i(t, \hat{y}(t), \hat{\mathbf{z}}(t)),$$
$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{Z}(t, \hat{\mathbf{z}}(t), \hat{y}(t)).$$

When all agents are connected, it becomes an extension of the Fitzhugh-Nagumo model, which exhibits bursting behavior. Such behavior is utilized in neuromorphic engineering, for instance, to emulate PWM (Pulse Width Modulation). Note that each individual agent can only converge to the equilibrium and only by interacting with each other a collective behavior emerges.

Our results are validated by the simulation as follows. Figure 3 shows that all output differences corresponding to an edge evolve inside the respective funnel (Theorem 1) and since the fractions $|\nu_{ij}(t)|/\psi_{ij}(t)$ are uniformly smaller than 1 the corresponding inputs are bounded (Theorem 2). As illustrated in Figure 2, the compact attractor (limit cycle) of the blended dynamics approximates the trajectory of the system (4) coupled via (6). This is the counterpart of Theorem 5 when the concept of input-to-state stability to a compact set is considered, cf. Remark 4 (v).

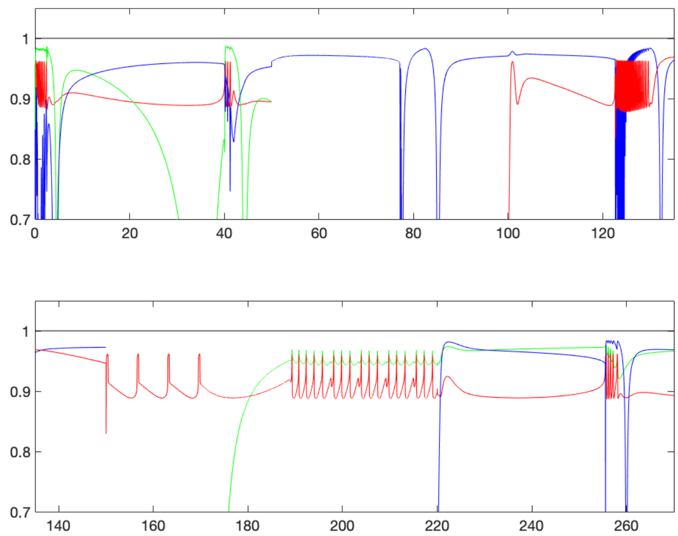

Figure 3: Illustration of all the fractions $|\nu_{ij}(t)|/\psi_{ij}(t)$.

5. Conclusion

In this paper, we introduced the edge-wise funnel coupling law, which retains all the benign properties of the node-wise funnel coupling law (2) from [1], but exhibits a more straightforward design of the emergent behavior, which is given exactly by the blended dynamics. Moreover, the emergent behavior can be realized without any restrictions and additional effort. The new coupling law is also better suitable for plug-and-play operation, which was illustrated by a simulation. Future research will focus on the extension of the results to systems with arbitrary relative degree and/or time-varying interaction topologies.

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Appendix A. Graph theoretical lemmas

For technical reasons, regardless of our assumption being that the underlying graph is undirected and connected (Assumption 4), in this section, we present two graph theoretical lemmas, that are essential for our proof of Theorem 1 outlined in Appendix B. The lemmas are concerned with directed graphs that have no loops. In Appendix B we will consider directed subgraphs of the original graph that have this property.

Recall that a tuple $(i_0, i_1, \ldots, i_l) \in \mathcal{N}^{l+1}$ is called a *path* (of length l) from i_0 to i_l, if $i_k \in \mathcal{N}_{i_{k+1}}$ for all $k = 0, \ldots, l-1$. If $i_1, \ldots, i_l$ are distinct, then it is called *elementary*. A *loop* is an elementary path with $i_0 = i_l$. A node is *isolated*, if it has no incoming/outgoing edges. A *source* (*sink*) is a node that has no incoming (outgoing) edge. An isolated node is regarded as a source. If a graph has no loop and $\mathcal{E} \neq \emptyset$, then there exist both a source and a sink. Note that if $\{(i,j),(j,i)\} \in \mathcal{E}$, then this "undirected edge" constitutes a loop (i,j,i) in $\mathcal{G}$.

Lemma 4. *Consider a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with non-empty $\mathcal{E}$. Then $\mathcal{G}$ has no loop if, and only if, there exists a vector $\boldsymbol{\chi} \in \mathbb{R}^N$ such that $\chi_j - \chi_i > 0$ for all $(j,i) \in \mathcal{E}$.*

Proof. (Sufficiency): If there is a loop $(i_0, i_1, \ldots, i_l)$ in $\mathcal{G}$ where $i_0 = i_l$, then we have

$$0 = \chi_{i_0} - \chi_{i_l} = \textstyle\sum_{p=0}^{l-1} (\chi_{i_p} - \chi_{i_{p+1}}) > 0$$

by the assumption, which is a contradiction.

(Necessity): Since there is no loop, every path in the graph is elementary and has a finite length. Thus, we can define $\tilde{\mathcal{N}}_k$ as the set of nodes to which a path of maximal length k from a source leads. Obviously, $\tilde{\mathcal{N}}_0$ is the set of the sources, and there is a maximal length K for all paths in $\mathcal{G}$. Then, $\{\tilde{\mathcal{N}}_k\}_{k=0}^K$ is a partition of $\mathcal{N}$. Now, for each $k = 0, \ldots, K$, let $\chi_i := -k$ for all $i \in \tilde{\mathcal{N}}_k$. Then, for all $(j,i) \in \mathcal{E}$, if $j \in \tilde{\mathcal{N}}_k$ for some $k \in \{0, ..., K-1\}$ (note that $k = K$ is not possible), then clearly $i \in \tilde{\mathcal{N}}_l$ for some $l \in \{k+1, ..., K\}$, thus $\chi_j = -k$ and $\chi_i = -l \le -(k+1)$, thus $\chi_j - \chi_i \ge 1 > 0$. □

Let $\mathcal{N}_\uparrow$ and $\mathcal{N}_\downarrow$ be the sets of the sources and the sinks, respectively. Further, let $\mathcal{E}_\uparrow := \{ (j,i) \in \mathcal{E} \mid j \in \mathcal{N}_\uparrow \}$ and $\mathcal{E}_\downarrow := \{ (j,i) \in \mathcal{E} \mid i \in \mathcal{N}_\downarrow \}$, which are the outgoing edges from the sources, and the incoming edges to the sinks, respectively.

Lemma 5. *Consider a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with non-empty $\mathcal{E}$. If $\mathcal{G}$ has no loop, then there exist constants $\xi_{ij} > 0$ associated with each edge $(j,i) \in \mathcal{E}$ such that, for all vectors $\boldsymbol{\sigma} \in \mathbb{R}^N$, we have*

$$\sum_{(j,i)\in\mathcal{E}} \xi_{ij}(\sigma_j - \sigma_i) \equiv \sum_{(j,i)\in\mathcal{E}_\uparrow} \xi_{ij}\sigma_j - \sum_{(j,i)\in\mathcal{E}_\downarrow} \xi_{ij}\sigma_i. \quad \text{(A.1)}$$

Proof. The graph theoretic interpretation of (A.1) is the existence of edge weights ξ_{ij}, such that for all nodes which are not sinks or sources the sum of the weights of the incoming edges is equal to the sum of the outgoing edges. We show that, by choosing appropriate edge weights starting from the sources the proof can be concluded.

In the following, we sequentially pick a node $j \in \mathcal{N}$ and determine ξ_{ij} for all outgoing edges from node j. To this end, let d_j be the out-degree of node $j \in \mathcal{N}$ (i.e., the number of all outgoing edges), and let $\mathcal{E}_k := \{(j,i) \in \mathcal{E} \mid j \in \tilde{\mathcal{N}}_k\}$ be the set of all outgoing edges from the nodes in $\tilde{\mathcal{N}}_k$, where $\tilde{\mathcal{N}}_k$ is as in the proof of Lemma 4. It is clear that $\{\mathcal{E}_k\}_{k=0}^K$ is a partition of $\mathcal{E}$. As the first step, for each $(j,i) \in \mathcal{E}_0$, assign $\xi_{ij} := 1/d_j$. Regarding ξ_{ij} as the amount of flow through the edge (j,i), this is interpreted as assigning the equally divided outgoing flow from the source. By this, the incoming flows for all nodes $j \in \tilde{\mathcal{N}}_1$ are determined, and thus, we can assign the outgoing flow ξ_{ij} for all $(j,i) \in \mathcal{E}_1$ as the amount of incoming flow divided by its out-degree:

$$\xi_{ij} := \frac{\sum_{l\in\mathcal{N}_j} \xi_{jl}}{d_j} > 0. \quad \text{(A.2)}$$

In this way, we sequentially assign all the outgoing flow for the nodes in $\tilde{\mathcal{N}}_k$, $k = 0, \ldots, K$, in the increasing order of k. Recalling that $\{\mathcal{E}_k\}_{k=0}^K$ is a partition of $\mathcal{E}$, this procedure determines the flow $\xi_{ij} > 0$ for all edges in $\mathcal{E}$. Then, by construction, (A.1) holds. □

Appendix B. Proof of Theorem 1

The proof technique is similar to that of the node-wise funnel coupling case, given in [1], hence we will keep the proof brief. In this section, we explain the main differences. For this purpose, we will cite equations from [1] as, for example, (3) in [1] as (N3). The full proof is available in the extended version of the paper on arXiv.

First, we show the existence of a unique (local) solution. Let $q := n_1 + \ldots + n_N$ and define the relatively open set

$$\Omega := \left\{ \begin{array}{l} (t, \mathbf{x}_1, \ldots, \mathbf{x}_N) \\ \in \mathbb{R}_{\ge 0} \times \mathbb{R}^q \end{array} \middle| \begin{array}{l} \forall\,(j,i) \in \mathcal{E}\ \forall\, p \in \mathcal{M} : \\ |\mathbf{H}_j(\mathbf{x}_j)^p - \mathbf{H}_i(\mathbf{x}_i)^p| < \psi_{ij}^p(t) \end{array} \right\}$$

and $\mathbf{R} : \Omega \to \mathbb{R}^q, (t, \mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto \big(\mathbf{R}_1(t, \Phi_1(\mathbf{x}_1)), \ldots, \mathbf{R}_N(t, \Phi_N(\mathbf{x}_N))\big)$ with

$$\mathbf{R}_i(t, \mathbf{y}_i, \mathbf{z}_i) = \begin{pmatrix} \tilde{\mathbf{F}}_i(t, \mathbf{y}_i, \mathbf{z}_i) + \sum_{j\in\mathcal{N}_i} \mathbf{u}_{ij}(t, \boldsymbol{\nu}_{ij}) \\ \mathbf{Z}_i(t, \mathbf{y}_i, \mathbf{z}_i) \end{pmatrix},$$

$i \in \mathcal{N}$. Then the system (4), (6) is equivalent to

$$\dot{\mathbf{x}}(t) = \mathbf{R}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathrm{col}\big(\mathbf{x}_1(t_0), \ldots, \mathbf{x}_N(t_0)\big).$$

By assumption we have $\mathbf{x}(t_0) \in \Omega$ and $\mathbf{R}$ is measurable and locally integrable in t and locally Lipschitz continuous in x. Therefore, by the theory of ordinary differential equations (see e.g. [17, § 10, Thm. XX]) there exists a unique maximal solution $\mathbf{x} : [t_0, \omega) \to \mathbb{R}^q$, $\omega \in (0, \infty]$, of (4) and (6) which satisfies $(t, \mathbf{x}(t)) \in \Omega$ for all $t \in [t_0, \omega)$. Furthermore, the closure of the graph of this solution is not a compact subset of Ω.

Assume that $\omega < \infty$. Then, different from [1], we find that

$$\mathcal{E}^p_+(\{\tau_k\}) := \left\{ (j,i) \in \mathcal{E} \;\middle|\; \lim_{k\to\infty} \frac{\nu^p_{ij}(\tau_k)}{\psi^p_{ij}(\tau_k)} = 1 \right\}$$

is non-empty for some $p \in \mathcal{M}$,

or

$$\mathcal{E}^p_-(\{\tau_k\}) := \left\{ (j,i) \in \mathcal{E} \;\middle|\; \lim_{k\to\infty} \frac{\nu^p_{ij}(\tau_k)}{\psi^p_{ij}(\tau_k)} = -1 \right\}$$

is non-empty for some $p \in \mathcal{M}$,

instead of that $\mathcal{I}_+(\{\tau_k\})$ is non-empty or $\mathcal{I}_-(\{\tau_k\})$ is non-empty. Assuming that $\mathcal{E}^p_+(\{\tau_k\})$ is non-empty, we will instead show that a contradiction occurs, if the graph $(\mathcal{N}, \mathcal{E}^p_+(\{\tau_k\}))$ has a loop. If $(\mathcal{N}, \mathcal{E}^p_+(\{\tau_k\}))$ has no loop, then we will show that it is possible to construct another time sequence $\{\bar\tau_k\}$ (based on $\{\tau_k\}$), such that $|\mathcal{E}^p_+(\{\tau_k\})| < |\mathcal{E}^p_+(\{\bar\tau_k\})|$, similar to (N9). By repeating the argument, we arrive at a graph $(\mathcal{N}, \mathcal{E}^p_+(\{\hat\tau_k\}))$, for some time sequence $\{\hat\tau_k\}$, that has a loop, which yields a contradiction as we will show. Therefore, we conclude that $\omega = \infty$ and (7) is achieved.

For convenience, we write $\mathcal{E}^p$ instead of $\mathcal{E}^p_+(\{\tau_k\})$ in the following. Note first that, by the definition of $\mathcal{E}^p$, there exists $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and all $(j,i) \in \mathcal{E}^p$ we have $y^p_j(\tau_k) - y^p_i(\tau_k) = \nu^p_{ij}(\tau_k) > 0$, because $\psi^p_{ij}(t) > 0$ for all $t \in [t_0, \omega)$. Hence, by Lemma 4 in Appendix A, the graph $(\mathcal{N}, \mathcal{E}^p)$ cannot have a loop.

The remainder of the proof follows as in [1], where we instead use the absolutely continuous function

$$W(t) := \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij}\nu^p_{ij}(t) = \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \cdot (y^p_j(t) - y^p_i(t))$$

where ξ_{ij} is given by Lemma 5 in Appendix A in terms of the graph $(\mathcal{N}, \mathcal{E}^p)$. The sequences $\{\varepsilon_q\}_{q\in\mathbb{N}}$, $\{\tau_{k_q}\}_{q\in\mathbb{N}}$ and $\{s_q\}_{q\in\mathbb{N}}$ are similarly defined as in [1], and similar to (N14) we may conclude that, for some $\bar\xi > 0$,

$$\forall\, q \in \mathbb{N} : \; \dot W(s_q) \geq -\bar\xi\theta_\psi. \tag{B.1}$$

The main difference appears when we arrive at the derivation of (N16). We have to instead invoke Lemma 1 together with Lemma 5 in Appendix A for the graph $(\mathcal{N}, \mathcal{E}^p)$ to obtain, for almost all $t \in [t_0, \omega)$,

$$\begin{aligned} \dot W(t) \leq M_0 &+ \sum_{(j,i)\in\mathcal{E}^p_\uparrow} \xi_{ij} \sum_{(l,j)\in\mathcal{E}} \mu^p_{jl}(t) \\ &- \sum_{(j,i)\in\mathcal{E}^p_\downarrow} \xi_{ij} \sum_{(l,i)\in\mathcal{E}} \mu^p_{il}(t), \end{aligned} \tag{B.2}$$

where $\mu^p_{kl}(t) = \mu^p_{kl}(\nu^p_{kl}(t)/\psi^p_{kl}(t))$ for $(l,k) \in \mathcal{E}$. Define the edge sets

$$\begin{aligned} \mathcal{E}^p_{\mathrm{large}} &:= \left\{ (l,i) \in \mathcal{E}^p \;\middle|\; \exists\, j \in \mathcal{N} : \; (j,i) \in \mathcal{E}^p_\downarrow \right\} = \mathcal{E}^p_\downarrow, \\ \mathcal{E}^p_{\mathrm{small}} &:= \left\{ (l,j) \in \mathcal{E} \;\middle|\; \exists\, i \in \mathcal{N} : \; (j,i) \in \mathcal{E}^p_\uparrow \right\} \\ &\cup \left\{ (i,l) \in \mathcal{E} \;\middle|\; (l,i) \in \mathcal{E} \setminus \mathcal{E}^p, \; \exists\, j \in \mathcal{N} : \; (j,i) \in \mathcal{E}^p_\downarrow \right\}. \end{aligned}$$

By definition of $\mathcal{E}^p_\uparrow$ and $\mathcal{E}^p_\downarrow$ in Appendix A, we have $\emptyset \neq \mathcal{E}^p_\downarrow = \mathcal{E}^p_{\mathrm{large}} \subseteq \mathcal{E}^p$ and $\emptyset \neq \{(i,j) | (j,i) \in \mathcal{E}^p_\uparrow\} \subseteq \mathcal{E}^p_{\mathrm{small}} \subseteq \mathcal{E} \setminus \mathcal{E}^p$. The latter holds because from $(j,i) \in \mathcal{E}^p_\downarrow$, node i is a sink of the graph $(\mathcal{N}, \mathcal{E}^p)$, hence $(i,l) \notin \mathcal{E}^p$ for all $l \in \mathcal{N}$. Similarly, if $(j,i) \in \mathcal{E}^p_\uparrow$, then j is a source of the graph $(\mathcal{N}, \mathcal{E}^p)$, hence $(l,j) \notin \mathcal{E}^p$ for all $l \in \mathcal{N}$.

Now, since $-\mu^p_{il}(t) = \mu^p_{li}(t)$ for any $(l,i) \in \mathcal{E}$ by Assumption 5, we can rewrite (B.2) as

$$\dot W(t) \leq M_0 + \sum_{(l,j)\in\mathcal{E}^p_{\mathrm{small}}} \zeta_{jl}\mu^p_{jl}(t) - \sum_{(l,i)\in\mathcal{E}^p_{\mathrm{large}}} \zeta_{il}\mu^p_{il}(t) \tag{B.3}$$

with positive constants

$$\zeta_{jl} = \begin{cases} \sum_{(j,k)\in\mathcal{E}^p_\uparrow} \xi_{kj} + \sum_{(k,i)\in\mathcal{E}^p_\downarrow} \xi_{ik}, & (l,j) \in \mathcal{E}^p_{\mathrm{small}}, \\ \sum_{(k,j)\in\mathcal{E}^p_\downarrow} \xi_{jk}, & (l,j) \in \mathcal{E}^p_{\mathrm{large}}. \end{cases}$$

Then, from (B.1) and (B.3), we may similarly conclude that $\sum_{(l,j)\in\mathcal{E}^p_{\mathrm{small}}} \max\{\mu^p_{jl}(s_q), 0\} \to \infty$ as $q \to \infty$. Therefore, invoking that $\mathcal{E} \setminus \mathcal{E}^p$ is finite, there exists a subsequence $\{\bar\tau_k\} = \{s_{q_k}\}$ and an edge $(j^*, i^*) \in \mathcal{E}^p_{\mathrm{small}} \subseteq \mathcal{E} \setminus \mathcal{E}^p$ such that $\nu^p_{i^*j^*}(\bar\tau_k)/\psi^p_{i^*j^*}(\bar\tau_k) \to 1$ as $k \to \infty$. Consequently, $\mathcal{E}^p_+(\{\tau_k\}) \subseteq \mathcal{E}^p_+(\{s_q\}) \subseteq \mathcal{E}^p_+(\{\bar\tau_k\})$. Since $(j^*, i^*) \in \mathcal{E}^p_+(\{\bar\tau_k\}) \setminus \mathcal{E}^p_+(\{\tau_k\})$, the proof concludes.

Appendix C. Detailed proof of Theorem 1 (only contained in arXiv-version)

First, we show the existence of a unique (local) solution. Let $q := n_1 + \ldots + n_N$ and define the relatively open set

$$\Omega := \left\{ \begin{array}{c} (t, \mathbf{x}_1, \ldots, \mathbf{x}_N) \\ \in \mathbb{R}_{\geq 0} \times \mathbb{R}^q \end{array} \middle| \begin{array}{c} \forall (j,i) \in \mathcal{E}\ \forall p \in \mathcal{M} : \\ |\mathbf{H}_j(\mathbf{x}_j)^p - \mathbf{H}_i(\mathbf{x}_i)^p| < \psi_{ij}^p(t) \end{array} \right\}$$

and $\mathbf{R} : \Omega \to \mathbb{R}^q, (t, \mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto \big(\mathbf{R}_1(t, \Phi_1(\mathbf{x}_1)), \ldots, \mathbf{R}_N(t, \Phi_N(\mathbf{x}_N))\big)$ with

$$\mathbf{R}_i(t, \mathbf{y}_i, \mathbf{z}_i) = \begin{pmatrix} \tilde{\mathbf{F}}_i(t, \mathbf{y}_i, \mathbf{z}_i) + \sum_{j \in \mathcal{N}_i} \mathbf{u}_{ij}(t, \boldsymbol{\nu}_{ij}) \\ \mathbf{Z}_i(t, \mathbf{y}_i, \mathbf{z}_i) \end{pmatrix},$$

$i \in \mathcal{N}$. Then the system (4), (6) is equivalent to

$$\dot{\mathbf{x}}(t) = \mathbf{R}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \operatorname{col}\big(\mathbf{x}_1(t_0), \ldots, \mathbf{x}_N(t_0)\big).$$

By assumption we have $\mathbf{x}(t_0) \in \Omega$ and $\mathbf{R}$ is measurable and locally integrable in t and locally Lipschitz continuous in x. Therefore, by the theory of ordinary differential equations (see e.g. [17, § 10, Thm. XX]) there exists a unique maximal solution $\mathbf{x} : [t_0, \omega) \to \mathbb{R}^q$, $\omega \in (0, \infty]$, of (4) and (6) which satisfies $(t, \mathbf{x}(t)) \in \Omega$ for all $t \in [t_0, \omega)$. Furthermore, the closure of the graph of this solution is not a compact subset of Ω.

The proof is done by a contradiction. Assume that $\omega < \infty$. This implies that there is a time sequence $\{\tau_k\}$ such that τ_k is strictly increasing and $\lim_{k\to\infty} \tau_k = \omega$, and

$$\mathcal{E}_+^p(\{\tau_k\}) := \left\{ (j,i) \in \mathcal{E} : \lim_{k\to\infty} \frac{\nu_{ij}^p(\tau_k)}{\psi_{ij}^p(\tau_k)} = 1 \right\}$$

is non-empty for some $p \in \mathcal{M}$,

or

$$\mathcal{E}_-^p(\{\tau_k\}) := \left\{ (j,i) \in \mathcal{E} : \lim_{k\to\infty} \frac{\nu_{ij}^p(\tau_k)}{\psi_{ij}^p(\tau_k)} = -1 \right\}$$

is non-empty for some $p \in \mathcal{M}$.

Let us first assume that $\mathcal{E}_+^p(\{\tau_k\})$ is non-empty for some $p \in \mathcal{M}$. Then, we will first show that a contradiction occurs if graph $(\mathcal{N}, \mathcal{E}_+^p(\{\tau_k\}))$ has a loop. If graph $(\mathcal{N}, \mathcal{E}_+^p(\{\tau_k\}))$ has no loop, we will then show that it is possible to construct another time sequence $\{\bar{\tau}_k\}$ (based on $\{\tau_k\}$), such that

$$|\mathcal{E}_+^p(\{\tau_k\})| < |\mathcal{E}_+^p(\{\bar{\tau}_k\})| \tag{C.1}$$

where the notation $|\cdot|$ implies the cardinality of the set. By repeating this argument (i.e., by replacing the role of $\{\tau_k\}$ with $\{\bar{\tau}_k\}$), we arrive after finitely many steps at the condition that graph $(\mathcal{N}, \mathcal{E}_+^p(\{\tau_k\}))$ has a loop (because $\mathcal{E}_+^p(\{\tau_k\}) \subseteq \mathcal{E}$ and the original graph $(\mathcal{N}, \mathcal{E})$ has a trivial loop represented by an undirected edge), which yields a contradiction. This means that there is no such sequence $\{\tau_k\}$ that makes $\mathcal{E}_+^p(\{\tau_k\})$ non-empty for each $p \in \mathcal{M}$. Similarly, it can be seen that there is no sequence that makes $\mathcal{E}_-^p(\{\tau_k\})$ non-empty for each $p \in \mathcal{M}$. Therefore, we conclude that there is no such finite time ω, and the control objective (7) is achieved.

Let us carry out the above described proof steps. For convenience, we write $\mathcal{E}^p$ instead of $\mathcal{E}_+^p(\{\tau_k\})$ in the following. Note first that, by the definition of $\mathcal{E}^p$, there exists $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and all $(j,i) \in \mathcal{E}^p$ we have $y_j^p(\tau_k) - y_i^p(\tau_k) = \nu_{ij}^p(\tau_k) > 0$, because $\psi_{ij}^p(t) > 0$ for all $t \in [t_0, \omega)$. Hence, by Lemma 4 in Appendix A, the graph $(\mathcal{N}, \mathcal{E}^p)$ cannot have a loop.

Now, we continue the proof for the case when the graph $(\mathcal{N}, \mathcal{E}^p)$ has no loop, hence becomes a directed subgraph of the original graph $(\mathcal{N}, \mathcal{E})$. For this purpose, let

$$W(t) := \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \nu_{ij}^p(t) = \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \cdot (y_j^p(t) - y_i^p(t))$$

where ξ_{ij} is given by Lemma 5 in Appendix A in terms of the graph $(\mathcal{N}, \mathcal{E}^p)$. Note that $W(t)$ is absolutely continuous,

$$W(t) < \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \psi_{ij}^p(t), \quad t \in [t_0, \omega),$$

and

$$\lim_{k\to\infty} W(\tau_k) = \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \psi_{ij}^p(\omega).$$

Let us now consider a strictly decreasing sequence $\{\varepsilon_q\} \subseteq (0,1)$ such that $\lim_{q\to\infty} \varepsilon_q = 0$ and $W(t_0) < (1 - \varepsilon_0) \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \psi_{ij}^p(t_0)$. Choose a subsequence $\{\tau_{k_q}\}_{q\in\mathbb{N}}$ of $\{\tau_k\}$ such that

$$\forall\, q \in \mathbb{N} : \ W(\tau_{k_q}) \geq \Big(1 - \frac{\varepsilon_q}{2}\Big) \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \psi_{ij}^p(\tau_{k_q}). \tag{C.2}$$

Based on this subsequence, we now construct a sequence $\{s_q\}_{q\in\mathbb{N}}$ such that

$$s_q := \max \left\{ s \in [t_0, \tau_{k_q}] \,\middle|\, W(s) = (1 - \varepsilon_q) \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \psi_{ij}^p(s) \right\}. \tag{C.3}$$

By (C.2) and (C.3), the sequence $\{s_q\}$ is strictly increasing and $\lim_{q\to\infty} s_q = \omega$. Moreover, since $\lim_{q\to\infty} W(s_q) / \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \psi_{ij}^p(s_q) = 1$,

$$\forall\, (j,i) \in \mathcal{E}^p : \ \lim_{q\to\infty} \frac{\nu_{ij}^p(s_q)}{\psi_{ij}^p(s_q)} = 1. \tag{C.4}$$

In addition, from Assumption 5 and from (C.2) and (C.3), it follows that $s_q < \tau_{k_q}$ and

$$\dot{W}(s_q) \geq (1 - \varepsilon_q) \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \dot{\psi}_{ij}^p(s_q) \geq -\bar{\xi}\theta_\psi \tag{C.5}$$

for all $q \in \mathbb{N}$, where $\bar{\xi} := \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij}$.[8]

On the other hand, if we compute $\dot{W}$, then we have

$$\begin{aligned} \dot{W}(t) = &\sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} (\tilde{f}_j^p(t, \mathbf{y}_j, \mathbf{z}_j) - \tilde{f}_i^p(t, \mathbf{y}_i, \mathbf{z}_i)) \\ &+ \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \sum_{(l,j)\in\mathcal{E}} \mu_{jl}^p(t) - \sum_{(j,i)\in\mathcal{E}^p} \xi_{ij} \sum_{(l,i)\in\mathcal{E}} \mu_{il}^p(t) \end{aligned}$$

[8] Without loss of generality, we may choose ε_q such that W is differentiable at s_q, as W is differentiable almost everywhere.

for almost all $t \in [t_0, \omega)$, where $\widetilde{\mathbf{F}}_i(t, \mathbf{y}_i, \mathbf{z}_i) = \mathrm{col}(\tilde{f}_i^1(t, \mathbf{y}_i, \mathbf{z}_i), \ldots, \tilde{f}_i^m(t, \mathbf{y}_i, \mathbf{z}_i))$ and $\mu_{kl}^p(t) = \mu_{kl}^p(\nu_{kl}^p(t)/\psi_{kl}^p(t))$, $k \in \mathcal{N}$, $(l, k) \in \mathcal{E}$, for simplicity. We can bound the first sum by $M_0 := \overline{\xi} M_f$, where the constant M_f is such that

$$\forall\, t \in [t_0, \omega) : \ |\tilde{f}_j^p(t, \mathbf{y}_j(t), \mathbf{z}_j(t)) - \tilde{f}_i^p(t, \mathbf{y}_i(t), \mathbf{z}_i(t))| \leq M_f, \tag{C.6}$$

whose existence follows from Lemma 1 and Assumption 1, because ω is finite. Invoking Lemma 5 for the edge set $\mathcal{E}^p$, we therefore have that

$$\begin{aligned} \dot{W}(t) \leq M_0 + &\sum_{(j,i)\in\mathcal{E}_\uparrow^p} \xi_{ij} \sum_{(l,j)\in\mathcal{E}} \mu_{jl}^p(t) \\ - &\sum_{(j,i)\in\mathcal{E}_\downarrow^p} \xi_{ij} \sum_{(l,i)\in\mathcal{E}} \mu_{il}^p(t) \end{aligned} \tag{C.7}$$

for almost all $t \in [t_0, \omega)$. Let $\mathcal{E}_{\text{large}}^p$ be the set of all edges $(l, i) \in \mathcal{E}$ in (C.7) such that $(l, i) \in \mathcal{E}^p$, hence $\lim_{q\to\infty} \mu_{il}^p(s_q) = \infty$;

$$\mathcal{E}_{\text{large}}^p := \left\{ (l, i) \in \mathcal{E}^p \ \middle|\ \exists\, j \in \mathcal{N} : \ (j, i) \in \mathcal{E}_\downarrow^p \right\} = \mathcal{E}_\downarrow^p.$$

By its construction, $\emptyset \neq \mathcal{E}_\downarrow^p = \mathcal{E}_{\text{large}}^p \subseteq \mathcal{E}^p$. Now, since $-\mu_{il}^p(t) = \mu_{li}^p(t)$ for any $(l, i) \in \mathcal{E}$ by Assumption 5, we can rewrite (C.7) as

$$\dot{W}(t) \leq M_0 + \sum_{(l,j)\in\mathcal{E}_{\text{small}}^p} \zeta_{jl}\mu_{jl}^p(t) - \sum_{(l,i)\in\mathcal{E}_{\text{large}}^p} \zeta_{il}\mu_{il}^p(t) \tag{C.8}$$

with positive constants

$$\zeta_{jl} = \begin{cases} \sum_{(j,k)\in\mathcal{E}_\uparrow^p} \xi_{kj} + \sum_{(k,i)\in\mathcal{E}_\downarrow^p} \xi_{ik}, & (l, j) \in \mathcal{E}_{\text{small}}^p, \\ \sum_{(k,j)\in\mathcal{E}_\downarrow^p} \xi_{jk}, & (l, j) \in \mathcal{E}_{\text{large}}^p, \end{cases}$$

where

$$\begin{aligned} \mathcal{E}_{\text{small}}^p := &\left\{ (l, j) \in \mathcal{E} \ \middle|\ \exists\, i \in \mathcal{N} : \ (j, i) \in \mathcal{E}_\uparrow^p \right\} \\ \cup &\left\{ (i, l) \in \mathcal{E} \ \middle|\ (l, i) \in \mathcal{E} \setminus \mathcal{E}^p, \ \exists\, j \in \mathcal{N} : \ (j, i) \in \mathcal{E}_\downarrow^p \right\}. \end{aligned}$$

Note that, by the definition of $\mathcal{E}_\uparrow^p$ and $\mathcal{E}_\downarrow^p$ in Appendix A, we have $\emptyset \neq \{(i, j) | (j, i) \in \mathcal{E}_\uparrow^p\} \subseteq \mathcal{E}_{\text{small}}^p \subseteq \mathcal{E} \setminus \mathcal{E}^p$. This is because from $(j, i) \in \mathcal{E}_\downarrow^p$, node i is a sink of the graph $(\mathcal{N}, \mathcal{E}^p)$, hence $(i, l) \notin \mathcal{E}^p$ for all $l \in \mathcal{N}$. Similarly, if $(j, i) \in \mathcal{E}_\uparrow^p$, then j is a source of the graph $(\mathcal{N}, \mathcal{E}^p)$, hence $(l, j) \notin \mathcal{E}^p$ for all $l \in \mathcal{N}$.

Now, (C.5) and (C.8) yield

$$\sum_{(l,j)\in\mathcal{E}_{\text{small}}^p} \zeta_{jl}\mu_{jl}^p(s_q) \geq \sum_{(l,i)\in\mathcal{E}_{\text{large}}^p} \zeta_{il}\mu_{il}^p(s_q) - M_0 - \overline{\xi}\theta_\psi =: M_q.$$

Thus, it follows from (C.4) that $M_q \to \infty$ as $q \to \infty$. Since

$$\sum_{(l,j)\in\mathcal{E}_{\text{small}}^p} \zeta_{jl}\mu_{jl}^p(s_q) \leq \overline{\zeta} \sum_{(l,j)\in\mathcal{E}_{\text{small}}^p} \max\{\mu_{jl}^p(s_q), 0\}$$

where $\overline{\zeta} := \max_{(l,j)\in\mathcal{E}_{\text{small}}^p} \zeta_{jl} > 0$, we have

$$\sum_{(l,j)\in\mathcal{E}_{\text{small}}^p} \max\{\mu_{jl}^p(s_q), 0\} \geq \frac{M_q}{\overline{\zeta}}. \tag{C.9}$$

Therefore, for each sufficiently large q, there is an edge $(j_q, i_q) \in \mathcal{E}_{\text{small}}^p \subseteq \mathcal{E} \setminus \mathcal{E}^p$ such that $\mu_{i_q j_q}^p(s_q) \geq M_q/(|\mathcal{E}_{\text{small}}^p|\overline{\zeta})$; hence

$$\mu_{i_q j_q}^p\left(\frac{\nu_{i_q j_q}^p(s_q)}{\psi_{i_q j_q}^p(s_q)}\right) \to \infty, \quad \text{i.e.} \quad \frac{\nu_{i_q j_q}^p(s_q)}{\psi_{i_q j_q}^p(s_q)} \to 1.$$

Since $\mathcal{E} \setminus \mathcal{E}^p$ is a finite set, there is a subsequence $\{\bar{\tau}_k\} = \{s_{q_k}\}$ such that $(j^*, i^*) = (j_{q_k}, i_{q_k}) \in \mathcal{E}_{\text{small}}^p \subseteq \mathcal{E} \setminus \mathcal{E}^p$ and $\nu_{i^*j^*}^p(\bar{\tau}_k)/\psi_{i^*j^*}^p(\bar{\tau}_k) \to 1$ as $k \to \infty$. Consequently,

$$\mathcal{E}_+^p(\{\tau_k\}) \overset{\text{(C.4)}}{\subseteq} \mathcal{E}_+^p(\{s_q\}) \subseteq \mathcal{E}_+^p(\{\bar{\tau}_k\}).$$

By construction, $(j^*, i^*) \in \mathcal{E}_+^p(\{\bar{\tau}_k\}) \setminus \mathcal{E}_+^p(\{\tau_k\})$ and we can conclude (C.1) as desired.