Optimal control of DAEs with unconstrained terminal costs

Paul Wijnbergen and Stephan Trenn

***Abstract*— This paper is concerned with the linear quadratic optimal control problem for impulse controllable differential algebraic equations on a bounded half open interval. Regarding the cost functional, a general positive semi-definite weight matrix is considered in the terminal cost. It is shown that for this problem, there generally does not exist an input that minimizes the cost functional. First it is shown that the problem can be reduced to finding an input to an index-1 DAE that minimizes a different quadratic cost functional. Second, necessary and sufficient conditions in terms of matrix equations are given for the existence of an optimal control.**

I. INTRODUCTION

In this paper, we consider the problem of finding an input u that minimizes the cost functional

$$J^-(x,u)=\int_{t_0}^{t_f}\left(\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]^\top \left[\begin{smallmatrix}Q & S\\ S^\top & R\end{smallmatrix}\right]\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\right)+x(t_f^-)^\top P x(t_f^-) \quad (1)$$

on a half open interval $[t_0,t_f)$ for some positive definite $R=R^\top$ and positive semi-definite $Q=Q^\top$ and $P=P^\top$, subject to a differential algebraic equation (DAE)

$$E\dot{x}=Ax+Bu, \qquad x(0^-)=x_0, \quad (2)$$

which is assumed to be impulse controllable and where $E,A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, for $n,m\in\mathbb{N}$, $x:[t_0,t_f)\to\mathbb{R}^n$ is the state and $u:[t_0,t_f)\to\mathbb{R}^m$ is the input. The weight matrix in (1) is assumed to be symmetric and positive semi definite such that it can be decomposed as

$$\left[\begin{smallmatrix}Q & S\\ S^\top & R\end{smallmatrix}\right]=\left[\begin{smallmatrix}C^\top\\ D^\top\end{smallmatrix}\right]\left[\begin{smallmatrix}C & D\end{smallmatrix}\right],$$

Hence the running cost in (1) can be regarded as the integral over the norm of the output $y(t)=Cx(t)+Du(t)$. Besides positive semi-definiteness of the weight matrix P, no further assumptions are made on the terminal cost. This is in contrast to the assumption commonly made in the literature that the weight matrix only penalizes differential states, *i.e.,* is of the form $x(t_f)^\top E^\top PEx(t_f)$ [1]–[3] or no terminal cost is considered with an infinite time horizon [4]–[8]. Also note that whereas commonly a closed interval is of interest, in this paper a half open interval is considered. Consequently, the terminal cost penalizes $x(t_f^-)$. However the algebraic states of (2) can possibly be controlled instantaneously and as a result $x(t_f^-)$ is not necessarily equal to $x(t_f)$ or even well defined such that an optimal solution might fail to exist as is shown by the following example.

Example 1: Consider the DAE given by

$$\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\dot{x}=\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]x+\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]u, \quad x(t_0^-)=\left[\begin{smallmatrix}x_0\\0\end{smallmatrix}\right], \quad (3)$$

for some $x_0\in\mathbb{R}$ and the cost functional

$$J^-(x,u)=\int_{t_0}^{t_f}(x_1^2+x_2^2+u^2)+(x_1(t_f^-)+x_2(t_f^-))^2. \quad (4)$$

Using the structure of the DAE and considering $t_0=0$ and $t_f=1$ this simplifies to

$$J^-(x,u)=x_0^2+2\int_0^1 u(\tau)^2d\tau+(x_0+u(1^-))^2\geqslant x_0^2.$$

By choosing $u(t)=0$ on $[0,1-\varepsilon)$ for some $\varepsilon>0$ and $u(t)=-x_0$ on $[1-\varepsilon,1)$, we obtain

$$J(x,u)=x_0^2+2\int_{1-\varepsilon}^1 u(t)^2=(1+2\varepsilon)x_0^2.$$

This shows that $\inf_u J(x,u)=x_0^2$. However for any input[1] for which $u(1^-)=-x_0\neq 0$ we have that $\inf_u J(x,u)<J(x,u)$, because $\int_0^1 u^2>0$. Hence there does not exists an optimal control.

However, in the case that the terminal cost only penalizes the algebraic states, *i.e.,* the cost functional is given by

$$J^-(x,u)=\int_0^1(x_1^2+x_2^2+u^2)+x_2(1^-)^2. \quad (5)$$

it is obvious that the optimal input is given by $u(t)=0$ for all $t\in[0,1)$. This shows that a terminal cost of the form $x(t_f)^\top E^\top PEx(t_f)$ is only sufficient for existence of an optimal control, but not necessary. ◇

The problem described above is motivated by the study of linear quadratic optimal control of *switched DAEs*. By assuming that (2) is the first mode of *e.g.,* a single switched DAE defined on $[t_0,\infty)$ and regarding t_f as the switching time, the terminal cost represents the cost originating from the system on the interval $[t_f,\infty)$. In that case, the weight matrix P has no direct structural relation to the DAE (2) and can only be assumed to be positive semi-definite.

The literature on optimal control of non-switched DAEs is quite mature, (besides the already mentioned literature) see *e.g.,* [9]–[12], and several structural properties of switched DAEs have been investigated recently [13], [14]. Also results on optimal control of switched ordinary differential equations have appeared *e.g.,* [15]. However, to the best of the authors knowledge, optimal control of switched DAEs has not been studied yet. This paper can thus be regarded as a step

The authors are with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, Nijenborgh 9, 9747 AG, Groningen, The Netherlands.
Email: p.wijnbergen@rug.nl; s.trenn@rug.nl
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[1]By writing $u(1^-)$ we implicitely assume that $t=1$ is a left-Lebesgue point of u, see Section II-B

towards the optimal control of switched DAEs. The main contributions can be summarized as follows.

The aim of this paper is to state conditions on the existence of inputs that minimizes (1). First, by applying a preliminary feedback, we will show that minimizing (1) subject to (2) is equivalent to minimizing a different quadratic cost functional subject to an index-1 DAE. Second, we will show that if there exists an optimal control, it is a feedback. Then, using a completion of the squares formula we show that the cost functional can be expressed in terms of a solution to a Riccati differential equation and is quadratic. Finally we state results on the existence of a solution that minimizes (1) in terms of matrix equations.

The remainder of this paper is structured as follows. The mathematical preliminaries and the main results are given in Section II and III, respectively. Conclusions and a discussion on future work are given in Section IV.

II. Mathematical preliminaries

In this section we recall some notation and properties related to the DAE (2).

A. Properties and definitions for regular matrix pairs

In the following, we call a matrix pair (E, A) and the associated DAE (2) *regular* iff the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form (QWF)* [16].

Proposition 2: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right), \tag{6}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.
The nilpotency index of N in the QWF (6) is called the *index* of the matrix pair (E, A) (or the DAE (2)); in particular, the DAE is called *index-1* if, and only if, $N = 0$ in the QWF. The matrices S and T can be calculated by using the so-called *Wong sequences* [16], [17]:

$$\begin{array}{lll} \mathcal{V}_0 := \mathbb{R}^n, & \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), & i = 0, 1, ... \\ \mathcal{W}_0 := \{0\}, & \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), & i = 0, 1, ... \end{array} \tag{7}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup_i \mathcal{W}_i. \tag{8}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (6) holds.

Based on the Wong sequences we define the following projector and selectors.

Definition 3: Consider the regular matrix pair (E, A) with corresponding QWF (6). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}.$$

Furthermore, the *differential selector* and *impulse selector* are respectively given by

$$\Pi^{\text{diff}}_{(E,A)} := T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S, \quad \Pi^{\text{imp}}_{(E,A)} := T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S.$$

In all three cases the block structure corresponds to the block structure of the QWF. Furthermore we define

$$\begin{aligned} A^{\text{diff}} &:= \Pi^{\text{diff}}_{(E,A)} A, & E^{\text{imp}} &:= \Pi^{\text{imp}}_{(E,A)} E, \\ B^{\text{diff}} &:= \Pi^{\text{diff}}_{(E,A)} B, & B^{\text{imp}} &:= \Pi^{\text{imp}}_{(E,A)} B. \end{aligned}$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the regular matrix pair (E, A). Furthermore if a classical solution to (2) exists, it satisfies

$$x(t) = x^{\text{diff}}(t) + x^{\text{imp}}(t) \tag{9}$$

where x^{diff} and x^{imp} are referred to as the *differential* and *algebraic* states, respectively and are defined as [18]

$$\begin{aligned} x^{\text{diff}}(t) &:= e^{A^{\text{diff}} t}\Pi x_0 + \int_0^t e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau)\, d\tau, \\ x^{\text{imp}}(t) &:= -\sum_{i=0}^{\nu} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \end{aligned}$$

B. The distributional solution framework

For studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [19] as the solution space, *i.e.*, we seek a solution $(x, u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+m}$ to the following initial-trajectory problem (ITP):

$$(E\dot{x})_{[0,\infty)} = (Ax + Bu)_{[0,\infty)}, \;\; x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{10}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise smooth distribution f to an interval $\mathcal{I}$. The ITP (10) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. Furthermore, the solution of (10) considered on $[0, \infty)$ is uniquely determined by $x^0(0^-)$ and is independent of the whole past trajectory. Hence, (2) with initial condition $x(0^-) = x_0$ can be interpreted as a short hand notation for the ITP (10) for some x^0 with $x^0(0^-) = x_0$.

When considering distributional solutions of (2), x can not simply be evaluated at some time $t \in \mathbb{R}$, as x operates on the space of test-functions. However, when restricting the solution space to piecewise-smooth distributions, the left-sided evaluation $x(t^-)$, right-sided evaluation $x(t^+)$ and impulsive evaluation $x[t]$ are well-defined for any $t \in \mathbb{R}$.

Furthermore, for index-1 DAEs it is not necessary to consider distributional solutions, instead any locally integrable function pair (x, u) is considered a solution if Ex is absolutely continuous and (2) is satisfied almost everywhere. In that case, the notation $x(t_f^-)$ (or $u(t_f^-)$) implicitly assumes that t_f is a left-Lebesgue point of x (or u), *i.e.,*

$$x(t_f^-) := \lim_{h\searrow 0} \frac{1}{h} \int_{t_f-h}^{t_f} x(\tau) d\tau \quad \text{is well defined.}$$

C. Properties of DAEs and optimal control

As for some DAEs Dirac impulses can be avoided by means of an input, the following definition is introduced.

Definition 4 ([20]): The DAE (2) is *impulse controllable* if for any $x_0 \in \mathbb{R}^n$ there exists a solution (x, u) of the ITP (10) such that $x(0^-) = x_0$ and $(x, u)[0] = 0$.

If Dirac impules can be avoided, it can be done by feedback.

Lemma 5 ([21]): The DAE (2) is impulse controllable if and only if there exists a feedback $u = Kx + v$ such that the closed-loop DAE $E\dot{x} = (A + BK)x + Bv$ is index-1.

In the following, consider the cost functional

$$J_E(x,u) = \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) + x(t_f)^\top E^\top PEx(t_f) \quad (11)$$

where $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$ and P are positive semidefinite and $R = R^\top$ is positive definite. Applying Pontryagin's maximum principle leads to the following necessary conditions.

Theorem 6 ([2]): Consider the impulse controllable DAE (2). Let (x^*, u^*) be a solution minimizing the cost functional (11). Then there exits a costate function $\mu^*(t)$ such that (x^*, u^*, μ^*) satisfies the boundary value problem

$$\begin{bmatrix} A & 0 & B \\ Q & A^\top & S \\ S^\top & B^\top & R \end{bmatrix} \begin{bmatrix} x \\ \mu \\ u \end{bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & -E^\top & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{bmatrix}, \quad (12)$$

$$Ex(t_0) = Ex_0, \quad E^\top \mu(t_f) = E^\top PEx(t_f). \quad (13)$$

Given these results, we can recall the following result.

Theorem 7 ([2]): Let $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix}$ and let U and V be matrices such that $U^\top EV = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $V^\top QV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$. If (2) is impulse-controllable and $\begin{bmatrix} C_2 & D \end{bmatrix}$ has full rank, then there exists a unique solution to (12) and (13) and hence a unique optimal control.

Finally we conclude this recapitulation by recalling that the optimal control yields a state-feedback.

Theorem 8 ([3]): Consider (2) with cost functional (11). If (2) is impulse controllable and stabilizable, and (12) is regular, impulse-free and no finite eigenvalue lies on the imaginary axis, there exists a solution $(X(t), Y(t))$ to the generalized Riccati differential equation (GRDE) given by

$$\begin{aligned} &E^\top \dot{X}(t) + Y(t)A + A^\top X(t) + Q \\ &\qquad - (S + Y(t)B)R^{-1}(B^\top X(t) + S^\top) = 0, \quad (14) \\ &Y(t)E = E^\top X(t), \end{aligned}$$

with terminal condition $E^\top X(t_f) = E^\top PE$. The optimal cost is quadratic and given by

$$J_E(x^*, u^*) = \frac{1}{2} x_0^\top E^\top X(0)x_0, \quad (15)$$

with optimal input $u^*(t) = -R^{-1}(B^\top X(t) + S^\top)x^*(t)$.

III. Optimal control with unconstrained terminal cost

Equipped with the mathematical preliminaries, we will return to the minimization of (1) subject to (2). Recall that (2) is assumed to be impulse controllable. Hence by Lemma 5 there exists a preliminary feedback $u = Kx + v$ ensuring that the resulting closed-loop DAE is index-1 and causes the cost functional (1) to take the form

$$\begin{aligned} J^-(x,u) &= \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ Kx+v \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ Kx+v \end{bmatrix} \right) + x(t_f^-)^\top Px(t_f^-) \\ &= \int_0^{t_f} \left(\begin{bmatrix} x \\ v \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \right) + x(t_f^-)^\top Px(t_f^-) =: \bar{J}^-(x,v), \quad (16) \end{aligned}$$

where $\bar{Q} = Q + SK + K^\top S^\top + K^\top RK$, $\bar{S} = S + K^\top R$, $\bar{R} = R$. Instead of aiming to prove the existence of an optimal input u to (2) directly, it suffices to find an input v that minimizes (16) subject to the DAE obtained after the preliminary feedback. This is formalized in the following lemma.

Lemma 9: There exists an input $u(\cdot)$ that minimizes (1) subject to the impulse controllable DAE (2) if and only if there exists an optimal input $v(\cdot)$ that minimizes $\bar{J}^-(\bar{x}, v)$ given by (16) subject to the index-1 DAE

$$E\dot{\bar{x}} = \bar{A}\bar{x} + Bv, \qquad \bar{x}(0^-) = x_0, \quad (17)$$

where $\bar{A} := A + BK$.

Proof: Applying a feedback to (2) can be regarded as a change of coordinates

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}. \quad (18)$$

Writing (2) as $\begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$ reveals

$$\begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{x}} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} = \begin{bmatrix} \bar{A} & B \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}.$$

Hence (x, u) solves (2) if and only if, $(\bar{x}, v)$ satisfying (18) solves (17). Furthermore, by (16) $J^-(x, u) = \bar{J}^-(\bar{x}, v)$. ■

Next, observe that as (17) is index-1, its solution can be decomposed as $\bar{x}(t) = \bar{x}^{\text{diff}}(t) + \bar{x}^{\text{imp}}(t) = \bar{\Pi}\bar{x}(t) - \bar{B}^{\text{imp}}v(t)$ and thus the cost functional

$$\begin{aligned} \bar{J}^-(\bar{x},v) &= \int_0^{t_f} \left(\begin{bmatrix} \bar{x} \\ v \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} \right) + \bar{x}(t_f^-)^\top P\bar{x}(t_f^-) \\ &= \int_0^{t_f} \left(\begin{bmatrix} \bar{\Pi}\bar{x} - \bar{B}^{\text{imp}}v \\ v \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} \bar{\Pi}\bar{x} - \bar{B}^{\text{imp}}v \\ v \end{bmatrix} \right) \\ &\qquad + \bar{x}(t_f^-)^\top P\bar{x}(t_f^-), \quad (19) \\ &= \int_{t_0}^{t_f} \left(\begin{bmatrix} \bar{x} \\ v \end{bmatrix}^\top \begin{bmatrix} \Pi^\top \tilde{Q}\Pi & \Pi^\top \tilde{S} \\ \tilde{S}^\top \Pi & \tilde{R} \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} \right) + \bar{x}(t_f^-)^\top P\bar{x}(t_f^-), \end{aligned}$$

where

$$\begin{aligned} &\tilde{Q} = \bar{Q}, \quad \tilde{S} = \bar{S} - \bar{Q}\bar{B}^{\text{imp}}, \\ &\tilde{R} = \bar{R} - \bar{B}^{\text{imp}\top}\bar{S} - \bar{S}^\top \bar{B}^{\text{imp}} + \bar{B}^{\text{imp}\top}\bar{Q}\bar{B}^{\text{imp}}. \end{aligned}$$

Altogether, the problem of finding an input u that minimizes (16) subject to (2) is thus equivalent to the problem of finding an input v that minimizes (19) subject to (17).

For analytical purposes, we will assume invertibility of $\tilde{R}$ throughout the remainder of the paper. This is not necessarily the case, as the invertibility might depend on the feedback matrix K used to reduce the index of (2). However, under the conditions of the following lemma, which are similar to one of the conditions of Theorem 7, positive definiteness of $\tilde{R}$ is guaranteed regardless of the choice of K.

Lemma 10: Let $\bar{W}$ be any full rank matrix with $\operatorname{im} \bar{W} = \ker E$. If $[\, C\bar{W} \;\; D \,]$ has full rank, then $\tilde{R}$ is positive definite.

Under the assumption that $\tilde{R}$ is invertible, we can confine our attention to the following simpler problem.[2]

Problem 1: Consider the DAE (2) and assume it is of index-1 and with consistency projector Π. Furthermore, consider the cost function

$$J_\Pi^-(x,u) = \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) + x(t_f^-)^\top P x(t_f^-), \tag{20}$$

where $R = R^\top$ is positive definite, $Q = Q^\top$ and $P = P^\top$ are positive semidefinite. Find an input $u(\cdot)$ that minimizes (20) subject to (2). We will call any such input an *optimal input* with corresponding *optimal solution* x.

Due to the quadratic nature of the cost functional (20) we can prove that if there exists an input that minimizes (20), it is linear in the optimal differential state.

Lemma 11: If there exists a solution $u(\cdot)$ that solves Problem 1, then $u(t) = K(t)x^{\text{diff}}(t)$ for some $K(t) \in \mathbb{R}^{m \times n}$.

Proof: First we will show that the map $x_0 \mapsto u$ is linear, where u is minimizing (20) subject to (2); in particular, we will show that λu is the optimal control for the initial value λx_0 and that for any optimal inputs u_x, u_z corresponding to any initial values $x_0, z_0 \in \mathbb{R}^n$ the input $u_x + u_z$ is optimal for the initial value $x_0 + z_0$. To that extent, let $V(x_0, t_0)$ be the value function defined by

$$V(x_0, t_0) = \min_u J_\Pi^-(x,u)$$

i.e., the cost for the optimal input input u and the corresponding trajectory x with initial condition $x(t_0^-) = x_0$. Applying the input λu to an initial condition λx_0 results in a trajectory λx, due to the linearity of solutions of (2). This means that $J_\Pi^-(\lambda x, \lambda u) = \lambda^2 J_\Pi^-(x,u)$ for any $\lambda \in \mathbb{R}$ and we can conclude that $\lambda^2 V(x_0,t_0) = V(\lambda x_0, t_0)$, which shows that λu is optimal for λx.

Furthermore, since $J_\Pi^-(x+z, u+v) + J_\Pi^-(x-z, u-v) = 2J_\Pi^-(x,u) + 2J_\Pi^-(z,v)$ we have

$$V(x_0+z_0,t_0) + V(x_0-z_0,t_0) = 2J_\Pi^-(x,u) + 2J_\Pi^-(z,v).$$

Which means that $V(x_0+z_0,t_0) + V(x-z,t_0) \leqslant 2V(x_0,t_0) + 2V(z_0,t_0)$. Conversely,

$$\begin{aligned} 2V(x_0,t_0) + 2V(z_0,t_0) &\leqslant 2J_\Pi^-(x,u) + 2J_\Pi^-(z,v) \\ &= J_\Pi^-(x+z,u+v) + J_\Pi^-(x-z,u-v) \end{aligned}$$

and hence we can conclude $V(x_0+z_0,t_0) + V(x_0-z_0,t_0) = 2V(x_0,t_0) + 2V(z_0,t_0)$. Furthermore, if u_x is the optimal input for x and u_z is the optimal input for z then

$$\begin{aligned} &V(x_0-z_0,t_0) + V(x_0+z_0,t_0) \\ &= 2V(x_0,t_0) + 2V(z_0,t_0) = 2J_\Pi^-(x,u_x) + 2J_\Pi^-(z,u_z) \\ &= J_\Pi^-(x+z,u_x+u_z) + J_\Pi^-(x-z,u_x-u_z). \end{aligned}$$

[2]Note that the matrices Q, S, R appearing in Problem 1 are not equal to the original matrices Q, S, R appearing in (1), but need to be adjusted according to the derivation of (16) and (19), furthermore, Π is the consistence projector *after* the application of the possible index-reducing feedback. However, the terminal cost matrix P is *not* effected by all these preliminary transformations.

Since $V(x_0+z_0,t_0) \leqslant J_\Pi^-(x+z,u_x+u_z)$ and similarly $V(x_0-z_0,t_0) \leqslant J_\Pi^-(x-z,u_x-u_z)$, it follows that

$$\begin{aligned} 0 &\leqslant J_\Pi^-(x+z,u_x+u_z) - V(x_0+z_0,t_0) \\ &= V(x_0-z_0,t_0) - J_\Pi^-(x-z,u_x-u_z) \leqslant 0, \end{aligned}$$

and thus $V(x_0+z_0,t_0) = J_\Pi^-(x+z,u_x+u_z)$. This shows that u_x+u_z is optimal for $x+z$. Hence there exists a linear map between the optimal trajectory and the optimal input. Particularly, the map $x(t_0^-) = x_0 \mapsto u(t_0)$ is linear, *i.e.,* there exists a $K(t_0) \in \mathbb{R}^{m \times n}$ such that $u(t_0) = K(t_0)x(t_0^-)$.

From the dynamic programming principle [22] it follows that $u_{[\tau,t_f)}$ is the optimal control for the cost function (19) considered on the interval $[\tau, t_f)$ for any $\tau \in [t_0, t_f)$, hence by replacing the initial time t_0 in the above argumentation by $\tau \in [t_0,t_f)$ we can conclude that for every $\tau \in [t_0,t_f)$ a matrix $K(\tau) \in \mathbb{R}^{m \times n}$ exists such that the optimal control satisfies $u(\tau) = K(\tau)x(\tau^-)$.

Noting that as $\Pi x = x^{\text{diff}}$ and $x^{\text{imp}} = -B^{\text{imp}}u$ the cost functional (20) can be written as a function of x^{diff} and u only, *i.e.,* $J_\Pi^-(x,u) = J_\Pi^-(x^{\text{diff}}, u)$. Hence it follows that the input $u(t) = K(t)x^{\text{diff}}(t)$. ■

The fact that if there exists an optimal control it is linear in x^{diff}, leads to the following implications.

Corollary 12: Assume that there exists an input u that solves Problem 1. Let $x = x^{\text{diff}} + x^{\text{imp}}$ be the corresponding optimal trajectory. If $x^{\text{diff}}(\tau^-) = 0$ for some $\tau \in [t_0,t_f)$ then $x^{\text{diff}}(t^-) \equiv 0$ on $[t_0,t_f)$. Consequently, $x^{\text{diff}}(\tau^-) = 0$ for some $\tau \in [t_0,t_f)$ if and only if $x^{\text{diff}}(t_0^-) = 0$.

Proof: Since $x(t)$ is a solution to (2) and the input $u = K(t)x^{\text{diff}}(t)$, in view of (9) the component x^{diff} solves the homogeneous time-varying ODE

$$\dot{x}^{\text{diff}}(t) = \left(A^{\text{diff}} + B^{\text{diff}}K(t)\right)x^{\text{diff}}(t),\; x^{\text{diff}}(0^-) = \Pi x_0.$$

Consequently, if $x^{\text{diff}}(\tau^-) = 0$ for some $\tau \in [t_0,t_f)$, $x^{\text{diff}}(t) = 0$ for all $t \in [\tau, t_f)$. Solving the ODE backwards in time yields that $x^{\text{diff}}(t_0^-) = 0$ and hence $x^{\text{diff}} \equiv 0$. ■

Next, we show that for any $p \in \operatorname{im}\Pi$ there exists an initial condition and an input such that $J_\Pi^-(x,u)$ is minimal and $x(t_f^-) = p$.

Corollary 13: Assume that Problem 1 is solvable. Then for any $p \in \operatorname{im}\Pi$, there exists an initial value x_0 such that the optimal trajectory satisfies $x(t_0^-) = x_0$ and $x^{\text{diff}}(t_f^-) = p$.

Next we will derive necessary conditions on the input u. We will show that the extension of u to $[t_0,t_f]$ by defining $u(t_f) = u(t_f^-)$ has to solve another optimization problem.

Lemma 14: An input u solves Problem 1 if, and only if, $\bar{u}$ defined on $[t_0,t_f]$ as $\bar{u}_{[t_0,t_f)} = u_{[t_0,t_f)}$ and $\bar{u}(t_f) = u(t_f^-)$ and its corresponding solution $\bar{x}$ to (2) on $[t_0,t_f]$ minimizes

$$J_\Pi(\bar{x},\bar{u}) = \int_{t_0}^{t_f} \left(\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right) + \bar{x}(t_f)^\top P \bar{x}(t_f), \tag{21}$$

Proof: ($\Leftarrow$) Let $\bar{x}$ be the corresponding optimal trajectory. Since $\bar{x}^{\text{diff}}$ is continuous on $[t_0,t_f]$ and $\bar{u}(t_f) = u(t_f^-)$ by assumption, it follows that

$$\begin{aligned} \bar{x}(t_f) &= \bar{x}^{\text{diff}}(t_f) - B^{\text{imp}}\bar{u}(t_f) \\ &= \bar{x}^{\text{diff}}(t_f^-) - B^{\text{imp}}\bar{u}(t_f^-) = \bar{x}(t_f^-). \end{aligned}$$

Consequently, given the input $u = \bar{u}_{[t_0,t_f)}$, the corresponding trajectory $x = \bar{x}_{[t_0,t_f)}$. Hence $J^-_\Pi(x,u) = J_\Pi(\bar{x},\bar{u})$.

Seeking a contradiction, suppose that there exists an input v and a corresponding trajectory y for which $J^-_\Pi(y,v) < J^-_\Pi(x,u)$ then the input $\bar{v}(t)$ with $\bar{v}(t_f^-) = \bar{v}(t_f)$ and $\bar{v}_{[t_0,t_f)} = v_{[t_0,t_f)}$ yields $J_\Pi(\bar{y},\bar{v}) = J^-_\Pi(y,v) < J^-_\Pi(x,u) = J_\Pi(\bar{x},\bar{u})$, which contradicts the optimality of $\bar{u}$. Hence for all inputs v and corresponding trajectories y we have $J^-_\Pi(x,u) \leqslant J^-_\Pi(y,v)$ and thus u minimizes (20).

($\Rightarrow$) For the input $\bar{u}$ defined as $\bar{u} = u$ on $[t_0,t_f)$ and $\bar{u}(t_f) = u(t_f^-)$ we obtain that $J^-_\Pi(x,u) = J_\Pi(\bar{x},\bar{u})$. Seeking a contradiction, suppose that there exists an input $\bar{w}$ and a corresponding trajectory $\bar{y}$ defined on $[t_0,t_f]$ for which $J_\Pi(\bar{y},\bar{w}) < J_\Pi(\bar{x},\bar{u})$. For $\delta > 0$ define $\bar{w}_\delta(t)$ as $\bar{w}_\delta(t) = \bar{w}(t)$ on $[t_0, t_f-\delta)$ and $\bar{w}_\delta(t) = \bar{w}(t_f)$ on $[t_f-\delta, t_f]$, then $J^-_\Pi(y_\delta, w_\delta) := J^-_\Pi((\bar{y}_\delta)_{[t_0,t_f)}, (\bar{w}_\delta)_{[t_0,t_f)}) = J_\Pi(\bar{y}_\delta, \bar{w}_\delta)$, where $\bar{y}_\delta$ is the solution corresponding to $\bar{w}_\delta$ on $[t_0,t_f]$. Furthermore, for every $\varepsilon > 0$ we find $\delta > 0$ such that $J_\Pi(\bar{y}_\delta,\bar{w}_\delta) < J_\Pi(\bar{y},\bar{w}) + \varepsilon$. Hence for sufficiently small $\varepsilon > 0$ and corresponding $\delta > 0$, we have $J^-_\Pi(y_\delta,w_\delta) = J_\Pi(\bar{y}_\delta,\bar{w}_\delta) < J_\Pi(\bar{y},\bar{w}) + \varepsilon < J_\Pi(\bar{x},\bar{u}) = J^-_\Pi(x,u)$; a contradiction to optimality of (x,u). ∎

Corollary 15: If there exists an input u that solves Problem 1, then the optimal trajectory $x = x^{\text{diff}} + x^{\text{imp}}$ and optimal input u satisfy

$$B^{\text{imp}\top} P x^{\text{diff}}(t_f^-) = B^{\text{imp}\top} P B^{\text{imp}} u(t_f^-).$$

Finally, using the completion of the squares formula the following result can be obtained.

Lemma 16: Consider Problem 1. Then for any input $u(\cdot)$ the cost (20) is given by

$$\begin{aligned} J^-_\Pi(x,u) &= x^{\text{diff}\top}(t_0^-) X(t_0^-) x^{\text{diff}}(t_0^-) \\ &+ \int_{t_0}^{t_f} \left(\left\| R^{\frac{1}{2}} u + R^{-\frac{1}{2}} (B^{\text{diff}\top} X + S^\top) x^{\text{diff}} \right\|_2^2 \right) \\ &+ x(t_f^-)^\top P x(t_f^-) - x^{\text{diff}\top}(t_f^-) X(t_f^-) x^{\text{diff}}(t_f^-), \end{aligned} \tag{22}$$

where $X(\cdot)$ is a solution to the Riccati equation

$$\begin{aligned} \dot{X} &= -A^{\text{diff}\top} X - X^\top A^{\text{diff}} - \Pi^\top Q \Pi \\ &+ (\Pi^\top S + X^\top B^{\text{diff}}) R^{-1} (B^{\text{diff}\top} X + S^\top \Pi), \end{aligned} \tag{23}$$

on $[t_0, t_f]$.

Combining all auxiliary results leads to the main result.

Theorem 17: Problem 1 is solvable if, and only if, there exist $X \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{m\times n}$ such that

$$K = -R^{-1}(B^{\text{diff}\top} X + S^\top) \tag{24a}$$

$$B^{\text{imp}\top} P \Pi = B^{\text{imp}\top} P B^{\text{imp}} K \Pi \tag{24b}$$

$$X = \Pi^\top (I - K^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} K) \Pi. \tag{24c}$$

Proof: ($\Rightarrow$) By Lemma 11 $u(t) = K(t) x^{\text{diff}}(t)$ and thus by Corollary 15 we have $B^{\text{imp}\top} P \Pi x^{\text{diff}}(t_f^-) = B^{\text{imp}} P B^{\text{imp}} K(t_f^-) x^{\text{diff}}(t_f^-)$. By Corollary 13 this needs to hold for any $x^{\text{diff}}(t_f) \in \operatorname{im}\Pi$, and thus

$$B^{\text{imp}\top} P \Pi = B^{\text{imp}} P B^{\text{imp}} K(t_f^-). \tag{25}$$

Consequently (24b) holds holds for $K = K(t_f^-)$. It follows from Lemma 16 that if $X(t)$ is a solution to the Riccati equation (23) with terminal condition

$$X(t_f^-) = \Pi^\top (I - K(t_f^-)^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} K(t_f^-)) \Pi,$$

and substituting $x(t_f^-) = (I - B^{\text{imp}} K(t_f^-)) \Pi x^{\text{diff}}(t_f^-)$, then the optimal cost is given by

$$\begin{aligned} &J^-_\Pi(x,u) = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} \\ &+ \int_{t_0}^{t_f} \left(\left\| \left(R^{\frac{1}{2}} K(\cdot) + R^{-\frac{1}{2}} \left(B^{\text{diff}\top} X(\cdot) + S^\top \right) \right) x^{\text{diff}} \right\|_2^2 \right). \end{aligned}$$

Now we will show that this implies $K(t) = -R^{-1}(B^{\text{diff}\top} X(t) + S^\top)$ and that $K(t_f^-) = K$, $X(t_f^-) = X$ solve (24a) and (24c). To seek a contradiction, assume that $K(t) \neq -R^{-1}(B^{\text{diff}\top} X(t) + S^\top)$. Then for some $M > 0$

$$J^-_\Pi(x,u) = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}}(t_0^-) + M.$$

Then we will show that there exists an input $\bar{u}$ that results in $J^-_\Pi(\bar{x},\bar{u}) < J^-_\Pi(x,u)$.

Consider $\varepsilon > 0$ arbitrarily small. Then the input $\bar{u}$, defined by $\bar{u}(t) = \bar{K}(t) x^{\text{diff}}(t)$, where

$$\bar{K}(t) = \begin{cases} -R^{-1}(B^{\text{diff}\top} X(\cdot) + S^\top) & t \in [t_0, t_f - \varepsilon) \\ K(t) & t \in [t_f - \varepsilon, t_f) \end{cases}$$

results in the following cost

$$\begin{aligned} &J^-_\Pi(\bar{x},\bar{u}) = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} \\ &+ \int_{t_f-\varepsilon}^{t_f} \left(\left\| \left(R^{\frac{1}{2}} K(\cdot) + R^{-\frac{1}{2}} (B^{\text{diff}\top} X(\cdot) + S^\top \right) \bar{x}^{\text{diff}} \right\|_2^2 \right) \\ &< x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} + M = J^-_\Pi(x,u). \end{aligned}$$

This contradicts the optimality of the input u. Hence $u(t) = -R^{-1}(B^{\text{diff}\top} X(t) + S^\top) x^{\text{diff}}(t)$. Consequently

$$-R^{-1}(B^{\text{diff}\top} X(t_f^-) + S^\top) x^{\text{diff}}(t_f^-) = K(t_f^-) x^{\text{diff}}(t_f^-),$$

implies that X, Y solve (24) if $K = K(t_f^-)$ and $X = X(t_f^-)$.

($\Leftarrow$) For any matrix K that satisfies

$$B^{\text{imp}\top} P \Pi = B^{\text{imp}\top} P B^{\text{imp}\top} K \Pi, \tag{26}$$

we have that for all $\bar{K}$

$$\begin{aligned} &x^{\text{diff}}(t_f^-)(I - B^{\text{imp}} K)^\top P (I - B^{\text{imp}} K) x^{\text{diff}}(t_f^-) \\ &\quad \leqslant x^{\text{diff}}(t_f^-)(I - B^{\text{imp}} \bar{K})^\top P (I - B^{\text{imp}} \bar{K}) x^{\text{diff}}(t_f^-), \end{aligned}$$

and that equality holds for any K that satisfies (26). Consequently, for any feedback matrix $\bar{K}$ we have for all $x^{\text{diff}}(t_f^-) \in \operatorname{im}\Pi$

$$\begin{aligned} 0 \leqslant x^{\text{diff}}(t_f^-)^\top (I - \bar{K}^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} \bar{K}) x^{\text{diff}}(t_f^-) \\ - x^{\text{diff}}(t_f^-)^\top X x^{\text{diff}}(t_f^-). \end{aligned}$$

For the feedback matrix $K(t) = R^{-1}(B^{\text{diff}\top} X(t) + S^\top) x^{\text{diff}}(t)$ where $X(t)$ is a solution to (23) with terminal condition $X(t_f) = X$, we have that $K(t_f) = K$ and thus satisfies (24b). Furthermore, as X satisfies (24c), the cost for the feedback $u(t) = K(t) x(t)$ is given by $J^-_\Pi(x,u) =$

$x_0^{\mathrm{diff}\top}X(t_0^-)x_0^{\mathrm{diff}}$. Note that for any other input $\bar{u}(t) = \bar{K}(t)x^{\mathrm{diff}}(t)$ the cost is given by

$$
\begin{aligned}
J_\Pi^-(\bar{x},\bar{u}) &= x_0^{\mathrm{diff}\top}X(t_0^-)x_0^{\mathrm{diff}} \\
&\quad + \int_{t_0}^{t_f} \left(\|R^{\frac{1}{2}}\bar{u} + R^{-\frac{1}{2}}(B^{\mathrm{diff}\top}X(t) + S^\top)\bar{x}^{\mathrm{diff}}\|_2^2\right) \\
&\quad + \bar{x}(t_f^-)^\top P\bar{x}(t_f^-) - \bar{x}^{\mathrm{diff}}(t_f^-)^\top X(t_f^-)\bar{x}^{\mathrm{diff}}(t_f^-), \\
&\geqslant x_0^{\mathrm{diff}\top}X(t_0^-)x_0^{\mathrm{diff}} - \bar{x}^{\mathrm{diff}}(t_f^-)^\top X(t_f^-)\bar{x}^{\mathrm{diff}}(t_f^-), \\
&\quad + x^{\mathrm{diff}}(t_f^-)^\top((I - \bar{K}^\top B^{\mathrm{imp}\top})P(I - B^{\mathrm{imp}}\bar{K})x^{\mathrm{diff}}(t_f^-) \\
&\geqslant x_0^{\mathrm{diff}\top}X(t_0^-)x_0^{\mathrm{diff}} \geqslant J_\Pi^-(x,u),
\end{aligned}
$$

and hence u minimizes $J_\Pi^-(x,u)$. ■

Corollary 18: Consider Problem 1 and assume it has a solution. Let X, K solve (24). Then the u that minimizes (20) is given by

$$u(t) = -R^{-1}\left(B^{\mathrm{diff}\top}X(t) + S^\top\right)x^{\mathrm{diff}}(t),$$

where $X(t)$ solves (23) with terminal condition $X(t_f^-) = X$

Corollary 19: Consider Problem 1 and let $\Pi^\top PB^{\mathrm{imp}} = 0$. Then Problem 1 is solvable if and only if

$$PB^{\mathrm{imp}}R^{-1}(B^{\mathrm{diff}\top}P + \bar{S}^\top)\Pi = 0. \tag{27}$$

Remark 20: In the case $P = E^\top\bar{P}E$ for some positive semi-definite $\bar{P}$ it follows that $\Pi^\top PB^{\mathrm{imp}} = \Pi^\top E^\top\bar{P}EB^{\mathrm{imp}} = 0$ as $EB^{\mathrm{imp}} = 0$. Condition (27) is clearly satisfied and thus there exist an optimal input u. Furthermore, the optimal input is given by

$$u(t) = -R^{-1}(B^{\mathrm{diff}}X(t) + S^\top)\Pi x^{\mathrm{diff}}(t),$$

where $X(t)$ is the solution of (23) with terminal constraint $X(t_f) = \Pi^\top P\Pi$.

Example 21 (Example 1 revisted): Returning to the example in the introduction, we have for (4) that $R = 1$, $S = 0$, $P = \left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]$, $\Pi = \left[\begin{smallmatrix}I & 0\\ 0 & 0\end{smallmatrix}\right]$, $B^{\mathrm{imp}} = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$ and $B^{\mathrm{diff}} = 0$. It follows that $\Pi^\top PB^{\mathrm{imp}} = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right] \neq 0$. This means that there exist an optimal input if and only if the conditions given in Theorem 17 are satisfied. However, according to (24a) $K = 0$, whereas (24b) states that K also satisfies

$$\left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right] = B^{\mathrm{imp}\top}P\Pi = B^{\mathrm{imp}\top}PB^{\mathrm{imp}}K\Pi$$

Hence the conditions are not satisfied and there indeed does not exist an optimal input u.

For the cost functional (5) we have $R = 1$, $S = 0$, $P = \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]$, $\Pi = \left[\begin{smallmatrix}I & 0\\ 0 & 0\end{smallmatrix}\right]$, $B^{\mathrm{imp}} = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$ and $B^{\mathrm{diff}} = 0$.This means that $\Pi^\top PB^{\mathrm{imp}} = 0$. However,

$$B^{\mathrm{imp}\top}PB^{\mathrm{imp}}R^{-1}(B^{\mathrm{diff}\top}P + S^\top)\Pi = 0,$$

which shows that there exists an optimal input u. ◇

IV. Conclusion

In this paper, the linear quadratic optimal control problem for DAEs with unconstrained terminal cost has been studied. It was shown that for a general weight matrix in the terminal cost, optimal input might fail to exist. Necessary and sufficient condition that guarantee the existence of an optimal solution in terms of matrix equations were formulated.

As motivated already in the introduction, the next step is now to apply our results to study optimal control of switched DAEs.

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