A solution theory for coupled systems of PDEs and switched DAEs

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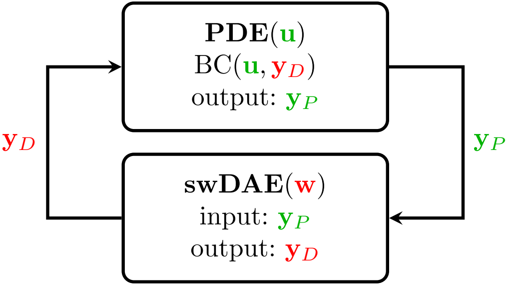
System class - overview



PDE for $x \in [a,b]$ and $t \geq t_0$:

$$\partial_t \boldsymbol{u}(t,x) + \boldsymbol{A}\partial_x \boldsymbol{u}(t,x) = 0$$

$$\boldsymbol{P}\begin{bmatrix} \boldsymbol{u}(a,t) \\ \boldsymbol{u}(b,t) \end{bmatrix} = y_D(t)$$

$$\boldsymbol{y}_P(t) = \boldsymbol{C}^P \begin{bmatrix} \boldsymbol{u}(a,t) \\ \boldsymbol{u}(b,t) \end{bmatrix}$$

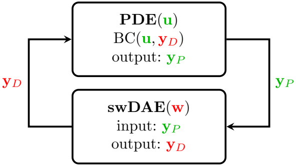
switched DAE for $t \geq 0$:

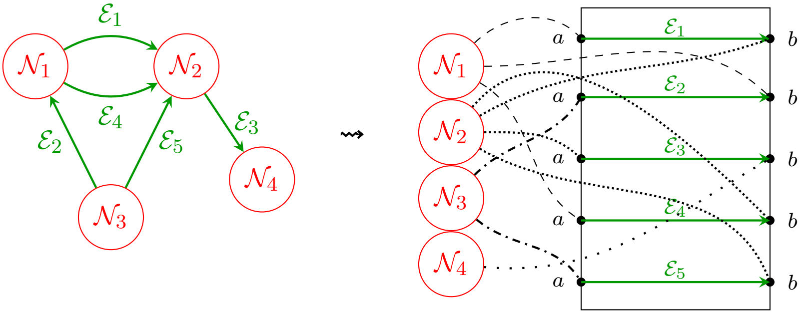
$$\boldsymbol{E}_{\sigma(t)}\dot{\boldsymbol{w}}(t) = \boldsymbol{H}_{\sigma(t)}\boldsymbol{w}(t) + \boldsymbol{B}_{\sigma(t)}\boldsymbol{y}_P(t) + \boldsymbol{f}_{\sigma(t)}(t)$$

$$\boldsymbol{y}_D(t) = \boldsymbol{C}^D_{\sigma(t)}\boldsymbol{w}(t)$$

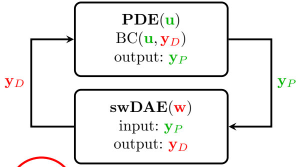
- Switching signal: $\sigma : [t_0, \infty) \to \{1, 2, \ldots, N\}$
- Coefficient matrices: $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{P} \in \mathbb{R}^{2n\times n}$, $\boldsymbol{C}_P \in \mathbb{R}^{\nu\times 2n}$, $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_N, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_N \in \mathbb{R}^{m\times m}$, $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_N \in \mathbb{R}^{m\times \nu}$, $\boldsymbol{C}^D_1, \ldots, \boldsymbol{C}^D_N \in \mathbb{R}^{n\times m}$
- Initial conditions: $\boldsymbol{u}(t_0, x) = \boldsymbol{u}^0(x)$ for all $x \in [a,b]$ and $\boldsymbol{w}(t_0) = \boldsymbol{w}^0$

System class - networks

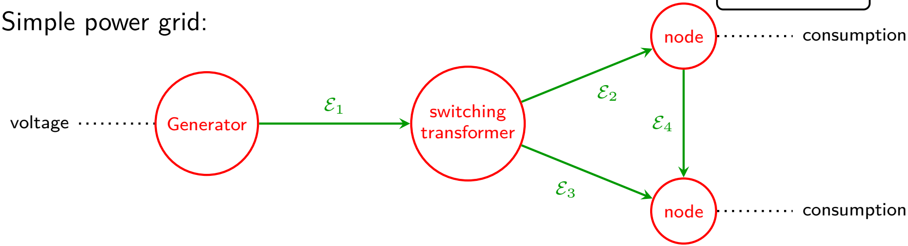




System class - example



Simple power grid:

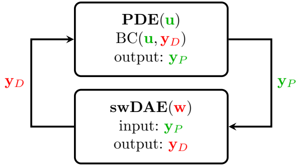


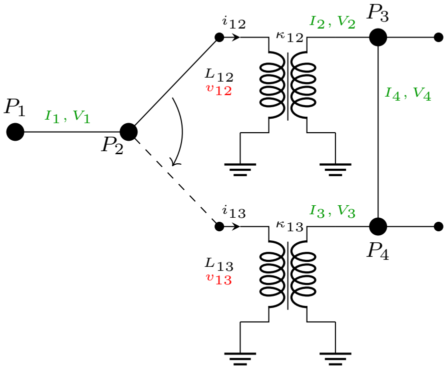
Each line modelled by telegraph equation:

$$\partial_t I(t,x) + \frac{1}{L}\partial_x V(t,x) = 0$$

$$\partial_t V(t,x) + \frac{1}{C}\partial_x I(t,x) = 0,$$

System class - example





Generator node: $0 = z_1 - v_G$, $y_D^G = z_1$

Consumption nodes:

$0 = y_D^{24} - R_{24}(I_4(\cdot, a) - I_2(\cdot, b))$,

$0 = y_D^{34} - R_{34}(I_3(\cdot, b) + I_4(\cdot, b))$,

Transformer:

$L_{12} \frac{\mathrm{d}}{\mathrm{d}t} i_{12} = v_{12}$, $L_{13} \frac{\mathrm{d}}{\mathrm{d}t} i_{13} = v_{13}$

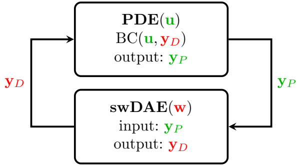
$y_D^2 = \kappa_{12} v_{12}$, $y_D^3 = \kappa_{13} v_{13}$

Switch dependent:

$0 = i_{12} - I_1(\cdot, b)$, $0 = i_{13}$ **or**

$0 = i_{13} - I_1(\cdot, b)$, $0 = i_{12}$

System class - example



Overall coupled model:

$$\partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} = 0$$
$$\boldsymbol{P}\boldsymbol{u}_{ab} = y_D$$
$$\boldsymbol{y}_P = \boldsymbol{C}^P \boldsymbol{u}_{ab}$$

$$\boldsymbol{E}_\sigma \dot{\boldsymbol{w}} = \boldsymbol{H}_\sigma \boldsymbol{w} + \boldsymbol{B}_\sigma \boldsymbol{y}_P + \boldsymbol{f}_\sigma$$
$$\boldsymbol{y}_D = \boldsymbol{C}_\sigma^D \boldsymbol{w}$$

with

$$\boldsymbol{u} = (I_1, V_1, I_2, V_2, I_3, V_3, I_4, V_4)$$
$$\boldsymbol{y}_P = (I_1(\cdot,a), I_2(\cdot,a), I_3(\cdot,a), I_4(\cdot,a), V_1(\cdot,b), V_2(\cdot,b), V_3(\cdot,b), V_4(\cdot,b))$$
$$\boldsymbol{w} = (z_1, i_{23}, i_{i3}, v_{12}, v_{13}, z_{24}, z_{34})$$
$$\boldsymbol{y}_D = (z_1, *, *, v_{13}, *, v_{13}, z_{34}, z_{34})$$

Applications and existing results

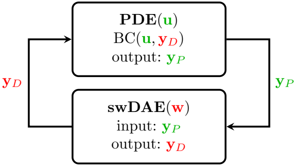
Applications

- **Large scale electrical circuits**
  - PDEs: telegraph equations of long power lines
  - swDAEs: transformers, consumers, generators, switches, Kirchhoff laws
- **Blood flow model**
  - PDEs: blood vessels to and from the heart
  - swDAE: simple heart model with valves
- **Gas networks**
  - PDEs: pipelines (Euler equations)
  - swDAEs: valves, pumps/compressors

Existing results

- Coupling hyperbolic PDEs with ODEs (Borsche et al., *Nonlinearity*, 2010; *JDE*, 2012)
- Switched PDEs (Hante et al., *AMO*, 2009; *JSCC*, 2010)

Ill-posed coupling



Example

$$\begin{aligned} \partial_t v + \partial_x v &= 0 \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} v(\cdot,a) \\ v(\cdot,b) \end{pmatrix} &= y_D \\ y_P &= v(\cdot,a) \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_P \\ y_D &= z \end{aligned}$$

PDE and DAE are both **well-posed individually**

However, when coupled, we have $y_P = v(\cdot,a) = y_D = z$, hence the DAE becomes:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix}$$

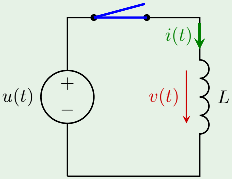
$\rightsquigarrow$ z is completely free $\rightsquigarrow$ **ill-posed coupling**

Distributional solutions

Non-standard solutions for switched DAEs (cf. TRENN 2009)

Solutions of switched DAEs may contain jumps and Dirac impulses

Example (Simple circuit producing a Dirac impulse)

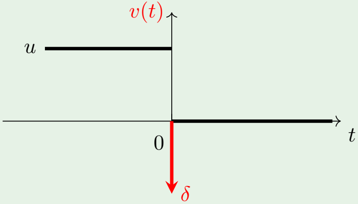


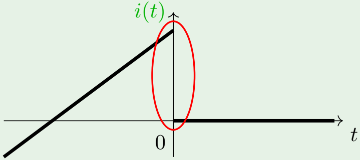
$t < 0:$

$$L\tfrac{\mathrm{d}}{\mathrm{d}t}i(t) = v(t)$$
$$0 = v(t) - u(t)$$

$t \geq 0:$

$$L\tfrac{\mathrm{d}}{\mathrm{d}t}i(t) = v(t)$$
$$0 = i(t)$$



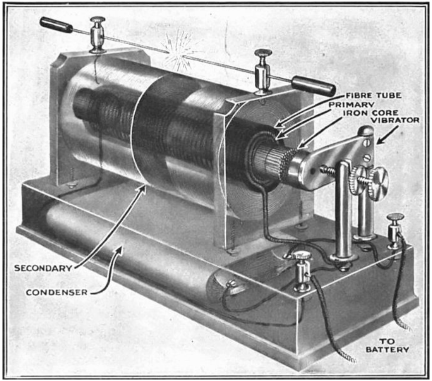


Dirac impulse is "real"

Dirac impulse

Not just a mathematical artefact!

Induction coil



Drawing: Harry Winfield Secor, public domain

Spark plug

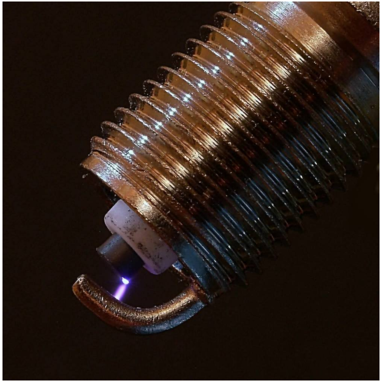


Photo: Ralf Schumacher, CC-BY-SA 3.0

Novel distributional solution framework

Distributional solutions for PDE

- How to evaluate distributional solutions at boundary or at initial time?
- Method of characteristics for distributions?

Definition (Piecewise-smooth distributions in 2D)

A distribution $D : \mathcal{C}_0^\infty(T \times X) \to \mathbb{R}$ is called piecewise-smooth $:\Longleftrightarrow$

$$D = \beta_{\mathbb{D}} + \sum_{j \in \mathcal{J}} \sum_{k,\ell} \alpha_j^{k,\ell} \partial_t^k \partial_x^\ell \delta_{L_j}$$

where

- $\beta_{\mathbb{D}}$ is a regular distribution induced by a piecewise-smooth function $\beta : T \times X \to \mathbb{R}$
- $\{L_j\}_{j \in \mathcal{J}}$ is a locally finite family of line segments in $T \times X$
- δ_L for a line segment $L \subseteq T \times X$ is called Dirac segment and is given by

$$\delta_L : \varphi \mapsto \int_L \varphi = \int_0^1 \varphi(t_0 + \alpha(t_1 - t_0), x_0 + \alpha(x_1 - x_0)) \sqrt{\Delta t^2 + \Delta x^2} \mathrm{d}\alpha$$

Some properties of piecewise-smooth distributions

- Closed under differentiation
- Trace-evaluation possible (resulting in 1D piecewise-smooth distribution):

$$D(t^\pm,\cdot) := \beta(t^\pm,\cdot)_{\mathbb{D}} + \sum_{j\in\mathcal{J}}\sum_{k,\ell}\alpha_j^{k,\ell}\partial_t^{(k)}\partial_x^{(\ell)}\left(\delta_{L_j}(t^\pm,\cdot)\right),$$

$$D(\cdot,x^\pm) := \beta(\cdot,x^\pm)_{\mathbb{D}} + \sum_{j\in\mathcal{J}}\sum_{k,\ell}\alpha_j^{k,\ell}\partial_t^{(k)}\partial_x^{(\ell)}\left(\delta_{L_j}(\cdot,x^\pm)\right).$$

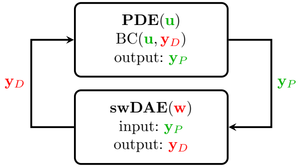
with

$$\delta_L(t^+,\cdot) := \begin{cases}\sqrt{1+\frac{\Delta x^2}{\Delta t^2}}\,\delta_{x_0+\frac{\Delta x}{\Delta t}(t-t_0)}, & t\in[t_0,t_1),\\ 0, & \text{otherwise,}\end{cases}$$

$$\delta_L(\cdot,x^+) := \begin{cases}\sqrt{1+\frac{\Delta t^2}{\Delta x^2}}\,\delta_{t_0+\frac{\Delta t}{\Delta x}(x-x_0)}, & x\in[x_0,x_1),\\ 0, & \text{otherwise,}\end{cases}$$

and left-sided evaluation analogously (with intervals $(t_0,t_1]$ and $(x_0,x_1]$)

Coupled system revisited



$$\partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} = 0$$
$$\boldsymbol{P}\begin{pmatrix} \boldsymbol{u}(\cdot,a^+) \\ \boldsymbol{u}(\cdot,b^-) \end{pmatrix} = y_D$$
$$\boldsymbol{u}(t_0^+,\cdot) = \boldsymbol{u}^0$$
$$\boldsymbol{y}_P = \boldsymbol{C}^P\begin{pmatrix} \boldsymbol{u}(\cdot,a^+) \\ \boldsymbol{u}(\cdot,b^-) \end{pmatrix}$$

$$\boldsymbol{E}_\sigma \dot{\boldsymbol{w}} = \boldsymbol{H}_\sigma \boldsymbol{w} + \boldsymbol{B}_\sigma \boldsymbol{y}_P + \boldsymbol{f}_\sigma$$
$$\boldsymbol{w}(t_0^-) = \boldsymbol{w}^0$$
$$\boldsymbol{y}_D = \boldsymbol{C}_\sigma^D \boldsymbol{w}$$

Well defined for 2D piecewise-smooth distribution $\boldsymbol{u}$ and 1D piecewise-smooth distributions $\boldsymbol{y}_P$, $\boldsymbol{u}^0$, $\boldsymbol{w}$, $\boldsymbol{y}_D$, $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_N$!

Equivalence to delay switched DAE

Assumption 1

The PDE is hyperbolic, i.e. $\boldsymbol{A} = [\boldsymbol{R}^-, \boldsymbol{R}^+] \begin{bmatrix} \Lambda^- & 0 \\ 0 & \Lambda^+ \end{bmatrix} [\boldsymbol{R}^-, \boldsymbol{R}^+]^{-1}$

Assumption 2

$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_a & 0 \\ 0 & \boldsymbol{P}_b \end{bmatrix}$ with $\ker \boldsymbol{P}_a \oplus \boldsymbol{R}^+ = \mathbb{R}^n$ and $\ker \boldsymbol{P}_b \oplus \boldsymbol{R}^- = \mathbb{R}^n$

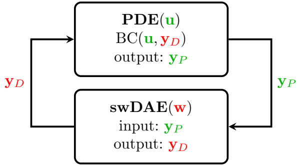
Theorem (Borsche, Kocoglu, Trenn, *MCSS*, 2020)

$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{u}_{ab} \end{bmatrix}$ *is solution of coupled system* $\iff$ $\boldsymbol{z}$ *solves*

$$\begin{bmatrix} \boldsymbol{E}_\sigma & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \dot{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{H}_\sigma & \boldsymbol{B}_\sigma \boldsymbol{C}^P \\ \boldsymbol{F}\boldsymbol{C}^D{}_\sigma & -\boldsymbol{I} \end{bmatrix} \boldsymbol{z} + \sum_{k=1}^{n} \left(\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_k \end{bmatrix} \mathcal{S}_{\text{time}}^{\tau_k} \boldsymbol{z} \right) + \begin{bmatrix} \boldsymbol{f}_\sigma \\ \boldsymbol{0} \end{bmatrix}$$

with suitable matrices $\boldsymbol{F}$ *and* $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ *and distributional time shift operator* $\mathcal{S}_{\text{time}}^{\tau_k}$

Well-posedness of coupled system



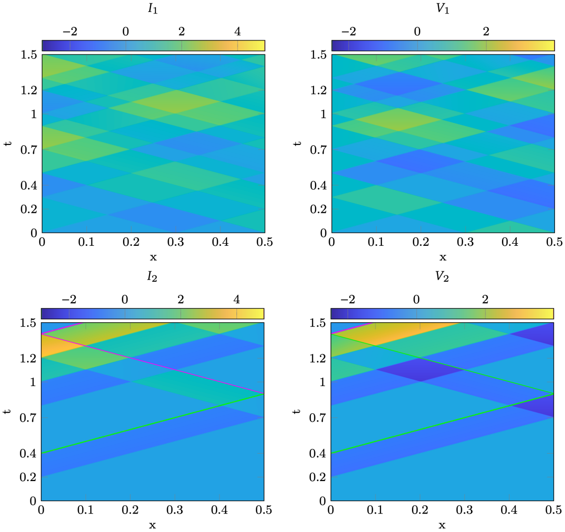
$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{A}\partial_x \boldsymbol{u} &= 0 \\ \boldsymbol{P}\begin{pmatrix} \boldsymbol{u}(\cdot, a^+) \\ \boldsymbol{u}(\cdot, b^-) \end{pmatrix} &= y_D \\ \boldsymbol{u}(t_0^+, \cdot) &= \boldsymbol{u}^0 \\ \boldsymbol{y}_P &= \boldsymbol{C}^P \begin{pmatrix} \boldsymbol{u}(\cdot, a^+) \\ \boldsymbol{u}(\cdot, b^-) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \boldsymbol{E}_\sigma \dot{\boldsymbol{w}} &= \boldsymbol{H}_\sigma \boldsymbol{w} + \boldsymbol{B}_\sigma \boldsymbol{y}_P + \boldsymbol{f}_\sigma \\ \boldsymbol{w}(t_0^-) &= \boldsymbol{w}^0 \\ \boldsymbol{y}_D &= \boldsymbol{C}_\sigma^D \boldsymbol{w} \end{aligned}$$

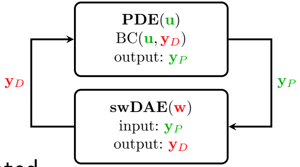
Corollary (Cf. Trenn & Unger, *CDC* 2019)

The coupled system is well-posed if the matrix pairs $(\boldsymbol{E}_\xi, \boldsymbol{H}_\xi + \boldsymbol{B}_\xi \boldsymbol{C}^P \boldsymbol{F} \boldsymbol{C}_\xi^D)$ *are regular for each* $\xi = 1, \ldots, N$.

Simulations for simple power grid example



Summary



- Coupling between PDEs and switched DAE well motivated
- Novel distributional solution framework to handle Dirac impulses
- Equivalence of coupled system with delay switched DAE
- Well-posedness result in terms of regularity-check of certain matrix pairs
- Simulations confirm solution theory

Open problems

- Adjusted numerical methods
- Extension to nonlinear case