A smooth model for periodically switched descriptor systems [⋆]

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Abstract

Switched descriptor systems characterized by a repetitive finite sequence of modes can exhibit state discontinuities at the switching time instants. The amplitudes of these discontinuities depend on the consistency projectors of the modes. A switched ordinary differential equations model whose continuous state evolution approximates the state of the original system is proposed. Sufficient conditions based on linear matrix inequalities on the modes projectors ensure that the approximation error is of linear order of the switching period. The theoretical findings are applied to a switched capacitor circuit and numerical results illustrate the practical usefulness of the proposed model.

Key words: Descriptor systems, differential algebraic equations, non-smooth and discontinuous problems, linear/nonlinear models, switched systems, power electronics.

1 Introduction

Switched descriptor systems represent the dynamic behavior of several physical apparatus, e.g. mechanical systems [20] and electronic circuits [17]. The dynamics of switched descriptor systems is determined by the switching among different modes, where each mode is characterized by a set of linear differential equations and algebraic constraints. A mathematical representation of this class of systems can be obtained in terms of switched linear differential algebraic equations (DAE).

Several modeling and control aspects related to switched DAE have been considered in the literature, e.g. observer design [25], stability [20,22], initial value problems [4,5], switched control systems with impulsive dynamics [1,2].

Switched DAE can exhibit specific behaviors, such as impulses and state discontinuities at the switching instants, which are not possible in switched ordinary differential equations (ODE). These complexities inspired studies which have investigated simpler models able to approximate the behaviour of the original system and to deduce properties. Each of these approaches is restricted to a specific class of switched descriptor systems. The reduced-order model proposed in [25] for observer design in the presence of unknown inputs assumes a common E matrix. Continuous ODE have been proposed for approximating the solution of switched DAE by using averaging techniques [13–15], but the conditions to be satisfied by the modes projectors do not hold for many practical systems [17]. In [20] reduced-order models of switched descriptor systems are used for deriving exponential stability conditions about the origin, but the results therein require nonsingular A matrices and the rank of the E matrices to be the same.

In this paper, we propose a switched ODE model which approximates the switched DAE system under milder assumptions with respect to those used in the previous literature. The use of switched ODE with a continuous state evolution approximating the dynamic behavior of a switched DAE has been shown to be useful for the analysis of switched descriptor systems. One of these situations is the simulation of descriptor systems with singularities, e.g. inconsistent initial conditions [21], where numerical issues could be amplified by the presence of switching modes. In this scenario switched ODE models could help to obtain numerical results by using standard software suited for systems with a continuous state evo-

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lution. The approximation of a switched descriptor system with a switched ODE has also been used for stability analysis [12] and observer design [18]. In particular, the analysis in [12] is based on a constant switching period, but in many practical applications, such as for power converters, the switching among the different modes are not repetitive in the sense that different duty cycles and different switching periods are required for the system operations [16].

The model proposed in this paper, by covering the situation when the switching period is not fixed a priori, can be considered as an extension of the model presented in [12]. The dynamic matrix of each mode of our switched ODE model depends on the switching period and we provide a design rule for this dependence. This technique allows us to prove an approximation result between the solution of the proposed model and that of the original switched DAE, showing that the difference of the two solutions is of the same order as the switching period. Moreover, we also provide operative sufficient conditions expressed in terms of linear matrix inequalities (LMIs) which allow one to verify the hypotheses of our main result. The approximation result is shown to be useful for asymptotic stability analysis of both the switched DAE and the proposed switched ODE systems.

The paper is organized in several sections. In Section 2 the class of switched descriptor systems of interest is presented. Section 3 presents the new switched ODE model. The main result of the paper is shown in Section 4. In Section 5 a numerical verification of the theoretical results obtained by considering a practical switched capacitor circuit is proposed. In Section 6 the results are summarized. All proofs of the lemmas and theorem proposed in the paper are collected in the Appendix.

2 Switched descriptor system

The switched descriptor system of interest can be represented as an homogeneous switched DAE with $\mathtt{q}$ modes, i.e.

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x \tag{1}$$

where $\sigma : \mathbb{R}_+ \to \Sigma$, with $\mathbb{R}_+$ the set of positive real numbers, is a piecewise constant right-continuous function, that selects at each time instant the index of the active mode from the finite index set $\Sigma := \{1, 2, \dots, \mathtt{q}\}$. We assume that each mode is given by a regular matrix pair (E_i, A_i), i.e. the polynomial $\det(sE_i - A_i)$ is not identically zero, and that the switching signal σ repeats the sequence of modes in any switching period $p > 0$, i.e.,

$$\sigma(t) = \begin{cases} 1, & t \in [t_k, s_{k,2}), \\ 2, & t \in [s_{k,2}, s_{k,3}), \\ \vdots & \\ \mathtt{q}, & t \in [s_{k,\mathtt{q}}, t_{k+1}) \end{cases} \tag{2}$$

with $k \in \mathbb{N}$, with $\mathbb{N}$ the set of positive integers, $i \in \Sigma$, the time instants t_k being the multiple of the period p, the switching time instants $s_{k,i}$ being the time instant when the i-th mode starts within the k-th period. In particular, we assume $s_{k,1} = t_k$ for all $k \in \mathbb{N}$. Then we have

$$t_k := kp, \qquad s_{k,i} := t_k + \sum_{j=1}^{i-1} d_{j,k}p, \tag{3}$$

where $d_{i,k} \in D$, $D = (0, 1)$, is the duty cycle of the i-th mode for the k-th period; in particular, $\sum_{i=1}^{\mathtt{q}} d_{i,k} = 1$, see Fig. 1.

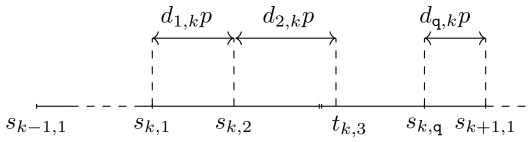

Fig. 1. Illustration of the switching times notation.

The solution of a switched DAE can also contain Dirac impulses, i.e., each mode can have impulsive modes of arbitrary degree. The impulse-free part of the solution is independent from the impulsive part (however the opposite does not hold, see [23]), which justifies the analysis of this paper that concentrates on the impulse-free part of the solution (which may still contain jumps). The impulse free part of the solution of the descriptor system can be obtained as the solution of a suitable switched ODE model. This model is defined by using the consistency projector and the flow matrix of each mode which can be obtained through a specific transformation. In particular, for any regular matrix pair (E_i, A_i) there exist transformation matrices S_i and T_i which put (E_i, A_i) into the quasi Weierstrass form, i.e.

$$(S_iE_iT_i, S_iA_iT_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N_i \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix} \right), \tag{4}$$

with $T_i = [V_i,\ W_i]$, $\quad S_i = [E_iV_i,\ A_iW_i]^{-1}$ where N_i is a nilpotent matrix, I is the identity matrix, J_i, V_i and W_i are matrices of appropriate size. Then, for any regular matrix pair (E_i, A_i) it is possible to define the consistence projector Π_i and the flow matrix F_i as follow:

$$\Pi_i = T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1}, \quad F_i = T_i \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1}. \tag{5}$$

It is easy to verify that the consistency projector is an idempotent matrix, i.e., $\Pi_i^2 = \Pi_i$, and that the projector and the flow matrix are commutative with their product equal to the flow matrix itself, i.e., $F_i\Pi_i = F_i = \Pi_iF_i$. The consistency projectors allow one to easily verify the impulse freeness of all distributional solution of (1).The solution of switched system (1) is impulse free under arbitrary switching if and only if for all $i \neq j \in \Sigma$ it holds $E_i(I - \Pi_i)\Pi_j = 0$.

In [15, Theorem 12] it is shown that $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the impulse-free part of any (distributional) solution of (1) if and only if it is a solution of the following switched ODE model with jumps

$$\dot{x}(t) = F_i x(t), \quad t \in (s_{k,i}, s_{k,i+1}) \tag{6a}$$
$$x(s_{k,i}^+) = \Pi_i x(s_{k,i}^-) \tag{6b}$$

with $x(0^-) = x_0$, for $k \in \mathbb{N}$, $i \in \Sigma$, where the matrices F_i are given by (5) and $s_{k,q+1} := t_{k+1} = s_{k+1,1}$.

The solution of (6) can be written by cascading the solutions of the different modes and by considering the jumps at the switching time instants. In particular, at the switching time instants one can write

$$x(s_{k,i}^-) = \prod_{m=1}^{k-1} \left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p} \Pi_j \right) \prod_{j=1}^{i} e^{F_j d_{j,m} p} \Pi_j x_0 \tag{7}$$

and at the time instants internal to the mode evolution it is

$$x(\tau) = e^{F_i(\tau - s_{k,i})} \Pi_i x(s_{k,i}^-) \tag{8}$$

for any $\tau \in (s_{k,i}, s_{k,i+1})$, $k \in \mathbb{N}$ and $i \in \Sigma$.

Note that throughout the paper the product of any q matrices $G_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, \mathtt{q}$ is defined as (note the order) $\prod_{i=1}^{\mathtt{q}} G_i = G_{\mathtt{q}} G_{\mathtt{q}-1} \cdot \dots \cdot G_2 G_1$.

3 Proposed approximating model

The main objective of this paper consists of finding a switched ODE model whose continuous solution approximates the discontinuous solution (7)-(8) with an error of order of the switching period p, except for small time intervals after the switching time instants.

3.1 Switched ODE model

The proposed switched model has the following modes dynamics

$$\dot{x}_s(t) = F_i^{\varepsilon_p} x_s(t), \quad t \in [s_{k,i}, s_{k,i+1}) \tag{9}$$

with $x_s(0) = x_0$, for $k \in \mathbb{N}$, $i \in \Sigma$. Each matrix $F_i^{\varepsilon_p}$, $i \in \Sigma$, is defined as the sum of the flow matrix of that mode and a suitable matrix $\Phi_i(p)$ which allows a smooth approximation of the possible state jump:

$$F_i^{\varepsilon_p} = F_i + \Phi_i(p) \tag{10}$$

$i \in \Sigma$, where the matrix F_i is defined in (5) and

$$\Phi_i(p) = T_i \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\varepsilon_p} I \end{bmatrix} T_i^{-1}. \tag{11}$$

In [12] a model similar to (9)–(11) was proposed but a constant parameter ε was considered therein. The interest in including a dependence on p comes from the fact that in many practical systems the switching period is not fixed a priori and a constant value for ε could lead to a weaker approximation result when the switching period varies. Specifically, we consider

$$\varepsilon_p = -\frac{\Delta_p p}{\log p^2} \tag{12}$$

where without loss of generality we assumed $p < 1$, and

$$\Delta_p \leq \Delta p, \quad 0 < \Delta \ll \min\{d_{i,k}\}_{i\in\Sigma,k\in\mathbb{N}} \tag{13}$$

for all $p < \bar{p}$. Since $p < 1$, the logarithm will be negative, so ε_p is actually positive.

The solution of (9) for any $\tau \in [s_{k,i}, s_{k,i+1}]$, $k \in \mathbb{N}$ and $i \in \Sigma$, can be written as

$$x_s(\tau) = e^{F_i^{\varepsilon_p}(\tau - s_{k,i})} x_s(s_{k,i}) \tag{14}$$

where

$$x_s(s_{k,i}) = \prod_{m=1}^{k-1} \left(\prod_{j=1}^{\mathtt{q}} e^{F_i^{\varepsilon_p} d_{j,m} p} \right) \prod_{j=1}^{i} e^{F_i^{\varepsilon_p} d_{j,m} p} x_0 \tag{15}$$

is the solution at the switching time instants.

3.2 Model motivation

In order to motivate the choice (12) it is useful to recall the following definition of an $\mathrm{O}(p)$ function

Definition 1 *For any finite integer $m \in \mathbb{N}$, a matrix function $G : \mathbb{R}_+ \to \mathbb{R}^{\mu\times\nu}$, $\mu \in \mathbb{N}$, $\nu \in \mathbb{N}$, is said to be an $\mathrm{O}(p^m)$ function as $p \to 0$ ($G(p) = \mathrm{O}(p^m)$ for short), if there exist positive constants α and $\bar{p}$ such that*

$$\|G(p)\| \leq \alpha p^m, \quad \forall p \in (0, \bar{p})$$

where $\|\cdot\|$ indicates the (induced) Euclidean norm.

In the following we show that the choice (12) implies that

$$x_s(s_{k,i} + \Delta_p p) = \Pi_i \bar{x} + \mathrm{O}(p^2) \tag{16}$$

where $\bar{x} = x_s(s_{k,i})$. In other words, by choosing ε_p as in (12), the solution of (9) after a time interval $\Delta_p p$ from the beginning of the i-th mode approximates with an error $\mathrm{O}(p^2)$ the jump that the solution of the switched DAE (1) would exhibit at the beginning of the i-th mode by starting from $\bar{x}$. Let us verify (16). With simple algebraic manipulations one can write

$$-\varepsilon_p \Phi_i(p) = \frac{\Delta_p p}{\log p^2}\Phi_i(p) = T_i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T_i^{-1}$$
$$= T_i \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} T_i^{-1} - T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1} = I - \Pi_i. \quad (17)$$

Therefore, by using (17) and (12) one can write:

$$\begin{aligned} e^{\Phi_i(p)\Delta_p p} &= e^{-\frac{\Delta_p p}{\varepsilon_p}(I-\Pi_i)} = e^{(I-\Pi_i)\log p^2} \\ &= \sum_{n=0}^{\infty} \frac{(I-\Pi_i)^n \log^n p^2}{n!} \\ &= I + \sum_{n=1}^{\infty} \frac{(I-\Pi_i)^n \log^n p^2}{n!} \\ &= I + (I-\Pi_i)\sum_{n=1}^{\infty} \frac{\log^n p^2}{n!} \\ &= I + (I-\Pi_i)\sum_{n=0}^{\infty} \frac{\log^n p^2}{n!} - (I-\Pi_i) \\ &= (I-\Pi_i)e^{\log p^2} + \Pi_i = \Pi_i + (I-\Pi_i)p^2. \end{aligned} \quad (18)$$

By using $x_s(s_{k,i}) = \bar{x}$, (14) with $\tau = s_{k,i} + \Delta_p p$ and (18) it follows that

$$\begin{aligned} x_s(s_{k,i} + \Delta_p p) &= e^{(\Phi_i(p)+F_i)\Delta_p p}\bar{x} \\ &\stackrel{a}{=} e^{\Phi_i(p)\Delta_p p} e^{F_i \Delta_p p}\bar{x} \\ &= (\Pi_i + (I-\Pi_i)p^2)(I + F_i\Delta_p p + \mathrm{O}(p^3))\bar{x} \\ &= \Pi_i \bar{x} + \mathrm{O}(p^2) \end{aligned} \quad (19)$$

where in $\stackrel{a}{=}$ has been used the commutativity property between F_i and $\Phi_i(p)$ which follows from

$$F_i(I-\Pi_i) = (I-\Pi_i)F_i = F_i - F_i\Pi_i = F_i - F_i = 0.$$

The confirmation of (16) through (19) is a preliminary step for proving the approximation of the solution of the switched descriptor system (1) by the solution of the proposed switched ODE model (9). This result, which is proved in next section, is not a straightforward implication of (16) because the error between the two solutions accumulates period by period.

4 Main result

In this section we provide sufficient conditions such that the solution (14)–(15) of the switched ODE (9) is an $\mathrm{O}(p)$ approximation of the solution (7)–(8) of the switched DAE (1). In particular, we show that $x(t) - x_s(t) = \mathrm{O}(p)$ holds uniformly for any $t \in [0,T] \setminus \{(s_{k,i}, s_{k,i} + \Delta_p p)\}_{k\in\mathbb{N}, i\in\Sigma}$. Note that in principle it is not possible to approximate a discontinuous function (solution of switched descriptor system) with a continuous function (solution of switched ODE) uniformly for all $t \in [0,T]$ unless the jump magnitude converges to zero, which we do not assume here. However, with the proposed approximation method we are able to show uniform convergence of order p outside a set (a union of small intervals following the switchings) whose measure is also of order p.

The proof of our main result combines some $\mathrm{O}(p)$ approximations of parts of the solutions (7)–(8) and (14)–(15). To do this, it is useful to provide some preliminary expressions for the exponential of the systems flow matrices which are proved through the following lemma.

Lemma 2 *Given a set of matrices defined as in* (5) *and* (10)–(13)*, the following relations hold*

$$e^{F_i d_i p} = I + F_i d_i p + O(p^2) \quad (20a)$$
$$e^{F_i^{\varepsilon_p} d_i p} = \Pi_i + F_i d_i p + O(p^2) \quad (20b)$$
$$\prod_{i=1}^{\mathsf{q}} e^{F_i^{\varepsilon_p} d_i p} = \prod_{i=1}^{\mathsf{q}} e^{F_i d_i p}\Pi_i + O(p^2) \quad (20c)$$

for all p, $d_i \in D = (0,1)$, $i \in \Sigma$.

The solutions (7)–(8) and (14)–(15) present repetitions, period by period, of products of exponential matrices. In order to analyze these terms, let us define the Lipschitz continuous matrix function $M : D^{\mathsf{q}} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, with $D = (0,1)$, as

$$M(\delta_k, p) = \prod_{i=1}^{\mathsf{q}} e^{F_i d_{i,k} p}\Pi_i \quad (21)$$

where $k \in \mathbb{N}$ and vector signal $\delta_k = [d_{1,k}, \ldots, d_{\mathsf{q},k}]^\top \in D^{\mathsf{q}}$ indicates the duty cycles of all modes over time. The solution (7) of the switched descriptor system at the k-th switching period involves the product of k matrices in the form (21). When p goes to zero over the time interval $(0,T)$, one should consider $k \in \{1, \ldots, \ell(p)\}$, where $\ell : \mathbb{R}_+ \to \mathbb{N}$ is the number of intervals of length p from 0 to T:

$$\ell(p) = \left\lfloor \frac{T}{p} \right\rfloor \quad (22)$$

with $\lfloor x \rfloor$ the largest integer less than or equal to $x \in \mathbb{R}$. Clearly, when p goes to zero $\ell(p)$ goes to infinity. In particular, it is $1/\ell(p) = \mathrm{O}(p)$. Indeed it is $1/\ell(p) \leq$

$p/(T-p) \leq \alpha p$ with $\alpha \geq 1/(T-\bar{p})$ where $\bar{p}$ is chosen according to Definition 1. In our case, the proof of the approximation result between $x(t)$ and $x_s(t)$, $t \in [0,T]$, requires some $\mathrm{O}(p)$ conditions on products of $\ell(p)$ terms in the form (21). This result is proved in the following lemma.

Lemma 3 *Consider a finite $T \in \mathbb{R}$, $\ell(p)$ as in* (22)*, a discrete time signal $(\delta_k)_{k\in\mathbb{N}}$ with values in $D^{\mathtt{q}} = (0,1)^{\mathtt{q}}$, and generic Lipschitz continuous matrix functions $M : D^{\mathtt{q}} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $G : D^{\mathtt{q}} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$. Assume that there exists a $\gamma_1 \geq 0$ such that*

$$\|M(\delta_k, p)\| \leq 1 + \gamma_1 p \tag{23a}$$

$$G(\delta_k, p) = \mathrm{O}(p^2), \tag{23b}$$

for all $k \in \{1, \ldots, \ell(p)\}$. Then

$$\prod_{k=1}^{\ell(p)} M(\delta_k, p) = \mathrm{O}(1) \tag{24a}$$

$$\prod_{k=1}^{\ell(p)} (M(\delta_k, p) + G(\delta_k, p)) = \prod_{k=1}^{\ell(p)} M(\delta_k, p) + \mathrm{O}(p). \tag{24b}$$

It is interesting to compare the results in Lemma 3 with those of Lemma 2 in [7]. Therein, by considering constant duty cycles, i.e. $\delta_k = \delta$ for all $k \in \mathbb{N}$, it is shown that if $M(p)^{\ell(p)} = \mathrm{O}(1)$ and $G(p) = \mathrm{O}(p)$ it is $(M(p) + G(p))^{\ell(p)} = \mathrm{O}(1)$. The assumption (23a) in the particular case that δ_k is constant, is more restrictive than $M(p)^{\ell(p)} = \mathrm{O}(1)$, so as it can be deduced from (A.1d). On the other hand, from Lemma 3 it follows that if (23) hold one can write the more explicit expression $(M(p) + G(p))^{\ell(p)} = M(p)^{\ell(p)} + \mathrm{O}(p)$.

The condition (23a) cannot be easily checked a priori from the structure of the model (9). The following lemma provides more operative conditions based on linear matrix inequalities which must be satisfied by the system projectors in order to let (23a) be satisfied.

Lemma 4 *Consider a set of matrices defined as in* (5) *and* (10)*–*(13)*. Assume that there exists a symmetric matrix P such that the following set of linear matrix inequalities*

$$P \succ 0 \tag{25a}$$

$$\Pi_i^\top P \Pi_i - P \preceq 0 \tag{25b}$$

with $i = 1, \ldots, \mathtt{q}$, has a solution, then there exists a $\gamma_1 \geq 0$ such that the following condition holds

$$\left|\!\left|\!\left| \prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p} \Pi_i \right|\!\right|\!\right| \leq 1 + \gamma_1 p \tag{26}$$

for any $k \in \{1, \ldots, \ell(p)\}$ and for all p with $|\!|\!|\cdot|\!|\!|$ being the norm induced by the matrix P.

Conditions (25) can be relaxed under the hypothesis that the sequence of the projectors is fixed. In particular one could replace (25b) with the weaker condition $\Pi_\cap^\top P \Pi_\cap - P \preceq 0$, where $\Pi_\cap = \prod_{i=1}^{\mathtt{q}} \Pi_i$. Note that if $\operatorname{im} \Pi_\cap \subseteq \operatorname{im} \Pi_i$ and $\ker \Pi_\cap \supseteq \ker \Pi_i$ then $\Pi_\cap$ is a projector himself, see [15].

By using the lemmas above, it is now possible to prove our main result.

Theorem 5 *Consider the switched DAE system* (1) *and the smooth model* (9) *with the same initial conditions $x(0^-) = x_s(0^-) = x_0$. Assume that there exists a symmetric matrix P such that the set of LMIs* (25) *is satisfied. Then*

$$x(t) - x_s(t) = O(p) \tag{27}$$

holds for any $t \in [0,T] \setminus \{(s_{k,i}, s_{k,i} + \Delta_p p)\}_{k\in\mathbb{N}, i\in\Sigma}$.

Theorem 5 can be useful for providing some stability properties of (1) and (9). In order to make some considerations on this, first note that the approximation result (27) is based on the existence of a norm $|\!|\!|\cdot|\!|\!|$ such that (26) holds. It is easy to see that this condition is not sufficient for having the asymptotic stability of the switched descriptor system (1). On the other hand, if some tighter conditions on the modes dynamics are assumed, one can obtain a sufficient condition which ensures the asymptotic stability of (1) and (9). In particular, from (7) it is easy to verify that if there exists a $\gamma_2 > 0$ such that

$$\left|\!\left|\!\left| e^{F_i d_{i,k} p} \Pi_i \right|\!\right|\!\right| \leq 1 - \gamma_2 p \tag{28}$$

for all $i \in \Sigma$, for any $k \in \{1, \ldots, \ell(p)\}$ and for all p, than (1) is asymptotically stable. On the other hand, if condition (28) hold, the proposed switched ODE model (9) is asymptotically stable either. Indeed, one can write

$$\begin{aligned} e^{F_i^{\varepsilon_p} d_{i,k} p} &= e^{F_i d_{i,k} p} e^{\Phi_i(p) d_{i,k} p} \\ &= e^{F_i d_{i,k} p} \left(\Pi_i + (I - \Pi_i) p^{\frac{2 d_{i,k}}{\Delta_p}} \right) \\ &= e^{F_i d_{i,k} p} \Pi_i + (I - \Pi_i) p^{\frac{2 d_{i,k}}{\Delta_p}}. \end{aligned} \tag{29}$$

Now, since $d_{i,k} < 1$ for all $i \in \Sigma$ and $k \in \mathbb{N}$, from (28) and (29) one obtains that there always exists a $\bar{p}$ such that

$$\begin{aligned} \left|\!\left|\!\left| e^{F_i^{\varepsilon_p} d_{i,k} p} \right|\!\right|\!\right| &\leq 1 - \gamma_2 p + \left|\!\left|\!\left| I - \Pi_i \right|\!\right|\!\right| p^{\frac{2}{\Delta_p}} \\ &\leq 1 - \gamma_2 p + \gamma_3 p \leq 1 - \gamma_4 p \end{aligned} \tag{30}$$

for any $p \in (0, \bar{p})$, with $\gamma_4 = \gamma_2 - \gamma_3 > 0$, for all $k \in \mathbb{N}$. By taking the norm in (14)–(15) and by using (30) the asymptotic stability of (9) directly follows.

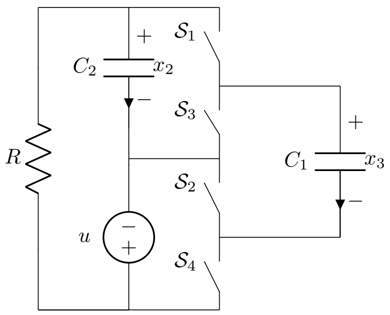

Fig. 2. Elementary cell of a ladder step-up switched capacitor converter.

In synthesis, if (28) hold then the switched descriptor system (1) and the proposed switched ODE system (9) with the design rule of ε_p expressed by (12), are both asymptotically stable. In this scenario, one can apply to the proposed model (9) some analysis and control design techniques which are standard for models which do not exhibit discontinuities and, thanks to the approximation result in Theorem 5, conclude corresponding results for the switched descriptor system (1).

5 Example

In this section we verify the approximation (27) in Theorem 5 by using numerical results obtained by considering a practical switched capacitor electrical circuit. Let us consider the typical elementary cell of a ladder step-up switched capacitor shown in Fig. 2. The circuit consists of two capacitors and four switches that are controlled in a complementary way. The analysis of linear electrical circuit by means of state space models has been extensively considered in the literature, e.g. [9,11,19,24]. The circuit in Fig. 2 can be represented in the form (1) with two modes corresponding to the pair $\{\mathcal{S}_1, \mathcal{S}_2\}$ turned on together with the pair $\{\mathcal{S}_3, \mathcal{S}_4\}$ turned off ($\sigma = 1$), and viceversa ($\sigma = 2$).

By considering as input a constant voltage source u the circuit can be modeled by adding a dummy state variable, say x_1, together with x_2 and x_3 being the state variables corresponding to the voltages on the capacitors C_1 and C_2, respectively. Then the matrices pairs and the consistence projectors of the two modes are:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_2 R & C_1 R \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_2 R & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$
$$Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 0 & C_2\rho & C_1\rho \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where $\rho = \frac{1}{C_1+C_2}$. It can be easily verified that the linear matrix inequalities (25) are satisfied, even though the projectors have euclidean norms larger than 1.

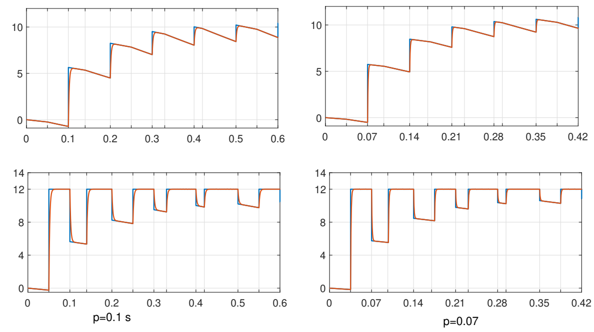

Fig. 3. Time evolution of the state variables (second top, third bottom) of the switched capacitor circuit with $p = 0.1\,\mathrm{s}$ and $p = 0.07\,\mathrm{s}$: switched DAE system (blue lines) and proposed model (red lines).

The matrices F_i and Φ_i, $i = 1, 2$, are:

$$F_1 = \frac{\rho^2}{R} \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{\rho} & -C_2 & -C_1 \\ -\frac{1}{\rho} & -C_2 & -C_1 \end{bmatrix} \qquad \Phi_1 = \frac{\rho \log p^2}{p\Delta_p} \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & -C_1 \\ 0 & -C_2 & C_2 \end{bmatrix}$$
$$F_2 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{C_2 R} & -\frac{1}{C_2 R} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Phi_2 = \frac{\log p^2}{p\Delta_p} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The simulation has been carried out by selecting the following parameters: $C_1 = C_2 = 120\,\mu\mathrm{F}$, $R = 10\,\mathrm{k}\Omega$ and $\Delta_p = 0.9p$. In Fig. 3 is shown the behavior of the state variables for different switching periods, i.e. $p = 0.1\,\mathrm{s}$ and $p = 0.07\,\mathrm{s}$ respectively, over six periods. Note that the duty cycles are different for each period. The state variable x_3 presents jumps when the system switches from mode 1 to mode 2 and viceversa. The state evolution of the error related to the second state variable together with the state evolution of the error obtained by reproposing the same scenario with the model presented in [12], where $\varepsilon = 0.004$ are shown in Fig. 4. At the switching time instants state jumps occur and the error becomes quite small after few switching periods. Clearly the amplitudes of the peak values of the errors at the switching time instants are the same for the two (continuous) models. Nevertheless the error decreases much faster with our model. Indeed, the root mean square of the state errors with respect to the solutions of (1) with $p = 0.1\,\mathrm{s}$ ($p = 0.07\,\mathrm{s}$) are 0.1178 (0.0752) for our model and 0.1983 (0.2324) for the model proposed in [12] with $\varepsilon = 0.004$. The integral of the error for the model in [12] is 23% (165%) larger than the error obtained with our model.

6 Conclusion

Many practical switched systems are characterized by a repetitive sequence of a finite number of modes and can be represented as descriptor systems. For switched descriptor systems which present jumps in the state at the switching time instants it is of practical interest to find

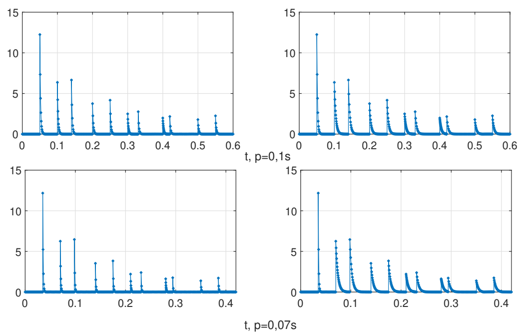

Fig. 4. Error time evolution over six periods for the state variable x_3 of the switched capacitor circuit for $p = 0.1\,\mathrm{s}$ (top) and $p = 0.07\,\mathrm{s}$ (bottom) by using the proposed method (left) and the model presented in [12] (right).

possible smooth models which approximate the behavior of the discontinuous system. In this paper a switched ODE model whose state continuous solution approximates the evolution of the switched descriptor system solution has been proposed. Linear matrix inequalities depending on the system projectors provide sufficient conditions for proving that the approximation error between the two models is of order of the switching period. The practical operating conditions of not constant duty cycles and varying switching periods have been considered. Numerical results obtained by considering a switched capacitor circuit have validated the theoretical results. In this paper we have considered homogeneous switched DAE and non-homogeneous systems that can be represented in this form too if inputs are such that the state can be enlarged by including a model of the input generator in the switched DAE model. Future step will extend this result for non-homogeneous switched DAE with more general inputs.

A Appendix

A.1 Properties on $\mathrm{O}(p)$ *functions*

We can state some properties of matrix functions which satisfy Definition 1. In particular, for any finite integer $m \in \mathbb{N}$, the following implications hold:

$$G(p) = \mathrm{O}(p^m) \implies G(p)\ell(p) = \mathrm{O}(p^{m-1}) \tag{A.1a}$$
$$G(p) = \mathrm{O}(p) \implies G(p)^{\ell(p)} = \mathrm{O}(p^m) \tag{A.1b}$$
$$G(p) = \mathrm{O}(p^2) \implies (G(p)\ell(p))^{\ell(p)} = \mathrm{O}(p^m) \tag{A.1c}$$
$$G(p)^{\ell(p)} = \mathrm{O}(1) \implies G(p) = \mathrm{O}(1). \tag{A.1d}$$

These properties can be verified as partially done in [6]. The implication (A.1a) follows from

$$\|G(p)\ell(p)\| \le \alpha p^m \left\lfloor \frac{T}{p} \right\rfloor \le \alpha p \frac{T}{p} = \alpha T p^{m-1}.$$

The implication (A.1b) follows from

$$\begin{aligned}\|G(p)^{\ell(p)}\| &\le \|G(p)\|^{\ell(p)} \le (\alpha p)^{\ell(p)}\\ &= \alpha^m p^m (\alpha p)^{\ell(p)-m} \le \alpha^m p^m\end{aligned}$$

and one can choose $\hat{p} \le \bar{p}$ such that $\alpha \hat{p} \le 1$, for some $\hat{p} \le \bar{p}$. The implication (A.1c) follows from

$$\begin{aligned}\|(G(p)\ell(p))^{\ell(p)}\| &\le \|G(p)\ell(p)\|^{\ell(p)} \le (\alpha p T)^{\ell(p)}\\ &= \alpha^m T^m p^m (\alpha p T)^{\ell(p)-m} \le \alpha^m T^m p^m\end{aligned}$$

and one can choose $\hat{p} \le \bar{p}$ such that $\alpha \hat{p} T \le 1$, for some $\hat{p} \le \bar{p}$.

The implication (A.1d) is a direct consequence of Definition 1. Note that the opposite of (A.1d) do not hold, in general.

Clearly, any linear combination of functions which are $\mathrm{O}(p^m)$ is an $\mathrm{O}(p^m)$ function itself.

A.2 Proof of Lemma 2

The condition (20a) is straightforward by using a Taylor expansion of the exponential matrix.

The condition (20b) is obtained as follows

$$\begin{aligned}e^{F_i^{\varepsilon p} d_{i,k} p} &= e^{F_i d_{i,k} p + \Phi_i(p) d_{i,k} p}\\ &\overset{a}{=} e^{F_i d_{i,k} p} e^{\Phi_i(p) d_{i,k} p}\\ &= (I + F_i d_{i,k} p + O(p^2)) e^{\Phi_i(p) d_{i,k} p}\\ &\overset{b}{=} (I + F_i d_{i,k} p + O(p^2))(\Pi_i + O(p^2))\\ &= \Pi_i + F_i d_{i,k} p + O(p^2)\end{aligned}$$

where in $\overset{a}{=}$ has been used the commutativity property between F_i and $\Phi_i(p)$ and in $\overset{b}{=}$ has been used the following result:

$$\begin{aligned}e^{\Phi_i(p) d_{i,k} p} &= \sum_{n=0}^{\infty} \frac{(I - \Pi_i)^n \log^n p^2}{n!} \frac{d_{i,k}^n}{\Delta_p^n}\\ &= I + \sum_{n=1}^{\infty} \frac{(I - \Pi_i)^n \log^n p^2}{n!} \frac{d_{i,k}^n}{\Delta_p^n}\\ &= I + (I - \Pi_i) \sum_{n=1}^{\infty} \frac{\log^n p^2}{n!} \frac{d_{i,k}^n}{\Delta_p^n}\\ &= I + (I - \Pi_i) \sum_{n=0}^{\infty} \left(\frac{d_{i,k} \log p^2}{\Delta_p} \right)^n \frac{1}{n!} - (I - \Pi_i)\\ &= (I - \Pi_i) e^{\frac{d_{i,k}}{\Delta_p} \log p^2} + \Pi_i\\ &= \Pi_i + (I - \Pi_i) p^{\frac{2 d_{i,k}}{\Delta_p}} = \Pi_i + O(p^2).\end{aligned}$$

The condition (20c) is obtained by applying (20a) and (20b). Indeed by using (20b) it follows that the left hand side of (20c) can be written as

$$\begin{aligned}\prod_{i=1}^{\mathtt{q}} e^{F_i^{\varepsilon_p} d_{i,k} p} &= \prod_{i=1}^{\mathtt{q}} (\Pi_i + F_i d_{i,k} p + O(p^2)) \\ &= \prod_{i=1}^{\mathtt{q}} (\Pi_i + F_i d_{i,k} p) + O(p^2).\end{aligned}$$

By using (20a) it follows that the first term in the right hand side of (20c) can be written as

$$\begin{aligned}\prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p} \Pi_i &= \prod_{i=1}^{\mathtt{q}} (I + F_i d_{i,k} p + O(p^2))\Pi_i \\ &= \prod_{i=1}^{\mathtt{q}} (\Pi_i + F_i d_{i,k} p) + O(p^2)\end{aligned}$$

where we exploited the property $F_i \Pi_i = F_i$ which holds for all $i \in \Sigma$. By combining the last two expressions the expression (20c) directly follows.

A.3 Proof of Lemma 3

For the sake of notation let us indicate $M(\delta_k, p)$ with $M_{k,p}$ and $G(\delta_k, p)$ with $G_{k,p}$, respectively.

Since the constant γ_1 is the same for all matrices $M_{k,p}$ one has $\left\|\prod_{k=1}^{\ell(p)} M_{k,p}\right\| \le \prod_{k=1}^{\ell(p)} \|M_{k,p}\| \le (1+\gamma_1 p)^{T/p} \le e^{\gamma_1 T}$. Then condition (24a) directly follows.

In order to obtain (24b) one can write

$$\prod_{k=1}^{\ell(p)} (M_{k,p} + G_{k,p}) = \prod_{k=1}^{\ell(p)} M_{k,p} + \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} H_{i,k,p} \quad \text{(A.2)}$$

with

$$N(\ell(p), k) = \frac{\ell(p)!}{k!(\ell(p)-k)!} = \frac{\prod_{i=0}^{k-1}(\ell(p)-i)}{k!}$$

and $H_{i,k,p}$ suitable linear combinations of matrices where each $H_{i,k,p}$ contains a product with k matrices $G_{j,p}$ with $j \in \mathbb{N}$. Therefore one can write

$$\begin{aligned}\left\|\sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} H_{i,k,p}\right\| &\le \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} \|H_{i,k,p}\| \\ &\le \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} (1+\gamma_1 p)^{\ell(p)-k} (\alpha_{i,k} p^2)^k \\ &\le \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} (1+\gamma_1 p)^{\ell(p)} (\alpha_{i,k} p^2)^k \\ &\le \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} e^{\gamma_1 T} (\alpha_{i,k} p^2)^k \\ &\le e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} N(\ell(p), k) (\bar{\alpha}_k p^2)^k \\ &= e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{\prod_{i=0}^{k-1}(\ell(p)-i)}{k!} (\bar{\alpha}_k p^2)^k \\ &\le e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{\ell(p)^k}{k!} (\bar{\alpha}_k p^2)^k \\ &= e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{(\bar{\alpha}_k \ell(p) p^2)^k}{k!} \le e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{(\bar{\alpha} p T)^k}{k!} \\ &\le e^{\gamma_1 T} p \sum_{k=1}^{\ell(p)} \frac{(\bar{\alpha} T)^k}{k!} \le e^{\gamma_1 T} p \sum_{k=1}^{\infty} \frac{(\bar{\alpha} T)^k}{k!} \\ &= e^{\gamma_1 T} \left(e^{\bar{\alpha} T} - 1\right) p \qquad \text{(A.3)}\end{aligned}$$

where for all k it is $\bar{\alpha}_k \ge \max\{\alpha_{i,k}\}_{i=1}^{N(\ell(p),k)}$, $\bar{\alpha} \ge \max\{\bar{\alpha}_k\}_{k=1}^{\ell(p)}$ and $p \ge p^k$ because without loss of generality one can assume $p \le 1$.

From (A.3) it follows that

$$\sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} H_{i,k,p} = \mathrm{O}(p)$$

and then by using (A.2) the condition (24b) directly follows.

A.4 Proof of Lemma 4

Let us consider the following

$$\begin{aligned}\prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p} \Pi_i &= \prod_{i=1}^{\mathtt{q}} (I + F_i d_{i,k} p + \mathrm{O}(p^2))\Pi_i \\ &= \prod_{i=1}^{\mathtt{q}} \Pi_i + \mathrm{O}(p) \qquad \text{(A.4)}\end{aligned}$$

Let us consider the difference equation $\xi_{k+1} = F_k \xi_k$ with $\xi_k \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $F_k \in \mathcal{F}$ and $\mathcal{F} = \{\Pi_1, \dots, \Pi_{\mathtt{q}}\}$. By using the piecewise quadratic stability based on Lyapunov theory [8] it follows that the existence of a matrix P which solves the LMIs (25) imply that the system is absolutely stable for any sequence of matrices in $\mathcal{F}$, see Sec. 5 in [3]. Then, by using Theorem 3 in [10] it follows that

$$\left|\left|\left|\Pi_i\right|\right|\right| \leq 1 \tag{A.5}$$

for all $i = 1, \dots, \mathtt{q}$ with $|||\cdot|||$ being the norm induced by the matrix P. Therefore, by applying such norm to (A.4) and by using (A.5) one to write

$$\begin{aligned}\left|\left|\left|\prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p}\Pi_i\right|\right|\right| &\leq \left|\left|\left|\prod_{i=1}^{\mathtt{q}} \Pi_i\right|\right|\right| + \gamma_1 p \\ &\leq \prod_{i=1}^{\mathtt{q}} \left|\left|\left|\Pi_i\right|\right|\right| + \gamma_1 p \leq 1 + \gamma_1 p \end{aligned} \tag{A.6}$$

which completes the proof.

A.5 Proof of Theorem 5

We first show that (27) holds at any switching time instant by considering for x the value before the possible jump, i.e.

$$x(s_{k,i}^-) - x_s(s_{k,i}) = O(p) \tag{A.7}$$

for any $k \in \{1, \dots, \ell(p)\}$ and $i \in \Sigma$.

By using Lemma 4 and Lemma 3 with $M(d_k, p) = \prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,k} p}\Pi_j$ it follows

$$\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,k} p}\Pi_j = \mathrm{O}(1) \tag{A.8a}$$

$$\prod_{j=1}^{\mathtt{q}} \left(e^{F_j d_{j,k} p}\Pi_j + O(p^2)\right) = \prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,k} p}\Pi_j + \mathrm{O}(p) \tag{A.8b}$$

for any $k \in \mathbb{N}_0$. Let us compute the solution of the smooth system:

$$\begin{aligned} x_s(s_{k,i}) &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j^{\varepsilon p} d_{j,m} p}\right)\prod_{j=1}^{i} e^{F_j^{\varepsilon p} d_{j,k} p} x_0 \\ &\overset{a}{=} \left[\prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j + \mathrm{O}(p^2)\right)\right. \\ &\quad \left. \cdot \prod_{j=1}^{i}(\Pi_j + F_j d_{j,k} p + O(p^2))\right] x_0 \\ &\overset{b}{=} \left[\prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\right. \\ &\quad \left. \cdot \prod_{j=1}^{i}(\Pi_j + F_j d_{j,k} p + \mathrm{O}(p^2)) + \mathrm{O}(p)\right] x_0 \\ &\overset{c}{=} \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i}\Pi_j x_0 + \mathrm{O}(p) \end{aligned} \tag{A.9}$$

where in $\overset{(a)}{=}$ has been used (20c) and (20b) in Lemma 2, in $\overset{(b)}{=}$ has been used (A.8b), in $\overset{(c)}{=}$ has been used (A.8a).

The solution of the switched DAE can be written as

$$\begin{aligned} x(s_{k,i}^-) &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i} e^{F_j d_{j,m} p}\Pi_j x_0 \\ &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i}(\Pi_j + O(p)) x_0 \\ &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i}\Pi_j x_0 + O(p). \end{aligned} \tag{A.10}$$

By subtracting (A.9) to (A.10) one obtains (A.7).

Now, it can be proven that $x(t) - x_s(t) = O(p)$ holds for time instants different from switching time instants except for the time intervals $\tau \in (s_{k,i}, s_{k,i} + \Delta_p p)$ for any $k \in \mathbb{N}$ and $i \in \Sigma$. Let us assume that $\tau \in [s_{k,i} + \Delta_p p, s_{k,i+1})$. Then the solution of the switched DAE system is given by

$$x(\tau) = e^{F_i(\tau - s_{k,i})}\Pi_i x(s_{k,i}^-).$$

In the same time interval, the solution of the smooth system is given by

$$x_s(\tau) = e^{F_i^{\varepsilon p}(\tau - \Delta_p p - s_{k,i})} x_s(s_{k,i}^-)$$

Then one can write

$$
\begin{aligned}
&x(\tau) - x_s(\tau) \\
&= e^{F_i(\tau - s_{k,i})}\Pi_i x(s_{k,1}^-) - e^{F_i^{\varepsilon p}(\tau - \Delta_p p - s_{k,i})} x_s(s_{k,i}^-) \\
&= (\Pi_i + O(\tau - s_{k,i}))x(s_{k,1}^-) \\
&- (\Pi_i + O((\tau - \Delta_p p - s_{k,i}))x_s(s_{k,i}^-) \\
&= \Pi_i(x(s_{k,i}^-) - x_s(s_{k,i}^-)) + O(p) = O(p) \qquad \text{(A.11)}
\end{aligned}
$$

for any $\tau \in [s_{k,i} + \Delta_p p, s_{k,i+1})$, $k \in \mathbb{N}$, $i \in \Sigma$, where in the last manipulation we used (A.7). By combining (A.7) and (A.11) the proof is complete.

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