Stability analysis of switched nonlinear differential-algebraic equations via nonlinear Weierstrass form

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***Abstract*— In this paper, we propose some sufficient conditions for checking the asymptotic stability of switched nonlinear differential-algebraic equations (DAEs) under arbitrary switching signal. We assume that each model of a given switched DAEs is externally equivalent to a nonlinear Weierstrass form. With the help of this form, we can define nonlinear consistency projectors and jump-flow solutions for switched nonlinear DAEs. Then we use a different approach from the paper [12] to study the stability of switched DAEs via a novel notion called the jump-flow explicitation, which attaches a nonlinear control system to a given nonlinear DAE and can be used to simplify the common Lyapunov function conditions for both the flow and the jump dynamics of switched nonlinear DAEs. At last, a numerical example is given to illustrate how to check the stability of a switched nonlinear DAE by constructing a common Lyapunov function.**

I. INTRODUCTION

We study switched nonlinear differential-algebraic equations (DAEs) of the following form

$$\Xi_\sigma : \quad E_\sigma(x)\dot{x} = F_\sigma(x), \tag{1}$$

where $\sigma : \mathcal{I} \to \mathcal{N}$ is a right continues switching signal with a locally finite number of jumps, $\mathcal{N} := \{1, \ldots, N\}$, the integer N is the number of DAE models and $\mathcal{I} \subseteq \mathbb{R}$ is a time interval. The variables $x \in X$ are called generalized state and X is an open subset of $\mathbb{R}^n$ (or an n-dimensional manifold). For each $p \in \mathcal{N}$, the maps $E_p : TX \to \mathbb{R}^n$ and $F_p : X \to \mathbb{R}^n$ are $\mathcal{C}^\infty$-smooth, where TX denotes the tangent bundle of X. In recent decades, there have been increased interests on the linear case of (1), i.e., a switched linear DAE

$$\Delta_\sigma : \quad E_\sigma \dot{x} = H_\sigma x, \tag{2}$$

where $E_p : \mathbb{R}^n \to \mathbb{R}^n$ and $H_p : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps for each $p \in \mathcal{N}$. Some results on the stability analysis of switched linear DAEs can be found in e.g., [11], [24], [26], [21], [16] using Lyapunov method and linear matrices inequalities (LMIs) technique, and in [24], [13], [25] with the help of commutativity conditions of matrix pencils.

Some results on the existence and uniqueness of $\mathcal{C}^1$-solutions of nonlinear DAEs (i.e., the non-switching case of (1)) can be consulted in e.g. [15], [14], and our recent papers [6], [3]. Unlike ordinary differential equations (ODEs), the $\mathcal{C}^1$-solutions of a DAE Ξ exist only on its *consistency space* $\mathfrak{C}$ (see section II below), which is in general a subset of X. So for any inconsistent initial point x_0, i.e., $x_0 \notin \mathfrak{C}$, there does not exist any $\mathcal{C}^1$-solutions, then there has to be a jump or an impulse which steers x_0 into a consistent point of $\mathfrak{C}$. Note that this inconsistent initialization problem can be frequently caused by switching behavior of switched DAEs because the switching point on the trajectory of the model before switching is usually not on the consistency space of the model after switching, so defining and analyzing jumps and impulses play important roles for studying solutions of switched DAEs.

Compared to the rich literature for switched linear DAEs, there are much less results concerning switched nonlinear DAEs. The challenges of studying switched nonlinear DAEs are multi-fold. For example, the distributional (generalized function) solution theory is a well-established framework for the discontinuity problems of (switched) linear DAEs, see [8], [18], [17], [20], while it may not be a suitable setting for nonlinear ODEs or DAEs, because e.g., the image of a nonlinear map applied to a Dirac impulse δ is not well-defined in general. Moreover, canonical forms as the Weierstrass form [22] (see (3)) decouple any linear DAE into its ODE part, which represents the flow dynamics, and its nilpotent part, which is related to its impulses and jumps. By using the Weierstrass form, it is always possible to construct the linear consistency projector, which defines a unique jump from any given inconsistent initial point to a consistent one. Nevertheless, obtaining a fully-decoupled normal form for nonlinear DAEs has been an open problem for decades, some efforts have been made for studying normal forms of nonlinear DAEs, see e.g. [2], [7], [3].

The first paper to rigorously study switched nonlinear DAEs of the form (1) is [12] and it is a significant inspiration for the present paper. However, the jump rule in [12] (see also (11) below) was shown in [4], [5] to be *not* consistent with nonlinear coordinates changes, i.e., it is not coordinate-free. In the present paper, we will use a nonlinear Weierstrass form from [3] (see Definition 2 below), by which we can define a coordinate-free nonlinear consistency projector (Definition 5) and obtain an impulse-free condition (see condition **(A2)** below). Moreover, we will introduce a novel notion called jump-flow explicitation of nonlinear DAEs, which allows to study the jump-flow behavior of nonlinear DAEs with the help of nonlinear control system theory. Our main result, i.e., stability analysis for switched DAEs, is given using jump-flow explicitation, the advantages of using this notion are discussed in Remark 17 below.

This paper is organized as follows. We recall the nonlinear

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Weierstrass form from [3] and discuss $\mathcal{C}^1$-solutions and jumps of nonlinear DAEs using such a normal form in Section II. The definition of jump-flow explicitation and the main results of the paper are given in Section III. We give an example in Section IV to illustrate our results of the stability conditions for switched DAEs. The conclusions and the proofs are put into Section V and the Appendix, respectively.

We will use the following notations for the present paper. We denote by $\mathcal{C}^k$ the class of k-times differentiable functions. For a map $A: X \to \mathbb{R}^{n\times n}$ and $\ker A(x)$, $\operatorname{Im} A(x)$ are the kernel and the image of A at x, respectively. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. We denote by $GL(n,\mathbb{R})$ the group of $n\times n$ invertible real matrices. We call a function $f: X \to \mathbb{R}$ positive-definite if $f(0) = 0$ and $f(x) > 0$, $\forall x \in X \backslash \{0\}$.

II. Nonlinear Weierstrass form for DAEs

Recall that for a linear DAE $\Delta : E\dot{x} = Hx$, the consistency space $\mathfrak{C}$ is a linear subspace that can be identified by the Wong sequence [23],

$$\mathscr{V}_1 := \mathbb{R}^n, \quad \mathscr{V}_{i+1} := H^{-1}E\mathscr{V}_i, \quad i \geq 1,$$

the consistency space $\mathfrak{C}$ of Δ coincides with the limits $\mathscr{V}^* := \mathscr{V}_n$ of the sequence $\mathscr{V}_i$. In particular, the DAE Δ is called *regular* if $\det(sE - H) \in \mathbb{R}^{n\times n}[s] \setminus \{0\}$. A linear *regular* DAE $\Delta = (E, H)$ is always (externally) equivalent (see Definition 1 below for nonlinear DAEs), via two constant invertible matrices Q and P, to the Weierstrass form [22], [1], given by $\tilde{\Delta} = (QEP^{-1}, QHP^{-1})$,

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3}$$

where $A_1 \in \mathbb{R}^{n_1\times n_1}$ and $N = \operatorname{diag}\{N_1, \ldots, N_a\} \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix, and where $N_i^{\nu_i - 1} \neq 0$, $N_i^{\nu_i} = 0$ for $\nu_i \geq 1$ and if x_2^i-subsystem is absent, then $\nu_i = 0$. The index ν of Δ is defined by $\nu := \max\{\nu_i, 1 \leq i \leq a\}$. The consistency projector of Δ [17], [11] is defined by

$$\Pi_{E,H} := P^{-1} \left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & 0 \end{smallmatrix}\right] P. \tag{4}$$

For a given inconsistent point $x_0^- \in \mathbb{R}^n \backslash \mathscr{V}^*$, the jumping point $x_0^+ \in \mathfrak{C}$ from x_0^- is *unique* and is defined by $x_0^+ = \Pi_{E,H}(x_0^-)$.

Now we recall and introduce some notions for nonlinear (non-switched) DAEs of the form

$$\Xi : \quad E(x)\dot{x} = F(x), \tag{5}$$

we denote shortly the above DAE by $\Xi = (E, F)$. We call a $\mathcal{C}^1$-curve $x: \mathcal{I} \to \mathbb{X}$ a $\mathcal{C}^1$*-solution* of Ξ if $E(x(t))\dot{x}(t) = F(x(t))$, $\forall t \in \mathcal{I}$. We call a $\mathcal{C}^1$-solution $x: \mathcal{I} \to X$ *maximal* if there is no other solution $\widetilde{x}: \widetilde{\mathcal{I}} \to X$ with $\mathcal{I} \subsetneq \widetilde{\mathcal{I}}$ and $x(t) = \widetilde{x}(t)$ for all $t \in \mathcal{I}$. A point $x_c \in X$ is called *consistent* if starting from x_c, there exists at least one $\mathcal{C}^1$-solution. The set of all consistent points is called the consistency space, denoted by $\mathfrak{C}$. We call Ξ *internally regular* if starting from any consistent point $x_c \in \mathfrak{C}$, there exists a unique maximal $\mathcal{C}^1$-solution. The consistency space of Ξ can be identified by constructing the sequence of submanifolds (by assuming[1] that each M_k is a smooth connected embedded submanifold on X): $M_1 = X$,

$$M_k := \{x \in M_{k-1} \,|\, F(x) \in E(x)T_xM_{k-1}\}, \ k \geq 1, \tag{6}$$

where $T_xM_k \subseteq \mathbb{R}^n$ denotes the tangent space of the submanifold M_k at $x \in M_k$. The last approach is called the geometric reduction method [15], [14], [6], [3], [7] and it is proved in [6], [3] that the limit $M^* := M_n$ of the sequence M_k coincides with the consistency space $\mathfrak{C}$ of Ξ.

Definition 1 (external equivalence [3], [4])**.** Two DAEs $\Xi = (E, F)$ and $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi: X \to \tilde{X}$ and a smooth map $Q: X \to GL(n,\mathbb{R})$ such that

$$\begin{aligned} \tilde{E}(\psi(x)) &= Q(x)E(x)\left(\frac{\partial\psi(x)}{\partial x}\right)^{-1}, \\ \tilde{F}(\psi(x)) &= Q(x)F(x). \end{aligned}$$

Note for two ex-equivalent systems Ξ and $\tilde{\Xi}$, a trajectory $x: \mathcal{I} \to \mathfrak{C}$ is a $\mathcal{C}^1$-solution of Ξ if, and only if, $\psi \circ x$ is a $\mathcal{C}^1$-solution of $\tilde{\Xi}$, however, since ψ is not only defined on the consistency space, the two systems are also expected to behave equivalent for inconsistent initial values.

Definition 2 (nonlinear Weierstrass form [3], [4])**.** We say that a nonlinear DAE $\tilde{\Xi}$ is represented in a nonlinear Weierstrass form if $\tilde{\Xi}$ is of the form

$$\textbf{(NWF)}: \quad \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} f^*(\xi_1) \\ \xi_2 \end{bmatrix},$$

where $\xi = (\xi_1, \xi_2) \in \tilde{X}_1 \times \tilde{X}_2 \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ are the generalized states, $\tilde{X}_2$ is a star domain with respect to the origin (i.e. for each $\xi_2 \in \tilde{X}_2$ also $\lambda\xi_2 \in \tilde{X}_2$ for all $\lambda \in [0, 1]$), $f^*(\xi_1)$ is a vector field on $\tilde{X}_1$ and $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent constant matrix as in (3). In particular, if $N = 0$, we say that $\tilde{\Xi}$ is in an index-1 nonlinear Weierstrass form **(INWF)**.

Remark 3. Necessary and sufficient conditions for the problem that when a nonlinear DAE is ex-equivalent to the **(NWF)** can be found in [3], but those conditions depend on a non-uniquely defined control system (explicitation) attaching to the DAE and are not easy to be checked. However, in [4], [5], we have shown that under some constant rank and regularity assumptions, a DAE $\Xi = (E, F)$ is (locally) ex-equivalent to the **(INWF)** if and only if Ξ is index-1 and the distribution defined by $\ker E$ is involutive, which are easily checkable conditions.

Lemma 4. *If a DAE Ξ is ex-equivalent to $\tilde{\Xi}$, represented in the* **(NWF)***, via a smooth map $Q: X \to GL(n,\mathbb{R})$ and a diffeomorphism $\psi = (\psi_1, \psi_2) = (\xi_1, \xi_2)$, then we have*

(i) *the ODE $\dot{\xi}_1 = f^*(\xi_1)$ has isomorphic $\mathcal{C}^1$-solutions with Ξ, i.e., a curve $x: \mathcal{I} \to X$ is a $\mathcal{C}^1$-solution of Ξ if and*

[1] Note that the assumption is simplified to adjust the purpose of the present paper, in general, M_k is a subset of M_{k-1} and we may need to take a neighborhood $U_k \subseteq X$ such that $M_k \cap U_k$ is a smooth connected embedded submanifold, see [3], [7]

only if $(\xi_1(t), \xi_2(t)) = (\psi_1(x(t)), 0)$ *is a solution of the* **(NWF)***;*

(ii) *the consistency space is*

$$\mathfrak{C}(\Xi) = M^*(\Xi) = \{x \in X \mid \psi_2(x) = 0\}\,; \qquad (7)$$

(iii) *the DAE* Ξ *is internally regular.*

Proof. The ex-equivalence defined by the invertible maps Q and the diffeomorphism ψ preserves $\mathcal{C}^1$-solutions. So $\xi(t) = (\xi_1(t), 0)$, where $\xi_1(t)$ is a solution of ODE $\dot{\xi}_1 = f^*(\xi_1)$, is a $\mathcal{C}^1$-solution of $\tilde{\Xi}$ if and only if $x(t) = \psi^{-1}(\xi(t))$ is a $\mathcal{C}^1$-solution of Ξ. It follows that the consistency spaces of Ξ and $\tilde{\Xi}$ coincide. By a directly calculation of the sequence $M_k(\tilde{\Xi})$ via (6), we get $\mathfrak{C}(\tilde{\Xi}) = M^*(\tilde{\Xi}) = \left\{\xi \in \tilde{X} \,\middle|\, \xi_2 = 0\right\}$, so item (ii) holds. Item (iii) is a direct consequence of item (i). ☐

Now in the same spirit of defining the linear consistent projector $\Pi_{E,H}$ of (4), we define the nonlinear consistency projector as follows.

Definition 5 (nonlinear consistency projector)**.** Consider a DAE $\Xi = (E, F)$ and assume that Ξ is ex-equivalent to the **(NWF)** via a Q-transformation and a diffeomorphism ψ. The nonlinear consistency projector $\Omega_{E,F} : X \to M^*$ of Ξ is then defined by

$$\Omega_{E,F} := \psi^{-1} \circ \pi \circ \psi,$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection attaching $(\xi_1, \xi_2) \mapsto (\xi_1, 0)$. In particular, the map $\Omega_{E,F}$ restricted to $\mathfrak{C} = M^*$ is the identity map.

Remark 6. The intuition behind the above definition is that, given a DAE in the **(NWF)**, an inconsistent initial point $\xi_0^- = (\xi_{10}^-, \xi_{20}^-) \notin M^*$ (i.e., $\xi_{20}^- \neq 0$) must jump into $\xi_0^+ = (\xi_{10}^-, 0) \in M^*$, i.e., only ξ_2-variables are allowed to jump, because any jump of ξ_1-variable will cause a Dirac impulse δ on the left-hand side of the ODE $\dot{\xi}_1 = f^*(\xi_1)$, then $f^*(\xi)$ has to produce the same impulse term to equalize the equation, which may not be possible, the jump $\xi_{20}^- \to 0$ of the ξ_2-variables also results in an impulse term on $N\dot{\xi}_2$, but we can still solve the linear DAE $N\dot{\xi}_2 = \xi_2$ in the distributional (generalized function) sense and the distributional solution (see e.g., [17]) at $t = 0$ can be expressed as

$$\xi_2[0] = -\sum_{i=0}^{\nu-2} N^{i+1}\xi_{20}^- \delta^{(i)}. \qquad (8)$$

However, the interpretation of this impulsive solution in the original coordinates is still unclear and outside the scope of this paper. Nevertheless, we can still call a solution *impulse-free* (in the original coordinates) if the expression (8) for a specific initial value evaluates to zero.

III. Stability analysis of switched DAEs via jump-flow explicitation

Consider a switched nonlinear DAE Ξ_σ, given by (1). We assume that

(SL) For all modes $\Xi_p = (E_p, F_p)$ of Ξ_σ we have that all local $\mathcal{C}^1$-solutions of Ξ_p on X can be extended to $\mathcal{I} = [0, \infty)$.

(NF) Each DAE model $\Xi_p = (E_p, H_p)$, $p \in \mathcal{N}$, of Ξ_σ is ex-equivalent to its **(NWF)** on X via a smooth map $Q_p : X \to GL(n, \mathbb{R})$ and a diffeomorphism $\psi_p = (\psi_{1p}, \psi_{2p}) : X \to \tilde{X}_p, x \mapsto (\xi_{1p}, \xi_{2p}) = (\psi_{1p}(x), \psi_{2p}(x))$.

Note that by the definition of the **(NWF)**, any jump $(\xi_{1p}, \xi_{2p}) \to (\xi_{1p}, 0)$ is well-defined and stays in $\tilde{X}_p \subseteq \mathbb{R}^n$. Then with the help of the nonlinear consistency projector of Definition 5, we define the jump-flow solution of the switched DAE Ξ_σ as follows.

Definition 7 (jump-flow solution)**.** Let σ be a switching signal with k switching times at $t_1, \ldots, t_k \in \mathcal{I}$, respectively, where $\mathcal{I} = (t_0, t_{k+1})$ is an open time interval. A jump-flow solution of Ξ_σ is a piecewise $\mathcal{C}^1$-curve $x : \mathcal{I} \to X$ such that for all $0 \leq i \leq k$, the curve x is a $\mathcal{C}^1$-solution of $\Xi_{\sigma(t_i^+)}$ on (t_i, t_{i+1}) and the jump $x(t_i^-) \to x(t_i^+)$ is defined by the nonlinear consistency projector of Definition 5, i.e., $x(t_i^+) = \Omega_{E_p,F_p}(x(t_i^-))$ with $p = \sigma(t_i^+)$. For convenience let $x(t_i) := x(t_i^+)$.

The following proposition is a straightforward consequence of Lemma 4 and Definitions 5 and 7.

Proposition 8 (existence and uniqueness of solutions)**.** *Consider a switched DAE* Ξ_σ*, given by (1), assume that conditions* **(SL)** *and* **(NF)** *are satisfied. Then there exists a unique jump-flow solution* $x : [0, +\infty) \to X$ *of* Ξ_σ *for any initial point* $x(0) = x_0 \in X$.

Similar as in [11], [12], impulse-freeness is relevant for the stability analysis of the switched nonlinear DAE (1). In view of (8), an initial value for (5) results in an impulse free solution if and only if $\xi_{20}^- \in \ker N$. Thus in the original coordinates, if the inconsistent initial point satisfies

$$x_0^- \in M_{IF}^* := \{x \in X \mid N\psi_2(x) = 0\}\,, \qquad (9)$$

then the jump $x_0^- \to x_0^+ = \Omega_{E,F}(x_0^-)$ results in an *impulse-free* solution. Extending this argument to the switched nonlinear DAE Ξ_σ satisfying **(SL)** and **(NF)**, impulse-free jump-flow solutions can be guaranteed by assuming the following condition:

(IF) the submanifolds M^* and M_{IF}^*, defined by (7) and (9), respectively, of each model Ξ_p of Ξ_σ satisfy

$$\forall p, q \in \mathcal{N}: \quad M^*(\Xi_p) \subseteq M_{IF}^*(\Xi_q).$$

Based on the analysis above, the result of the following proposition is clear.

Proposition 9 (impulse-freeness)**.** *For a switched DAE* Ξ_σ *under the assumptions* **(SL)** *and* **(NF)***. If condition* **(IF)** *holds, then for any initial point* $x_0 \in M_{IF}^*(\Xi_{\sigma(0)})$*, any jump-flow solution* $x : [0, +\infty) \to X$ *of* Ξ_σ *is impulse-free.*

Remark 10. Note that even for a *stable* (see Definition 12) switched nonlinear DAE satisfying **(SL)**, **(NF)**, **(IF)**, jump-flow solutions can still be *impulsive* because if the initial point $x_0^- \notin M_{IF}^*(\Xi_{\sigma(0)})$, the initial jump $x_0^- \notin M_{IF}^*(\Xi_{\sigma(0)}) \to x_0^+ \in M^*(\Xi_{\sigma(0)})$ can still produce impulsive terms. So if we want all jump-flow solutions are impulse-free, we must choose an initial point $x_0^- \in M_{IF}^*(\Xi_{\sigma(0)})$.

Remark 11. For a switched linear DAE Δ_σ, given by (2), the distributional solution is impulse-free [11], [12] if

$$\forall p,q \in \mathcal{N} : E_q(I-\Pi_{E_q,H_q})\Pi_{E_p,H_p} = 0, \qquad (10)$$

where Π_{E_p,H_p} is the linear consistency projector of Δ_p, defined by (4). Let Q_q and P_q be the two invertible matrices such that Δ_q is ex-equivalent its Weierstrass form via Q_q and P_q, then we have $E_q(I-\Pi_{E_q,H_q}) = Q_q^{-1}\left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & N_q \end{smallmatrix}\right] P_q(P_q^{-1}P_q - P_q^{-1}\left[\begin{smallmatrix} I_{n_1} & 0 \\ 0 & 0 \end{smallmatrix}\right] P_q) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & N_q \end{smallmatrix}\right] P_q = \left[\begin{smallmatrix} 0 \\ N_q P_q^2 \end{smallmatrix}\right]$, where $P_q = \left[\begin{smallmatrix} P_q^1 \\ P_q^2 \end{smallmatrix}\right]$. Notice that $\operatorname{Im}\Pi_{E_p,H_p} = \mathscr{V}^*(\Delta_q)$, the latter is the limit of the Wong sequence $\mathscr{V}_i$ of Δ_q. It follows that (10) is equivalent to

$$\forall p,q \in \mathcal{N} : \mathscr{V}^*(\Delta_p) \subseteq \ker N_q P_q^2.$$

It is seen that condition **(IF)** is a nonlinear generalization of (10). A different impulse-free condition for switched nonlinear DAEs is given by the jump rule

$$\forall p,q \in \mathcal{N}\ \forall x_0^- \in \mathfrak{C}(\Xi_p) : x_0^+ - x_0^- \in \ker E_q(x_0^+), \qquad (11)$$

shown in assumption A4 of [12]. We have illustrated in [4], [5] that even for non-switched DAEs, the jump rule (11) is not consistent with nonlinear coordinates transformations, i.e., not coordinate-free, we may get different jumping point x_0^+ depending on different coordinates chosen for the DAE.

Given any internally regular DAE $\Xi = (E,F)$, if $F(0) = 0$, then $x_c = 0$ is clearly consistent and is also an *equilibrium* of Ξ, because $x(t) = 0$ is the only $\mathcal{C}^1$-solution passing through $x_c = 0$. Consider a switched DAE Ξ_σ, the following assumption assures that $x_c = 0$ is a common equilibrium for all models Ξ_p.

(EQ) For all modes $\Xi_p = (E_p, F_p)$ of Ξ_σ we have $F_p(0) = 0$.

Definition 12. The equilibrium $x_c = 0$ of Ξ_σ is called *stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that all jump-flow solution x with $\|x(0)\| < \delta$ satisfy $\|x(t)\| < \varepsilon$ for all $t > 0$; the DAE Ξ_σ is called globally asymptotically stable if $x_c = 0$ is stable and all jump-flow solutions converge to zero.

Before giving the main result of the present paper, we introduce the following notion of jump-flow explicitation, which is a nonlinear control system, associated with any (non-switched) DAE being ex-equivalent to its **(NWF)**.

Definition 13 (jump-flow explicitation of DAEs)**.** Consider a DAE $\Xi = (E,F)$, given by (5), assume that Ξ is ex-equivalent to the **(NWF)** via a smooth map $Q : X \to GL(n,\mathbb{R})$ and a diffeomorphism $\psi = (\psi_1, \psi_2)$, the *jump-flow explicitation* of Ξ is the following nonlinear control system

$$\Sigma^e : \begin{cases} \dot{x} = f^e(x) + \sum_{i=1}^{m} g_i^e(x) v_i = f^e(x) + g^e(x) v, \\ y = h^e(x), \end{cases} \qquad (12)$$

denoted by $\Sigma^e = (f^e, g^e, h^e)$, where $v \in \mathbb{R}^m$ is an input vector and $m = n_2$ is the dimension of ξ_2-subsystem in the **(NWF)**, and

$$f^e := \left(\frac{\partial \psi}{\partial x}\right)^{-1} \left[\begin{smallmatrix} f^* \circ \psi_1 \\ 0 \end{smallmatrix}\right],\ g^e := \left(\frac{\partial \psi}{\partial x}\right)^{-1} \left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right],\ h^e := \psi_2.$$

Note that condition **(EQ)** implies that $f^e(0) = 0$ because $Q(0)F(0) = \left[\begin{smallmatrix} f^*(\psi_1(0)) \\ \psi_2(0) \end{smallmatrix}\right] = 0$ and $f^e(0) = \left(\frac{\partial \psi}{\partial x}\right)^{-1} \left[\begin{smallmatrix} f^*(\psi_1(0)) \\ 0 \end{smallmatrix}\right] = 0$.

Remark 14. (i) It can be seen that any $\mathcal{C}^1$-solution of the DAE Ξ coincides with a solution of the control system Σ under the constraints $0 = y = h^e(x) = \psi_2(x)$, i.e., a solution of the zero dynamics (see e.g., [9]) of Σ. Observe that the zero dynamics of Σ is actually the ODE $\dot{\xi}_1 = f^*(\xi_1)$, which has isomorphic $\mathcal{C}^1$-solutions with Ξ (see Lemma 4) and can be written as $\dot{x} = f^e(x)$ in x-coordinates. A jump-flow solution $x(t)$ of Ξ starting from any initial point (consistent or not) x_0 can be expressed as $x(t) = \Phi_t^{f^e} \circ \Omega_{E,F} \circ x_0$, where $\Phi_t^{f^e}$ is the flow map of the vector field f^e, i.e., $f^e(x) = \frac{d\Phi_t^{f^e}(x)}{dt}|_{t=0}$. Note that the vector field f^e plays the same role as the flow matrix $A^{\text{diff}} = P^{-1}\left[\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\right] P$ for a linear DAE Δ, see e.g., [11], [19].

(ii) The nonlinear consistency projector $\Omega_{E,F}$ can be seen as the flow map $\Phi_\tau^{v^e}(x)$ of the vector field $v^e := -g^e h^e$ for $\tau = +\infty$. Indeed, the canonical projection π attaching $(\xi_1,\xi_2) \mapsto (\xi_1, 0)$ coincides with the flow map $\Phi_\tau^{\tilde{v}^e}(\xi)$ of the vector field $\tilde{v}^e := -\left[\begin{smallmatrix} 0 \\ I_m \end{smallmatrix}\right]\xi_2$ for $\tau = +\infty$. Observe that $\tilde{v}^e = \frac{\partial \psi}{\partial x} v^e$, thus by, e.g., Proposition 1.1 of [10], their flow maps satisfy $\Phi_\tau^{v^e}(x) = \psi^{-1} \circ \Phi_\tau^{\tilde{v}^e}(x) \circ \psi = \psi^{-1} \circ \pi \circ \psi = \Omega_{E,F}$.

Theorem 15. *For a switched nonlinear DAE Ξ_σ, given by (1), assume that conditions* **(SL)***,* **(NF)***,* **(EQ)** *are satisfied. Let a control system $\Sigma_p^e = (f_p^e, g_p^e, h_p^e)$ be the jump-flow explicitation of the model Ξ_p for each $p \in \mathcal{N}$. Then the switched DAE Ξ_σ is globally asymptotically stable, uniformly for arbitrary switching signal σ if there exists a common $\mathcal{C}^1$-positive definite (Lyapunov) function $V : X \to \mathbb{R}$ such that the level set $\mathcal{L}_a := \{x \in X \,|\, V(x) \le a\}$ is compact for every $a \in V(X)$ and $\forall p,q \in \mathcal{N}$:*

$$\frac{\partial V(x)}{\partial x} f_p^e(x) < 0, \quad \forall x \in M^*(\Xi_p) \backslash \{0\}, \qquad (13)$$

$$\frac{\partial V(x)}{\partial x} v_p^e(x) \le 0, \quad \forall x \in M^*(\Xi_q), \qquad (14)$$

where $v_p^e := -g_p^e h_p^e$ is a vector field on X.

The following lemma is crucial for the proof of Theorem 15 and shows a connection between the above results and those in [12].

Lemma 16. *Condition (14) is equivalent to the following condition (it is also condition (14) in Theorem 4.1 of [12])*

$$V(\Omega_{E_p,F_p}(x)) - V(x) \le 0, \quad \forall x \in M^*(\Xi_q). \tag{15}$$

The proofs of Theorem 15 and Lemma 16 are given in Appendix. For the proof of Theorem 15, we first show the stability of $x_c = 0$ and then the asymptotic stability of Ξ_σ, which is different than the proofs in [12]. We now discuss the connections and differences between Theorem 15 of the present paper and Theorem 4.1 of [12]. Both of the two theorems provide sufficient conditions for asymptotically stability of switched nonlinear DAEs from a similar intuition, i.e., to find a common Lyapunov function $V(x)$ decreasing along both the flow and the jump dynamics of each DAE model Ξ_p. The main difference is that Theorem 15 gives conditions based on the jump-flow explicitation $\Sigma = (f, g, h)$ rather on the original DAE $\Xi = (E, F)$, the advantages of using explicitation are explained in the following remark.

Remark 17. (i) The construction of the Lyapunov function $V(x)$ for the flow dynamics ($\mathcal{C}^1$-solutions) of each DAE model Ξ_p shown in Definition 2.5 of [12] requires the existence of a map $\mathcal{F}_p$ such that $\frac{\partial V(x)}{\partial x} = \mathcal{F}_p(x, E_p(x)z)$, $\forall x \in \mathfrak{C}(\Xi_p)$ and $z \in T_x\mathfrak{C}(\Xi_p)$ to define $\dot V(x) := \mathcal{F}(x, F_p(x))$. While we can see from (13) that the differentiation of $V(x)$ along the flow dynamics of Ξ_p can be explicitly expressed as $\dot V(x) = \frac{\partial V(x)}{\partial x} f_p^e(x)$ since the flow dynamics of Ξ_p coincides with the zero dynamics $\dot x = f_p^e(x)$ of the explicitation Σ_p (see Remark 14(i)).

(ii) Condition (14) is shown to be equivalent to (14) of Theorem 4.1 in [12] by Lemma 16. Both conditions mean that $V(x)$ decreases along any jump $x_0^- \in M^*(\Xi_q) \to x_0^+ = \Omega_{E_p,F_p}(x_0^-) \in M^*(\Xi_q)$. Note that $v_p^e = -g_p^e h_p^e$ is a vector field whose flow map $\Phi_a^{v_p^e}$ coincides with nonlinear consistency projector Ω_{E_p,H_p} (see Remark 14(ii)), so condition (15), which is not involved with the set value map Ω_{E_p,H_p}, is actually a stability condition for the ODE $\dot x = v_p^e(x)$. Since both f_p^e and v_p^e are both vector fields, a straightforward corollary of Theorem 15 is that if $V(x)$ is a common Lyapunov function for a switched ODE with both $\dot x = f_p^e(x)$ and $\dot x = v_q^e(x)$ as switching models, then $V(x)$ is also a common Lyapunov function for the switched DAE Ξ_σ.

IV. Examples

Consider a switched nonlinear DAE Ξ_σ with the generalized states $x = (x_1, x_2, x_3) \in X = \mathbb{R}^3$ and two models $\Xi_1 = (E_1, F_1)$ and $\Xi_2 = (E_2, F_2)$, where

$$E_1(x) = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 0 & 0 \\ x_3 & 1 & x_1 \end{bmatrix}, \quad F_1(x) = \begin{bmatrix} x_2 + x_1(x_3 - (e^{x_3}x_1)^2) \\ x_2 + x_1x_3 \\ x_3 \end{bmatrix},$$
$$E_2(x) = \begin{bmatrix} (x_1)^2+1 & 0 & 0 \\ (x_1)^2+1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} -x_1 \\ x_2 + x_1(x_3 - 1) \\ x_3 + (x_1)^2 \end{bmatrix}.$$

The DAE Ξ_1 is ex-equivalent to

$$\tilde\Xi_1 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde x}_1 \\ \dot{\tilde x}_2 \\ \dot{\tilde x}_3 \end{bmatrix} = \begin{bmatrix} -(\tilde x_1)^3 \\ \tilde x_2 \\ \tilde x_3 \end{bmatrix},$$

via the diffeomorphism $\psi_1(x) = (\tilde x_1, \tilde x_2, \tilde x_3) = (e^{x_3}x_1, x_2 + x_1x_3, x_3)$ and $Q_1(x) = \begin{bmatrix} e^{x_3} & -e^{x_3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; the DAE Ξ_2 is ex-equivalent to

$$\bar\Xi_2 : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar x}_1 \\ \dot{\bar x}_2 \\ \dot{\bar x}_3 \end{bmatrix} = \begin{bmatrix} \frac{-\bar x_1}{(\bar x_1)^2+1} \\ \bar x_2 \\ \bar x_3 \end{bmatrix},$$

via the diffeomorphism $\psi_2(x) = (\bar x_1, \bar x_2, \bar x_3) = (x_1, x_2 + x_1x_3, x_3 + (x_1)^2)$ and $Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Observe that both $\tilde\Xi_1$ and $\bar\Xi_2$ are in the **(NWF)**. It follows by (7) and (9) that

$$\begin{aligned} M^*(\Xi_1) &= \left\{x \in \mathbb{R}^3 \,|\, x_2 = x_3 = 0\right\}, \\ M^*(\Xi_2) &= \left\{x \in \mathbb{R}^3 \,|\, x_2 + x_1x_3 = x_3 + (x_1)^2 = 0\right\}, \\ M^*_{IF}(\Xi_1) &= \left\{x \in \mathbb{R}^3 \,|\, x_2 + x_1x_3 = 0\right\}, \ M^*_{IF}(\Xi_2) = \mathbb{R}^3. \end{aligned}$$

It can been seen that conditions **(SL)**, **(NF)**, **(EQ)** and **(IF)** are all satisfied. The jump-flow explicitations of Ξ_1 and Ξ_2, constructed by Definition 13, are $\Sigma_1^e = (f_1^e, g_1^e, h_1^e)$ and $\Sigma_2^e = (f_2^e, g_2^e, h_2^e)$, respectively, where

$$\begin{aligned} f_1^e(x) &= e^{2x_3}\begin{bmatrix} -(x_1)^3 \\ (x_1)^3x_3 \\ 0 \end{bmatrix}, & g_1^e(x) &= \begin{bmatrix} 0 & -x_1 \\ 1 & x_1x_3 - x_1 \\ 0 & 1 \end{bmatrix}, \\ h_1^e(x) &= \begin{bmatrix} x_2 + x_1x_3 \\ x_3 \end{bmatrix}, & f_2^e(x) &= \frac{-x_1}{(x_1)^2+1}\begin{bmatrix} 1 \\ 2(x_1)^2 - x_3 \\ -2x_1 \end{bmatrix}, \\ g_2^e(x) &= \begin{bmatrix} 0 & 0 \\ 1 & -x_1 \\ 0 & 1 \end{bmatrix}, & h_2^e(x) &= \begin{bmatrix} x_2 + x_1x_3 \\ x_3 + (x_1)^2 \end{bmatrix}. \end{aligned}$$

A direct calculation gives

$$v_1^e = -g_1^e h_1^e = \begin{bmatrix} x_1x_3 \\ -x_2 - x_1(x_3)^2 \\ -x_3 \end{bmatrix}, \ v_2^e = -g_2^e h_2^e = \begin{bmatrix} 0 \\ (x_1)^2 - x_2 \\ (x_1)^2 - x_3 \end{bmatrix}.$$

We construct the following common Lyapunov function candidate:

$$V(x) = \frac{1}{2}(x_1)^2 + \frac{1}{4}(x_1)^4 + \frac{1}{2}(x_2 + x_1x_3)^2 + \frac{1}{2}(x_3)^2.$$

For a function $V : X \to \mathbb{R}$ and a vector field $f : X \to \mathbb{R}^n$, denote $L_fV(x) := \frac{\partial V(x)}{\partial x}f(x)$, then we have $L_{f_1^e}V(x)|_{M^*(\Xi_1)} = -e^{2x_3}((x_1)^4 + (x_1)^6) < 0$, $\forall x \in M^*(\Xi_1)\setminus\{0\}$; $L_{v_1^e}V(x)|_{M^*(\Xi_2)} = ((x_1)^2 + (x_1)^4)x_3 - (x_3)^2 = -(x_1)^6 - 2(x_1)^4 \le 0$, $\forall x \in M^*(\Xi_2)$; $L_{f_2^e}V(x)|_{M^*(\Xi_2)} = -(x_1)^2 + \frac{2(x_1)^2x_3}{(x_1)^2+1} = -(x_1)^2 - \frac{2(x_1)^4}{(x_1)^2+1} < 0$, $\forall x \in M^*(\Xi_2)\setminus\{0\}$; $L_{v_2^e}V(x)|_{M^*(\Xi_1)} = 0$, $\forall x \in M^*(\Xi_1)$. Hence conditions (13) and (14) are satisfied and the switched DAE Ξ_σ is globally asymptotically stable under arbitrary switching signal.

Consider an initial point $x_0^- = (-1.5, 0.75, 0.5) \in M^*_{IF}(\Xi_1)$ and a switched signal σ with $\sigma(0) = 1$ and switches at $t = 0.3, 0.6, 1.2, 1.8, \ldots$, respectively. Observe that $x_0^- \notin M^*(\Xi_1)$, so we have an initial jump $x_0^- \to x_0^+ \in M^*(\Xi_1)$, given by $x_0^+ = \Omega_{E_1,H_1}(x_0^-) = (-1.5, 0, 0)$, where Ω_{E_1,H_1} is the nonlinear consistency projector of Ξ_1. We draw the jump-flow solution of Ξ_σ starting from x_0^- in the following figure, this jump-flow solution is impulse-free (because $x_0^- \in M^*_{IF}(\Xi_1)$ and **(IF)** holds) and converges to zero.

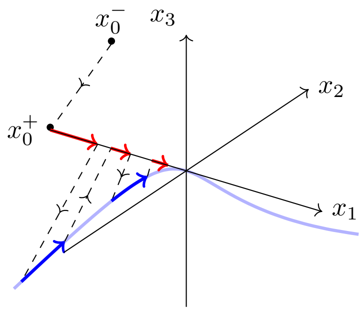

Fig. 1: x_1-axis: $M^*(\Xi_1)$, light blue curve: $M^*(\Xi_2)$, dark red curve: $\mathcal{C}^1$-solutions of Ξ_1, dark blue curve: $\mathcal{C}^1$-solutions of Ξ_2, dashed lines: jumps.

V. CONCLUSIONS

In this paper, we use a nonlinear Weierstrass form to define the nonlinear consistency projectors for nonlinear DAEs. The nonlinear consistency projector can be used to calculate the consistent initialization jump and to formulate an impulse-free condition for switched nonlinear DAEs. New sufficient conditions are provided for a switched nonlinear DAE being globally asymptotically stable under arbitrary switching signal, those conditions are derived with the help of the jump-flow explication of each model of the switched DAE.

APPENDIX

Proof of Theorem 15. Step 1: We prove that $t \mapsto V(x(t))$ is monotonically decreasing for any jump-flow solution $x : [0, \infty) \to X$ of Ξ_σ. For any interval $\mathcal{I}_i \subseteq [0, \infty)$ without switching times, $x(t)$ is a $\mathcal{C}^1$-solution of the model Ξ_p, where $p = \sigma(t)$ for any $t \in \mathcal{I}_i$, so $x(t)$ is also a solution of the ODE $\dot{x} = f_p^e(x)$ defined on $M^*(\Xi_p)$ (see Remark 14(i)). By (13), we have $\dot{V}(x(t)) = \frac{\partial V}{\partial x} f_p^e(x(t)) < 0$, $\forall t \in \mathcal{I}_i$. For any switching time t_i, denote $q = \sigma(t_i^-)$ and $p = \sigma(t_i^+)$, then $x(t_i^-) \in M^*(\Xi_q)$ and $x(t_i^+) = \Omega_{E_p,F_p}(x(t_i^-)) \in M^*(\Xi_p)$, thus by (15) of Lemma 16, we have $V(x(t_i^+)) - V(x(t_i^-)) \leq 0$. Hence $t \mapsto V(x(t))$ is decreasing on the whole interval $[0, \infty)$.

Step 2: We show $x_c = 0$ is stable. Fix $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r := \{x \in X \,|\, ||x|| \leq r\}.$$

Then we prove that there exists $\beta_r > 0$ depending on r such that the set $\mathcal{L}_{\beta_r} := \{x \in X \,|\, V(x) \leq \beta_r\}$ is strictly contained in B_r, i.e., $\mathcal{L}_{\beta_r} \subsetneq B_r$. Assume the contrary, i.e., for all $\beta_r > 0$, there exists $x \in \mathcal{L}_{\beta_r}$ satisfying $||x|| \geq r$, which implies that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{L}_{\frac{1}{n}}$ such that $||x_n|| \geq r$. By construction, $\lim\limits_{n\to\infty} V(x_n) = 0$. Moreover, since $\mathcal{L}_{\frac{1}{n}}$ is compact by assumption (for sufficiently large n), there exists a subsequence of (x_n) whose limit x^* exists and satisfies $||x^*|| \geq r$. Then we get $\lim\limits_{n\to\infty} V(x_n) = V(x^*) > 0$, which is a contradiction. Thus we have $\mathcal{L}_{\beta_r} \subsetneq B_r$.

Now we choose $\beta_r > 0$ such that $\mathcal{L}_{\beta_r} \subsetneq B_r$. Recall from Step 1 that $t \mapsto V(x(t))$ is decreasing, it follows that $\mathcal{L}_{\beta_r}$ is an invariant set for any jump-flow solution $x(t)$ starting from $x(0) = x_0 \in \mathcal{L}_{\beta_r}$ because $V(x(t)) \leq V(x(0)) \leq \beta_r$ implies that $x(t) \in \mathcal{L}_{\beta_r}$, $\forall t \geq 0$. Since $V(x)$ is continues and $V(0) = 0$, there exists $\delta > 0$ such that $B_\delta \subsetneq \mathcal{L}_{\beta_r}$. We thus have $x(0) \in B_\delta \subsetneq \mathcal{L}_{\beta_r} \Rightarrow x(t) \in \mathcal{L}_{\beta_r} \subsetneq B_r$, which implies that $||x(0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon$, hence $x_c = 0$ is stable.

Step 3: Let $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ be the switching time of a signal σ. We prove that all jump-flow solutions $x(t)$ converge to zero. Seeking a contradiction, assume that $x(t)$ does not converge to zero. Then, since $V(x(t))$ is nonnegative and decreasing, we have $\lim\limits_{t\to\infty} V(x(t)) = c > 0$. Notice that The set $\mathcal{L}_{c,d} = \{x \in X \,|\, c \leq V(x) \leq V(x(0)) = d\}$ is compact by assumption, it follows that, for each $p \in \mathcal{N}$, the continues function $\frac{\partial V(x)}{\partial x} f_p^e(x)$ attains its maximum $s_p < 0$ within $\mathcal{L}_{c,d}$. Then with $s = \max_{p\in\mathcal{N}} s_p$, we have that $\dot{V}(x(t)) \leq s < 0$ for all $t \in I_i$ for any interval $\mathcal{I}_i = (t_i, t_{i+1}) \subseteq [0, +\infty)$ without switching times. Consequently, for any $k \geq 1$,

$$V(x(t_k^-)) = V(x(t_0^-)) + \sum_{i=0}^{k-1} \int_{t_i^+}^{t_{i+1}^-} \dot{V}(x(t))dt + \sum_{i=0}^{k-1} \left(V(x(t_i^+)) - V(x(t_i^-))\right) \leq V(x(t_0^-)) + st_k.$$

So for $t_k > -\frac{V(x(0^-))}{s}$, the above relation results in $V(x(t_k^-)) < 0$, which is a contradiction. Hence all jump-flow solutions $x(t)$ converge to zero. □

Proof of Lemma 16. Recall that Ξ_p is ex-equivalent to its **(NWF)** via a diffeomorphism ψ_p and an invertible matrix Q_p. We have $\frac{\partial \psi_p(x)}{\partial x} g_p^e(x) = \begin{bmatrix} 0 \\ I_{m_p} \end{bmatrix}$ and $h_p^e(\psi_p^{-1}(\xi_p)) = \xi_{2p}$, where $\xi_p = (\xi_{1p}, \xi_{2p}) = \psi_p(x)$ are the coordinates of the **(NWF)** of Ξ_p. It follows that

$$\frac{\partial V(x)}{\partial x} g_p^e(x) h_p^e(x) = \frac{\partial V(\psi^{-1}(\xi_p))}{\partial \xi_p} \frac{\partial \psi(x)}{\partial x} g_p^e(x) h_p^e(\psi^{-1}(\xi_p)) = \frac{\partial \tilde{V}(\xi_{1p}, \xi_{2p})}{\partial \xi_{2p}} \xi_{2p}$$

where $\tilde{V}(\xi_{1p}, \xi_{2p}) = V(\psi^{-1}(\xi_p))$. By Definition 5, we have $V \circ \Omega_{E_p,F_p} \circ x = V \circ \Omega_{E_p,F_p} \circ \psi_p^{-1} \circ \xi_p = V \circ \psi_p^{-1} \circ (\xi_{1p}, 0)$. So $V(x) - V(\Omega_{E_p,F_p}(x)) = \tilde{V}(\xi_{1p}, \xi_{2p}) - \tilde{V}(\xi_{1p}, 0)$. Then by the mean value theorem, we have

$$\tilde{V}(\xi_{1p}, \xi_{2p}) - \tilde{V}(\xi_{1p}, 0) = \frac{\partial \tilde{V}}{\partial \xi_{2p}}(\xi_{1p}, c\xi_{2p})(\xi_{2p} - 0) = \frac{1}{c} \frac{\partial \tilde{V}}{\partial \xi_{2p}}(\xi_{1p}, c\xi_{2p}) c\xi_{2p},$$

for some $c \in [0, 1]$. It follows that $\tilde{V}(\xi_{1p}, \xi_{2p}) - \tilde{V}(\xi_{1p}, 0) \geq 0$ if and only if $\frac{\partial \tilde{V}(\xi_{1p}, \xi_{2p})}{\partial \xi_{2p}} \xi_{2p} \geq 0$, which implies that (14) is equivalent to (15). □

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