Observability and Determinability Characterizations for Linear Switched Systems in Discrete Time

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***Abstract*— In this article, we study the observability and determinability for discrete-time linear switched systems. Studies for the observability for this system class are already available in literature, however, we use assume here that the switching signal is known. This leads to less conservative observability conditions (e.g. observability of each individual mode is not necessary for the overall switched system to be observable); in particular, the dependencies of observability on the switching times and the mode sequences are derived; these results are currently not available in the literature on discrete-time switched systems. In addition to observability (which is concerned with recovering the state from the initial time onwards), we also investigate the determinability which is concerned with the ability to reconstruct the state value at the end of the observation interval. We provide several simple examples to illustrate novel features not seen in the continuous time case or for unswitched systems.**

I. Introduction

We study the observability and determinability of a class of switched systems where each mode is a discrete-time linear system. We consider the following general form:

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1a}$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) \tag{1b}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$ is the input, $y(k) \in \mathbb{R}^p, p \in \mathbb{N}$ is the output, $\sigma : \mathbb{N} \to \{0, 1, 2, ..., \mathtt{p}\}, \mathtt{p} \in \mathbb{N}$ is the switching signal determining which mode $\sigma(k)$ is active at a time instant k, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, and $D_i \in \mathbb{R}^{p\times m}$.

The system class (1) indicates that we assume the switching is triggered only by the time and neither by the state nor the input nor the output. We furthermore assume in this study that the switching signal σ is fully known and fixed. This assumption means that (1) can be seen as a specially structured time-varying linear system, and there are many switched systems which can be modelled in that framework (see e.g. [1], [2], [3], [4], [5], [6]).

To be specific, since there are several observability notions for switched systems studied in literature, for example, path-wise observability that requires observability for every path (switching signal) with some length [7], and mode observability that recovers a certain number of first modes in a switching signal [8], in this paper, we focus on the state observability notion, i.e. the ability to reconstruct the initial value from output measurement for a fixed and fully known switching signal.

In continuous time domain, studies about observability for linear switched systems have been addressed in an extensive number of papers (see e.g. [9], [10], [11], [12]). In discrete time domain, not so many papers are found in the literature. In some of existing studies, the observability condition for system (1) is commonly written in a Gramian form by checking the kernel of the corresponding Gramian observability matrix which is a huge matrix and is not computational-friendly form (see e.g. [13], [14]). Another form has been presented in a geometrical approach but is also not in a computational-friendly form (see e.g. [15]). An important point in discrete time systems is that, in general, it cannot be observable within too short observation time and hence a sufficiently large dwell time for each mode is needed. However, this is not enough to preserve the observability if the switching time(s) is changed and thus it becomes dependent on the switching time(s). Apart from our own study of the single-switch case [16], the dependency of observability on (multiple) switching times seems not to be discussed yet in any reference and therefore is a novel aspect we study in this paper.

Furthermore, the observability characterizations we will present are based on a geometric approach and are more friendly in terms of computational aspects. Additionally, a new geometric approach for determinability characterizations will be presented.

This paper is structured as follows. In Section II, we first observe that it suffices to consider the homogeneous version of (1); some new notations will also be introduced here. The main results will be presented in Section III for observability characterizations, and in Section IV for determinability characterizations including some counter- and illustrating examples.

II. Preliminaries

A. System's description and its solution

Under the assumption that the switching signal is known, it is easily seen that the ability to recover the state of (1) from the values of the external signals (input and output) is independent on how the input influences the state-dynamics. The forthcoming observability and determinability notions therefore do not depend in the input coefficient matrices B_i and D_i and, consequently, we will only consider the homogeneous version of the Linear Switched System (LSS)

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(1) given by

$$x(k+1) = A_{\sigma(k)} x(k), \quad k = 0, 1, ... \tag{2a}$$

$$y(k) = C_{\sigma(k)} x(k), \quad k \in \mathbb{N} \tag{2b}$$

where the initial value $x(0)$ is unknown and $y(k)$ is the output measurement. By $\{(A_0, C_0), (A_1, C_1), \ldots (A_{\mathtt{p}}, C_{\mathtt{p}})\}$ we denote the family of the system's matrix pairs of all modes involved in (2). All solutions of (2) satisfy

$$x(k) = \Phi_\sigma(k, h) x(h), \quad \forall k, h \in \mathbb{N} \text{ with } k \geq h,$$

where $\Phi_\sigma(k, h) = A_{\sigma(k-1)} \cdots A_{\sigma(h)}$ is so called the state transition matrix. With initial value $x(0) = x_0 \in \mathbb{R}^n$, system (2) has the unique solution

$$x(k) = \Phi_\sigma(k, 0) x_0, \quad k \in \mathbb{N}. \tag{3}$$

For the rest of this paper, we restrict the switching signal to a fixed and known switching signal given via a fixed mode sequence $(\sigma_j)_{j \in \mathbb{N}}$ as follows (see also Fig. 1)

$$\sigma(k) = \sigma_j \text{ if } k \in [k_j^s, k_{j+1}^s), \ j \in \{0, 1, 2, ...\}, \tag{4}$$

where $k_j^s \in \mathbb{N}$ denotes the switching time with $k_0^s = 0$, $k_{J+1}^s = K + 1$, and $\sigma_j \in \{0, 1, ..., \mathtt{p}\}$. Here $K > 0$ denotes the final time of interest and $J > 0$ is the number of switches in the interval $[0, K]$.

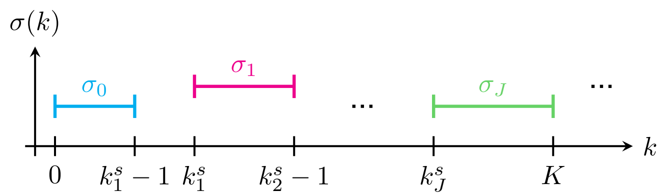


Fig. 1: Mode sequence (4)

We refer to σ_0 as the initial mode and we assume that the switching times k_j^s are strictly increasing, i.e. $k_{j+1}^s > k_j^s \ \forall j$, which means that each mode is active at least one time instant when it is active. This also means that there is no mode that is never active (e.g. due to collapsed) over the time interval $[0, K]$. One may wonder that $\{(A_0, C_0), (A_1, C_1), \ldots (A_{\mathtt{p}}, C_{\mathtt{p}})\}$ may produce many possible mode sequences; however, by the definition of the switching signal (4), the study in this paper corresponds to only one fixed mode sequence and therefore to examine a different mode sequence, one should repeat the calculation except in some specific situations. This is a limitation of the study in this paper.

We also use the term so-called "dwell-time" which was introduced in [17]. For a positive constant τ_D, let $\mathbb{S}^{[\tau_D]}$ be the set of all fixed and known mode sequences given by (4) with interval between two consecutive switching times no smaller than τ_D. We refer to τ_D as the (fixed) dwell-time. Furthermore, by $\mathbb{S}_{[0,K]}^{[\tau_D]}$ we denote the set of all switching signals $\sigma \in \mathbb{S}^{[\tau_D]}$ defined on the time interval $[0, K]$.

Under a fixed switching signal we can write the solution at any switching time k_j^s by using the formula given in the following lemma.

Lemma 2.1: Under a fixed switching signal (4), the solution of linear switched systems (2) at any switching time k_j^s is given by

$$x(k_j^s) = \psi_\sigma(j, 0) x(0) \tag{5}$$

where for $j = 0, 1, ...$

$$\psi_\sigma(j, 0) = A_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s} A_{\sigma_{j-2}}^{k_{j-1}^s - k_{j-2}^s} \cdots A_{\sigma_0}^{k_1^s - k_0^s}. \tag{6}$$

Proof: Extracting the solution from (3) at the switching time k_j^s proves (5). ■

We refer to matrix (6) as the state transition matrix at the switching time k_j^s. Moreover, for every j, (6) can be rewritten in a recursive form as

$$\psi_\sigma(j, 0) = A_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s} \psi_\sigma(j-1, 0) \tag{7}$$

with $\psi_\sigma(0, 0) = I_n$. This is more computational-friendly compared to (6).

B. Observability and Determinability Definitions

In this paper, we study the observability (and also the determinability) for a finite observation time interval $[0, K]$ where the number of switches is as many as J and the number of the output measurements available to reconstruct the state is as many as $K + 1$.

Intuitively, system (2) is observable on $[0, K]$ with respect to (w.r.t.) a fixed switching signal σ if the knowledge of the output measurements $\{y(0), y(1), ..., y(K)\}$ is sufficient to determine the state on this interval. This can be defined mathematically as follows.

Definition 2.2 (Observability): Linear switched system (2) is called observable on $[0, K]$ w.r.t. a fixed switching signal given by (4) if for all solutions on $[0, K]$ the following implication holds:

$$y^1 \equiv y^2 \Rightarrow x^1 \equiv x^2. \tag{8}$$

Due to linearity, we can reduce the observability condition (8) as zero-observability as in the following proposition.

Proposition 2.3 (Zero-observability): Linear switched system (2) is observable on $[0, K]$ w.r.t. a fixed switching signal given by (4) if, and only if, for all solutions on $[0, K]$ the following implication holds:

$$y \equiv 0 \Rightarrow x \equiv 0. \tag{9}$$

Proof: The proof is straightforward and is omitted. ■

By the fact that using (2), the knowledge of the state on $[0, K]$ can be derived recursively if we know $x(0)$, then we can rewrite the observability definition as follows: (2) is observable on $[0, K]$ w.r.t a fixed switching signal σ if the knowledge of the output sequence $\{y(0), y(1), ..., y(K)\}$ is sufficient to determine $x(0)$. Furthermore, since $x \equiv 0$ on $[0, K]$ if, and only if, $x(0) = 0$, then the observability condition (9) can be reduced to

$$y(k) = 0 \ \forall k \in [0, K] \Rightarrow x(0) = 0. \tag{10}$$

Finally, we say system (2) is (globally) observable if there exists such positive integer K.

For non-switched systems it is well known that even if the pair (A, C) is observable (i.e. the corresponding Kalman observability matrix has full rank) the initial state may not be observable if the output is not measured long-enough. It is however easily seen that never more than n output measurements are necessary to extract all the information about the n-dimensional state. For switched systems this means that if the dwell time is smaller than the state-dimension, we can expect a loss of observability just because of the fact, that we didn't remain long enough in a mode to extract all available information of the state. A novel aspect of switched systems is however, that even if each mode remains active long enough (i.e. no more information about the state can be obtained by staying longer in that mode) the observability may still depend on how long each mode remains active. Switched systems where this dependence does *not* occur are of special interest and justifies the following definition.

Definition 2.4 (Constant Observability): Consider the linear switched system (2) with fixed mode seqeunce (σ_j). The observability of this system is called *constant* (under slow switching) if it is either observable or unobservable for every $\sigma \in \mathbb{S}^{[n]}_{[0,K]}$ with fixed mode sequence (σ_j).

In other words, constant observability means that the observability does not depend on the switching time(s), and changing the switching time(s) does not change the observability (provided each mode remains active long enough). In particular, constant observability indicate a certain robustness of the fundamental system property of observability with respect to the switching times. Furthermore, from the definition, if $K = Jn$ i.e. there is only one possible switching signal with the dwell time at least n on $[0, K]$, then the observability is trivially constant; however, for $K > Jn$ constant observability depends in general on the specific matrices (A_i, C_i) and the mode sequence (σ_j).

In a situation where the state on the time interval $[0, K]$ cannot be reconstructed by the given output measurement, one may want to reconstruct the state at the final time instant K. Once we know the state at K, we could iterate the system's model to obtain the solutions at the future time instants, for instance to design a state feedback. Based on this motivation, we study in the following the determinability concept which was initially introduced in [18] for continuous time. Note that in this study, the switching signal is already known and fixed. If the switching signal is fully or partially unknown, one may refer to the switch observability/determinability study discussed in [19], [20].

Once we have a time instant $K > 0$ where we can determine $x(K)$ from the output measurement, we say that the system is determinable. To be precise, we define the determinability in a mathematically intuitive form by defining the determinability on a time interval $[0, K]$ as follows.

Definition 2.5: The linear switched system (2) is called determinable on $[0, K]$ w.r.t. a fixed switching signal given by (4) if the knowledge of the output measurements $\{y(0), y(1), ..., y(K)\}$ is sufficient to determine $x(K)$.

Let $y_{[0,K]} := \{y(0), y(1), ..., y(K)\}$. The determinability definition above can be brought into the following final-state-sustainability definition.

Definition 2.6: The linear switched system (2) is *determinable* on $[0, K]$ w.r.t. a fixed switching signal given by (4) if the following implication holds:

$$y^1_{[0,K]} \equiv y^2_{[0,K]} \Rightarrow x^1(K) = x^2(K). \tag{11}$$

Similar to the observability, we can also bring the determinability definition above into a zero-determinability condition as follows.

Proposition 2.7: The linear switched system (2) is determinable on $[0, K]$ w.r.t. a fixed switching signal given by (4) if, and only if, the following implication holds:

$$y_{[0,K]} \equiv 0 \Rightarrow x(K) = 0. \tag{12}$$

Proof: The proof is straightforward and is omitted. ■

Thus, the system (2) is determinable if, and only if there exists $K \in \mathbb{N}$ such that (11) holds. In other words, it is determinable if, and only if there exists $K \in \mathbb{N}$ such that (12) holds. We use the latter condition in the characterizations.

Remark 2.8: When the matrices A_i for all i in (2) are invertible, then determinability and observability are equivalent since (2) can be rewritten in a backward dynamical system. However, in general, observability implies determinability on the same time interval but the converse is not always true. Furthermore, if the switched system is determinable, i.e. $x(K)$ can be reconstructed, then the state of the future time instants can be determined. This is why it is also called as "forward observable" in [21].

III. Observability

Under a single switch switching signal, the observability characterization for (2) has been covered in [16]. We present in the following the results regarding the observability characterizations for general switching signals.

Theorem 3.1: The linear switched system (2) is observable on $[0, K], K \in [k^s_J, k^s_{J+1})$ w.r.t. to the fixed switching signal (4) if, and only if,

$$\bigcap_{j=0}^{J} [\psi_\sigma(j, 0)]^{-1} (\mathcal{O}^{k^s_{j+1} - k^s_j - 1}_{\sigma_j}) = \{0\} \tag{13}$$

where $\psi_\sigma(j, 0)$ is given by (6) and

$$\mathcal{O}^k_i := \ker[C_i^\top, (C_i A_i)^\top, \ldots, (C_i A_i^k)^\top]^\top. \tag{14}$$

Proof: Taking the kernel of the observability matrix over the time interval $[0, K]$ and using the fact that $\ker \left[\begin{smallmatrix} M_1 \\ M_2 M_3 \end{smallmatrix}\right] = \ker M_1 \cap [M_3]^{-1} \ker M_2$ for any matrices M_i proves the observability condition (13). ■

Note that $[*]^{-1}$ denotes the preimage and not the inverse. The observability condition (13) above for any sufficient slow switching signal $\sigma \in \mathbb{S}^{[n]}_{[0,K]}$ can be reduced to

$$\bigcap_{j=0}^{J} [\psi_\sigma(j, 0)]^{-1} (\mathcal{O}_{\sigma_j}) = \{0\} \tag{15}$$

where $\mathcal{O}_{\sigma_j} := \mathcal{O}^{n-1}_{\sigma_j}$. This means that if each mode is active long enough (at least n time steps) and there is no switch after K then the observability will depend only on the switching time and thus (15) is the condition for global observability.

Remark 3.2: In general, system (2) is defined on $[0, \infty)$. In this situation, the observability condition is not dependent on K anymore and it is dependent only on the switching times k^s_j. Moreover, it is clear that an observable initial mode on $[0, k^s_1 - 1]$ implies global observable but an observable subsequent mode on the corresponding time interval doesn't always imply global observable.

Next, we study the dependence of the observability on the switching times k^s_j. For any dimension of the state x, if the first subspace in (15) equals $\{0\}$ then clearly the observability is constant i.e. observable for any k^s_j. We consider first the result for one-dimensional states in the following proposition.

Proposition 3.3: The observability of (2) with one-dimensional states is constant for any switching signal.

Proof: Let (a_i, c_i) be the individual mode. Its unobservable space is

$$\ker[c_1, c_2 a_1^{k_1^s-1}, c_3 a_2^{k_2^s-k_1^s-1} a_1^{k_1^s-1}, \dots]^\top$$
$$= \ker[c_1] \cap \ker[c_2 a_1^{k_1^s-1}] \cap \ker[c_3 a_2^{k_2^s-k_1^s-1} a_1^{k_1^s-1}] \cap \dots.$$

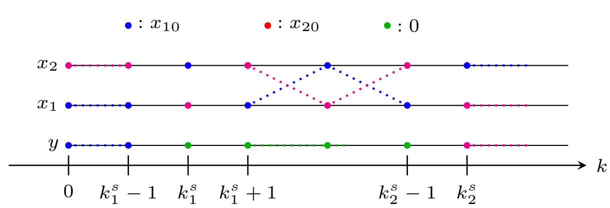
For any k_j^s, the unobservable space is equal to $\mathbb{R}$ if $c_i = 0\ \forall i$, or is equal to $\{0\}$ if $c_i \neq 0$ for some i i.e. it is either unobservable or observable for any given k_j^s. Thus the observability is always constant. ■

Proposition 3.3 can be explained intuitively by the fact that, in one-dimensional space, it is impossible to have different unobservable spaces with different switching times since we will always get either $\{0\}$ or $\mathbb{R}$ from (13). However, already in two dimension this argument is not valid anymore and indeed the following example shows that the observability property depends in general on the switching times.

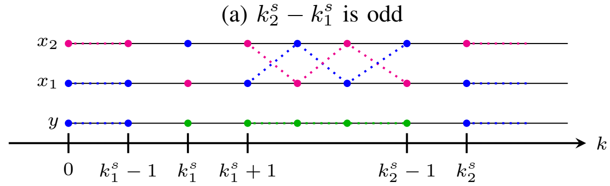
Example 3.4: Consider the linear switched system (2) composed by the following two modes

$$(A_1, C_1) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right)$$
$$(A_2, C_2) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right).$$

The system starts from mode-1 and switches to mode-2 at time instant k_1^s and switches again to mode-1 at time instant k_2^s. When $k_2^s - k_1^s$ is odd then mode-2 is active for an odd number of time instants. In this situation, all information can be completely deduced from the output measurements (see Fig. 2a). On the other hand, if $k_2^s - k_1^s$ is even, some information will be lost (see Fig. 2b). This means that when $k_2^s - k_1^s$ is even, the switched system is unobservable, because the initial value $x_2(0) = x_{20}$ will never be visible in the output.



(a) $k_2^s - k_1^s$ is odd



(b) $k_2^s - k_1^s$ is even

Fig. 2: Solution of the switched system in Example 3.4

Moreover, when the system starts from mode-2 and switches to mode-1 and switches again to mode-2, the switched system is always unobservable for arbitrary switching times k_1^s and k_2^s i.e. its observability is constant. This showed that the constant observability property is not preserved under permutation of the switching sequence. ◇

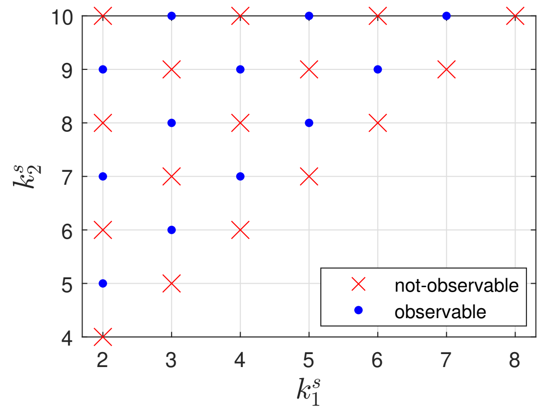


Fig. 3: Switching time vs observability Example 3.4

This example showed that the switching time dependence in the observability characterization (15) even for dwell-time switching signals cannot be removed in general. For multiple-switching this switching time-dependence is also present in the continuous time case and we believe that it is possible to derive sufficient or necessary conditions for observability in a similar way as in [18, Sec. IV], however, for switching signals without a dwell time, these conditions may be more complicated or may not exist at all; this is ongoing research.

From the observability condition (13), and confirmation derived from the Example 3.4, in general, the observability of the LSS (2) depends on the switching times and on how long each mode is active. This is similar to the result for LSSs in continuous time as discussed in [11]. Indeed the dependence occurs even for LSSs with only two modes as illustrated by the following example.

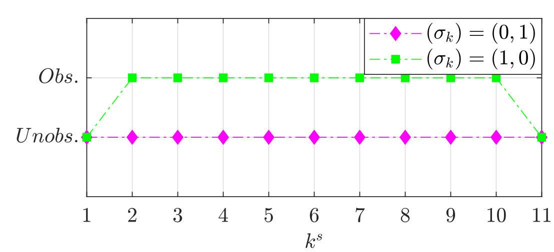


Fig. 4: Observability characterization results of Example 3.5

Example 3.5: Consider the linear switched system (2) composed by two modes with

$$(A_0, C_0) = \left(\begin{bmatrix} 0&0&0&1 \\ 1&1&0&0 \\ 0&1&1&0 \\ 0&1&1&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&1 \end{bmatrix} \right)$$
$$(A_1, C_1) = \left(\begin{bmatrix} 1&0&0&0 \\ 1&1&1&0 \\ 0&0&1&1 \\ 1&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&1 \end{bmatrix} \right).$$

We observe here for the time interval $[0, 12]$. As individual systems, both modes are not-observable since

$$\mathcal{O}_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \mathcal{O}_1 = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The observability property of the switched system w.r.t. the single switch switching signal given by the mode sequence $(\sigma_j) = (0, 1)$ or $(1, 0)$ with varying switching times $k_s \in [1, 11]$ are illustrated in Fig. 4.

For the first mode sequence $(\sigma_j) = (0, 1)$ the switched system remains unobservable independently of the switching time. However, for the reversed switching sequence $(\sigma_j) = (1,0)$ it turns out the switched system is observable if the switching times are sufficiently far away from the time-interval boundaries. In fact, if the switching time is too early so the initial mode is active too short, or too late (close enough to the end time K), the switched system gets unobservable also for the switching sequence $(1,0)$. ⋄

Although the observability property of the previous example clearly depends on the switching time, it does satisfy the definition of constant-observability because the observability properties remains constant when each mode is active long enough. Inspired by the single-switch result for the continuous time case, one may conjecture that also in the discrete time case the dependence of the observability property on the switching time in the single-switch case can only come from not obtaining enough information from the individual modes. We have already investigated this question in [16] in the context of singular systems, but we were only able to show the constant observability property under some additional subspace assumptions. Here we are now able to state the constant observability without any further assumptions; the result is based on the following technical lemma.

Lemma 3.6: For any invertible matrix $A \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{m\times n}$, the following equation holds for every $k \in \mathbb{Z}$ and every $\hat{n} \geq n$

$$\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \ker \begin{bmatrix} CA^k \\ CA^{1+k} \\ \vdots \\ CA^{\hat{n}-1+k} \end{bmatrix}. \tag{16}$$

Proof: Employing the Cayley-Hamilton theorem and some basic algebra proves this lemma. ■

Proposition 3.7: Consider the LSS (2) defined on $[0,K]$ under a single switch switching signal, and assume that each mode is active for at least n time steps. Then, its observability is constant, i.e. the observability property does not depend on the switching time $k_s \in [n, K-n+1]$.

Proof: Assume first A_0 is nonsingular, the observability condition (15) is equivalent to

$$A_0^{k^s}\mathcal{O}_0 \cap \mathcal{O}_1 = \{0\}. \tag{17}$$

where $A_0^k\mathcal{O}_0 = \mathcal{O}_0\ \forall k \in \mathbb{N}$ with $k_s \geq n$ (by Cayley-Hamilton), i.e. the observability does not depend on k^s.

Assume now A_0 is singular. Then we can rewrite A_0 in the Jordan canonical form $A_0 = S_0 \begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{A}_0 \end{bmatrix} S_0^{-1}$ where $S_0 \in \mathbb{R}^{n\times n}$ is invertible, $N_0 \in \mathbb{R}^{n_0\times n_0}$ is a nilpotent with nilpotency index at most n, and $\widetilde{A}_0 \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$ is invertible. By state transformation $\widetilde{x}(k) = S_0^{-1}x(k)$ the observability condition becomes

$$\ker \begin{bmatrix} C_0S_0 \\ C_0S_0\begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{A}_0 \end{bmatrix} \\ \vdots \\ C_0S_0\begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{A}_0 \end{bmatrix}^{n-1} \end{bmatrix} \cap \ker \left(O_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_0^{k^s} \end{bmatrix} \right) = \{0\} \tag{18}$$

Utilizing Lemma 3.6 proves that it is not possible to have different observability properties with different switching times, which is omitted due to space limitation. ■

This result is indeed stronger than the result presented in [16, Corr. 3.4] where we have now confirmed that the observability of LSSs, with any dimensional states and under a single switch switching signal, is always constant as long as each mode is long enough active.

IV. Determinability

We present in this section the determinability characterization for systems (2). We first characterize the determinability for single switch switching signals. The result is given by the following lemma. The understanding for this simple case is used to characterize the cases with multiple switches.

For the single switch switching signal given by the mode sequence $(0,1)$, we define the following sequence of subspaces on $[0,K]$ with the initial subspace $\mathcal{Q}^0 = \ker C_0$

$$\mathcal{Q}^k = \ker C_{\sigma(k)} \cap A_{\sigma(k-1)}\mathcal{Q}^{k-1}, \quad k = 1,2,\ldots,k^s,\ldots K. \tag{19}$$

The interpretation of this sequence is that $x_k \in \mathcal{Q}^k$ if, and only if, there exists a solution with $y(i) = 0$ for $i \in [0,k]$ and $x(k) = x_k$.

Lemma 4.1: The linear switched system (2) is determinable on $[0,K], K \geq k^s$ w.r.t. the mode sequence $(0,1)$ if, and only if,

$$\mathcal{Q}^K = \{0\}. \tag{20}$$

Proof: Necessity: By construction of $\mathcal{Q}^k$ it follows that for all $x_k \in \mathcal{Q}^k$ we have $C_{\sigma(k)}x_k = 0$ and that there exists $x_{k-1} \in \mathcal{Q}^{k-1}$ with $x_k = A_{\sigma(k-1}x_{k-1}$. Hence by assuming that $\mathcal{Q}^K \neq \{0\}$ we can pick $x_K \in \mathcal{Q}^K \setminus \{0\}$ and a sequence $x_K, x_{K-1},\ldots,x_2,x_1,x_0$ with $x_k \in \mathcal{Q}^k$ such that $x(\cdot)$ given by $x(k) := x_k$ is a solution of (2) on $[0,K]$ with $y(k) = C_{\sigma(k)}x_k = 0$. This shows that (2) is not determinable.
Sufficiency: Consider a solution $x(\cdot)$ of (2) and assume that $y(k) = 0$ for all $k \in [0,K]$. We will show that then $x(k) \in \mathcal{Q}^k$ for all $k \in [0,K]$ and hence determinability follows from $\mathcal{Q}^K = \{0\}$. It is clear that $y(k) = 0$ implies $x(k) \in \ker C_{\sigma(k)} \forall k$. Next, from $y(0) = 0$ it follows that $x(0) \in \ker C_0 = \mathcal{Q}^0$. Inductively, assume that $x(k) \in \mathcal{Q}^k$, then for $k < K$ we have that $x(k+1) = A_{\sigma(k)}x(k) \in A_{\sigma(k)}\mathcal{Q}^k$ and hence $x(k+1) \in \ker C_{\sigma(k)} \cap A_{\sigma(k)}\mathcal{Q}^k = \mathcal{Q}^{k+1}$. Thus we can conclude that $x(K) \in \mathcal{Q}^K = \{0\}$ as desired. ■

Remark 4.2: If $T < k^s$ then the condition for the determinability on $[0,T]$ is equivalent to non-switched systems since we have only the initial mode that actives on that time interval. Finally, (2) is determinable if we can find such T so that it is determinable on $[0,T]$.

The following example illustrates the determinability.

Example 4.3: Consider the linear switched system (2) with the following system's matrices

$$(A_0, C_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right)$$
$$(A_1, C_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \right).$$

With the mode sequence $\sigma = (0,1)$, the switched system is unobservable on $[0,12]$ for $2 \leq k^s \leq 9$. Surprisingly, it is always determinable. This shows that even though the switched system is not-observable, it could be determinable. Moreover, the smallest time instant K such that the switched system is determinable on $[0, K]$ depends on the switching time; it needs to be two time instants after the switching time in order to be determinable. ⋄

We study now the determinability for a multiple switches switching signal given by (4). By straightforward generalization from the single switch case above, we now define the sequence of subspaces (19) for $j = 1, 2, \ldots, J$ and for $k \in (k_j^s, k_{j+1}^s - 1)$. We can now characterize the determinability through the following theorem.

Theorem 4.4: The linear switched system (2) is determinable on $[0, K], K \in [k_J^s, k_{J+1}^s)$ w.r.t. the fixed switching signal (4) if, and only if,

$$\mathcal{Q}^K = \{0\} \tag{21}$$

where $\mathcal{Q}^k$ is given by (19).

Proof: The proof is straightforward generalization from the single switch case given by the Lemma 4.1 and is therefore omitted. ■

As in the observability, the determinability, in general, depends on the switching time, and furthermore, also depends on the number of the modes occuring on $[0, K]$. We probably have some situations where the determinability does not depend on the switching times; this is our future study.

Example 4.5: Recall the Example 3.4. We check the determinability on $[0, 20]$ and the result with some various switching times is shown in Fig. 5. In this example, the determinability characterization result is just the same to the result in the observability characterization (see Fig. 3). Compared to the result in the Example 4.3, in contrast, the determinability property in this Example depends on the switching time. ⋄

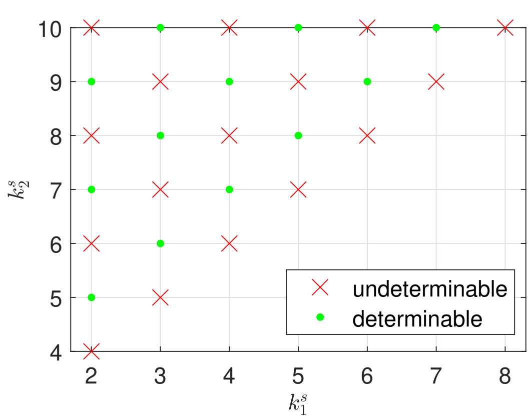


Fig. 5: Switching time vs determinability Example 4.5

V. Summary and Future Works

The observability and determinability characterizations were considered in this paper. Necessary and sufficient conditions for observability and determinability both under single switch and multiple switches switching signals were presented. Moreover, some specific situations where the observability is constant were also investigated.

In our upcoming works, we will extend the concepts presented in this paper to singular linear switched systems. We expect that some results in non-singular systems will also occur in singular systems. Moreover, some new studies will also be discussed for singular systems because of the presence of the one-step-map which is a special feature of singular system.

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