Observability of Singular Linear Switched Systems in Discrete Time: Single Switch Case

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***Abstract*— In this paper, we investigate the observability of singular linear switched systems in discrete time. As a preliminary study, we restrict ourselves to systems with a single switch switching signal, i.e. the system switches from one mode to another mode at a certain switching time. We provide two necessary and sufficient conditions for the observability characterization. The first condition is applied for arbitrary switching time and the second one is for switching times that are far enough from the initial time and the final time of observation. These two conditions explicitly contain the switching time variable that indicates that in general, the observability is dependent on the switching time. However, under some sufficient conditions we provide, the observability will not depend on the switching time anymore. Furthermore, the observability of systems with two-dimensional states is independent of the switching time. In addition, from the example we discussed, an observable switched system can be built from two unobservable modes and different mode sequences may produce different observability property; in particular, swapping the mode sequence may destroy observability.**

I. Introduction

We consider in this paper a discrete-time Singular Linear Switched System (SLSS) of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{1a}$$
$$y(k) = C_{\sigma(k)}x(k) \tag{1b}$$

where $k \in \mathbb{N}$ is the time instant; $x(k) \in \mathbb{R}^n$, $n \in \mathbb{N}$, is the state at time k; $y(k) \in \mathbb{R}^p, p \in \mathbb{N}$ is the output; $\sigma : \mathbb{N} \to \{1, 2, ..., \mathtt{p}\}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant k; $E_i, A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{p\times n}$, $i \in \{1, 2, \dots, \mathtt{p}\}$; E_i may be singular. The presence of singular matrices E_i occurs in some dynamical processes which are subject to algebraic constraints, see e.g. [1]. If all E_i are nonsingular then (1) is called non-singular and studies about their solvability, controllability, and observability are well established [2], [3], [4]. Note that some authors denote the state of a singular system as internal variable [5], [6] or semi-state [7]. In many references, singular systems are also known as differential algebraic equations, strong coupling systems, incomplete state systems, generalized systems, algebro-differential systems, descriptor systems, and implicit differential equations [8], [9].

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In the continuous time domain, SLSSs have been studied extensively. The solvability issue is well established in [10], [11] while the stability issue is comprehensively studied in [10], [12], [13], [14]. Furthermore, the controllability and observability characterizations are satisfactorily formulated in [15], [16], [17], and some strategies are already developed to control such systems (see e.g. [18], [19], [20]). In particular, some studies involving physical systems, such as power system [21], have been presented in the literature.

In the discrete time domain, some studies about SLSSs are already available in the literature. Most of them study the stability property as can be seen in [22], [9], [23], [24], [25], [26], [27], [28]. Some other studies have proposed control methods, such as iterative learning approach for trajectory tracking purposes [29] and state feedback approach [30]. The existence and uniqueness of solutions of SLSS has recently been revisited in [31] where a one-step map was introduced and which can be utilized in analysing stability [23].

In studying observability, we expect significant differences for the results in discrete time compared to the results in continuous time. In the latter, the observability property under a single switch switching signal is independent of the switching time [16], while this cannot be expected in discrete time. In fact, for non-singular systems, the observability property is dependent on the switching time as illustrated in the following example.

Example 1: Consider a (non-singular) linear switched system with two modes

$$x(k+1) = A_{\sigma(k)}x(k), \tag{2a}$$
$$y(k) = C_{\sigma(k)}x(k) \tag{2b}$$

with $\sigma : \mathbb{N} \to \{1, 2\}$, $A_1 = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right], A_2 = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$, $C_1 = [1\ 1]$, $C_2 = [1\ 0]$, $x(k) \in \mathbb{R}^2$ is the state, and $y(k) \in \mathbb{R}$ is the output at time $k \in \mathbb{N}$. Both individual modes are observable because their observability matrices

$$\begin{bmatrix} C_1 \\ C_1A_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_2 \\ C_2A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

have full rank. In particular, if the systems remains in one mode, then for both modes the state can be recovered by observing the output for two time steps, i.e. on the interval $[0, K]$ with $K = 1$. However, when considering now the switched system on the same time interval $[0, K]$ with $K = 1$, where mode 1 is active at time $k = 0$ and mode 2 is active at time $k = 1$, observability is lost. This follows from

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C_1x(0) \\ C_2A_1x(0) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2A_1 \end{bmatrix} x(0),$$

where clearly $\left[\begin{smallmatrix} C_1 \\ C_2 A_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ has a non-trivial kernel, i.e. we *cannot* uniquely determine the initial state from the output on the time interval $[0, K]$.

Furthermore, if we reverse the switching signal, i.e. at $k = 0$ mode 2 is active and at $k = 1$ mode 1 is active, we see that the initial-state-output matrix $\left[\begin{smallmatrix} C_2 \\ C_1 A_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ allows recovering $x(0)$ uniquely from $y(0)$ and $y(1)$. Hence observability of a switched system also depends on the mode-seqeunce of the switching signal. $\diamond$

In this paper, we study the observability characterization for the single switch cases of singular linear switched systems in the discrete time domain. It is assumed that the mode switching is triggered only by the time. Other discussions in the literature may include the mode switching that is triggered by the state and/or the output. The discussion is structured as follows. We recall the existence and the uniqueness of the solution as well as the observability definition in the Preliminaries section. We present the main results in Section III containing the observability theorem. Last, we present an illustrative example in Section IV.

II. Preliminaries

We are revisiting the solution and the observability characterization of singular linear (non-switched) systems and the solution of singular linear switched systems in this section as the foundations to characterize the observability notion discussed in this paper.

A. Singular Linear Systems

Consider a discrete-time homogeneous Singular Linear System (SLS) in the form of

$$Ex(k+1) = Ax(k), \tag{3a}$$

$$y(k) = Cx(k) \tag{3b}$$

for $k \in \mathbb{N}$ and where $E, A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{p \times n}$ are known, E is singular with $\operatorname{rank} E = r < n$, $x : \mathbb{N} \to \mathbb{R}^n$ is the state, and $y : \mathbb{N} \to \mathbb{R}^p$ is the output. We omit the input term because the observability of linear systems does not depend on the input.

If (3a) is regular, i.e. $\det(sE - A)$ is not identically zero, then there exist invertible matrices $S, T \in \mathbb{R}^{n \times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{4}$$

where $N \in \mathbb{R}^{n_N \times n_N}$ is a nilpotent, for some $J \in \mathbb{R}^{r \times r}$. Following [32], we call (4) a quasi Weirstrass form (QWF) of the matrix pair (E, A). If the nilpotency index of N is 1 (i.e. $N = 0$), then the system (3a) is called an index-1 SLS.

Lemma 2.1 ([31]): Consider the matrix pair (E, A) with $E, A \in \mathbb{R}^{n \times n}$ and let $\mathcal{S} := A^{-1}(\operatorname{im} E) = \{ \xi \in \mathbb{R}^n \mid A\xi \in \operatorname{im} E \}$. Then (E, A) is regular and index-1 if, and only if,

$$\mathcal{S} \cap \ker E = \{0\}.$$

Furthermore, choose full rank matrices V and W such that $\operatorname{im} V = \mathcal{S}$ and $\operatorname{im} W = \ker E$, then if (E, A) is regular and index-1, then $T = [V, W]$ and $S = [EV, AW]^{-1}$ transform (E, A) into QWF (4).

Corollary 2.2 ([31]): Consider (3a), let $\mathcal{S} := A^{-1}(\operatorname{im} E)$ and assume (E, A) is regular and index-1 with QWF (4). Then x is a solution of (3a) if, and only if, $x(0) \in \mathcal{S}$ and

$$x(k+1) = \Phi_{(E,A)} x(k),$$

where

$$\Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}. \tag{5}$$

Note that $\Phi_{(E,A)}$ is independent from the specific choice of T and is called one-step-map for system (3a). Furthermore, if (3a) is actually non-singular (i.e. $E = I$), then $\Phi_{(E,A)} = A$.

In view of the forthcoming extension to switched system, we want to highlight that the interpretation of $\Phi_{(E,A)}$ as a one-step-map for the SLS (3a) is only valid if (3a) holds for at least two time steps, see [31, Rem. 2.6].

Definition 2.3: SLS (3) is observable on the interval $[0, K]$, $K \in \mathbb{N}$, if its state x is uniquely determined on $[0, K]$ by its output y on $[0, K]$.

By using linearity, the observability definition of SLS (3) is equivalent to zero distinguishability; i.e. it is observable if, and only if, the following implication holds on $[0, K]$:

$$y \equiv 0 \quad \Longrightarrow \quad x \equiv 0.$$

Furthermore, in view of Corollary 2.2, if (3) is index-1, then $x \equiv 0$ on $[0, K]$ if, and only if, $x(0) = 0$, in particular, the observability condition can be reduced to

$$y(k) = 0 \ \forall k \in [0, K] \quad \Longrightarrow \quad x(0) = 0. \tag{6}$$

From Corollary 2.2 we can also derive that

$$y(k) = C\Phi^k_{(E,A)} x(0) \quad k \in [0, K]$$

as well as $x(0) \in \mathcal{S}$. Hence for index-1 SLS we immediately have the following observability characterization:

Lemma 2.4: Index-1 SLS (3) is observable on $[0, K]$ if, and only if,

$$\mathcal{S} \cap \mathcal{O}^K = \{0\}, \tag{7}$$

where $\mathcal{O}^K := \ker[C^\top, (C\Phi_{(E,A)})^\top, \ldots, (C\Phi^K_{(E,A)})^\top]^\top$.

Note that due to Cayley-Hamilton, $\mathcal{O}^K = \mathcal{O}^{n-1}$ if $K \geq n - 1$, but $\mathcal{O}^K \supsetneq \mathcal{O}^{n-1}$ is possible; in particular, the unobservable space (i.e. the subspace of all initial values $x(0)$ which produce a zero output) depends on the length K of the considered interval when K is small compared to the system dimension n. This is a major difference to the continuous time case, where the unobservable space (given by $\mathcal{O}^{n-1}$) is independent from the length of the observation interval. This makes the observability analysis for switched systems more challenging in the discrete time case compared to the continuous time case.

B. Singular Linear Switched Systems

To ensure existence and uniqueness of solutions of (1) for general switching signals it is in general not enough to assume that each matrix pair (E_p, A_p) is regular (in contrast to the continuous time case), even assuming that each matrix pair is index-1 is not sufficient, see the example in the introduction of [31]. In order to have a well-posed SLSS it is actually necessary to assume that the family of matrix pairs (E_p, A_p) is jointly index-1 in the following sense:

Definition 2.5 ([31]): A family of matrix pairs $\{(E_1, A_1), \ldots, (E_{\mathtt{p}}, A_{\mathtt{p}})\}$ or the corresponding system (1) is called (jointly) index-1 if, and only if,

$$\mathcal{S}_i \cap \ker E_j = \{0\}, \quad \forall i, j \in \{1, 2, \ldots, \mathtt{p}\},$$

where $\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i) := \{\xi \in \mathbb{R}^n : A_i\xi \in \operatorname{im} E_i\}$.

As shown in [31], a consequence of the (jointly) index-1 assumption for (1) is that $\operatorname{rank} E_i = \text{constant} =: r$ and that $\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n$ for all $i, j \in \{1, 2, \ldots, \mathtt{p}\}$ where the notation $\oplus$ represents the direct sum of two subspaces. Furthermore, for every index-1 SLSS, the individual modes are also index-1 (just choose $i = j$ in the definition and apply Lemma 2.1) hence we find nonsingular matrices S_i and T_i that satisfy

$$(S_iE_iT_i, S_iA_iT_i) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix}\right), \tag{8}$$

for some $J_i \in \mathbb{R}^{r\times r}$.

Moreover, it was shown in [31] that x is a solution of the index-1 system (1) if, and only if, $x(0) \in \mathcal{S}_{\sigma(0)}$ and

$$x\,(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x\,(k)\,, \forall k \in \mathbb{N}$$

where $\Phi_{i,j}$ is the *one-step map* from mode j to mode i given by

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j}\Phi_j \tag{9}$$

where $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is a unique projector onto $\mathcal{S}_i$ along $\ker E_j$ and

$$\Phi_j := \Phi_{(E_j,A_j)} = T_j\begin{bmatrix} J_j & 0 \\ 0 & 0 \end{bmatrix}T_j^{-1}. \tag{10}$$

Remark 2.6: Under the assumption that the switching signal is fixed and known (as is the case in our observability study), it may not be necessary to assume that (1) is jointly index-1 to have existence and uniqueness of solutions, because not all mode combination (i, j) will occur. This extension is a topic of future research. $\diamond$

III. Single Switch Results

In the following we will restrict our attention to the single switch case, i.e. we consider (1) with

$$\sigma(k) = \begin{cases} 1, & 0 \le k < k_s, \\ 2, & k_s \le k \le K, \end{cases} \tag{11}$$

see also Figure 1.

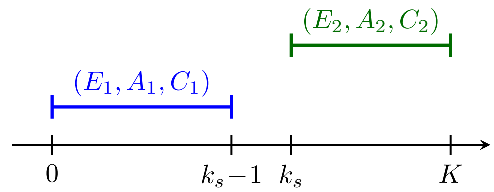

Fig. 1. Single switch linear singular system (1)

The observability definition from the nonswitched case carries over without change. Furthermore, under the index-1 assumption for (1) existence and uniqueness of solutions for all initial values $x(0) \in \mathcal{S}_1$ is guaranteed, hence the observability characterization via implication (6) remains valid.

We can now formulate our main result about characterizing observability of an SLSS with a single switch.

A. Arbitrary Switching Time and Observation Time

Theorem 3.1: Consider the SLSS (1) with the single switch switching signal (11) and assume it is index-1 with corresponding one-step maps Φ_1, Φ_2 given by (5) and $\Phi_{2,1}$ as in (9). Then (1) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_1 \cap \mathcal{O}_1^{k_s-1} \cap \left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}\left(\mathcal{O}_2^{K-k_s}\right) = \{0\}, \tag{12}$$

where, for $i = 1, 2$ and $k \in \mathbb{N}$,

$$\mathcal{O}_i^k := \ker[C_i^\top, (C_i\Phi_i)^\top, \ldots, (C_i\Phi_i^k)^\top]^\top.$$

Proof: The output of the switched system can be expressed in terms of the initial value $x(0) = x_0$ as follows:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k_s-1) \\ y(k_s) \\ y(k_s+1) \\ \vdots \\ y(K) \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 \\ C_1\Phi_1 \\ \vdots \\ C_1\Phi_1^{k_s-1} \\ C_2\Phi_{2,1}\Phi_1^{k_s-1} \\ C_2\Phi_2\Phi_{2,1}\Phi_1^{k_s-1} \\ \vdots \\ C_2\Phi_2^{K-k_s}\Phi_{2,1}\Phi_1^{k_s-1} \end{bmatrix}}_{O^{k_s,K}} x_0 = O^{k_s,K}x_0.$$

Then we have

$$\ker O^{k_s,K} = \mathcal{O}^{k_s,K} = \mathcal{O}_1^{k_s-1} \cap \left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}\left(\mathcal{O}_2^{K-k_s}\right),$$

where we use the fact, that $\ker(O\Phi) = \Phi^{-1}(\ker O)$ for any matrices O and Φ of appropriate size. Note that here Φ^{-1} denotes the preimage and not the inverse matrix (see the Definition 2.5).

Sufficiency: Assume $0 \neq x_0 \in \mathcal{S}_1 \cap \mathcal{O}^{k_s,K}$. Then there exists a unique, non-trivial solution x with $x(0) = x_0$. Since $x(0) \in \mathcal{O}^{k_s,K}$ then $y(k) = 0$, $0 \le k \le K$. This means that there exists a non-trivial solution of x with zero output. Hence, (3) is not observable.

Necessity: Consider a solution of (1) then $x(0) \in \mathcal{S}_1$. Furthermore, if $y(k) = 0$ for all $k \in [0, K]$, then $x(0) \in \mathcal{O}^{k_s,K}$. Hence $x(0) \in \mathcal{S}_1 \cap \mathcal{O}^{k_s,K} = \{0\}$, which shows the desired implication (6). $\blacksquare$

In general, the second and the third subspaces in the observability condition (12) depend on the switching time as k_s is explicitly appeared on them. This means that in discrete time, changing the switching time might change the observability property (see Example 2 for illustration). In contrast, the observability condition in continuous time does not depend on the switching time in the single switch case (see [16, Theorem 9]).

B. Large Enough Observation Time

The dependence on the switching time k_s in the observability characterization (12) can be reduced by exploiting the Cayley-Hamilton-Theorem as follows:

Corollary 3.2: Consider the index-1 SLSS (1) with the switching signal (11) and assume $n \leq k_s \leq K - n$. Then (1) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_1 \cap \mathcal{O}_1 \cap \left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}(\mathcal{O}_2) = \{0\} \tag{13}$$

where for $i = 1, 2$

$$\mathcal{O}_i := \ker[C_i^\top, (C_i\Phi_i)^\top, \ldots, (C_i\Phi_i^{n-1})^\top]^\top.$$

Furthermore, if Φ_1 is idempotent, then (1) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_1 \cap \mathcal{O}_1 \cap \Phi_{2,1}^{-1}(\mathcal{O}_2) = \{0\}. \tag{14}$$

Note that the observability characterization (14) is almost identical to the one obtained in continuous time (under an impulse-free, i.e. index-1, assumption), however, the assumption that Φ_1 is idempotent (i.e. Φ_1 is a projector) is extremely restrictive (and implies that the first mode has constant state trajectories). Nevertheless, we do actually believe that despite the explicite presence of the switching time k_s in the general observability condition (13) the observability does *not depend* on the switching time; this is ongoing research.

Due to the current lack of a general result about the independence of the observability condition (13) from the switching time k_s, we provide some further sufficient conditions, which are based on the following properties.

Lemma 3.3: Let $M, \Phi \in \mathbb{R}^{n\times n}$.

(i) If $\ker M \cap \operatorname{im}\Phi = \{0\}$ then there is a k, $0 \leq k \leq n$, such that

$$\ker(M\Phi^k) = \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

(ii) If, for some k, $\ker(M\Phi^k) \supseteq \ker(M\Phi^{k+1})$ then

$$\ker(M\Phi^k) \supseteq \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

(iii) If, for some k, $\ker(M\Phi^k) \subseteq \ker(M\Phi^{k+1})$ then

$$\ker(M\Phi^k) \subseteq \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

(iv) If, for some k, $\ker(M\Phi^k) = \ker(M\Phi^{k+1})$ then

$$\ker(M\Phi^k) = \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

Proof: The proofs easily follow from basic linear algebra and are therefore omitted. ∎

From Lemma 3.3 part (i) and (iv), we can derive the following corollary explaining the independence of observability on the switching time under some certain conditions.

Corollary 3.4:

(i) Assume that $\ker(O_2\Pi_{\mathcal{S}_2}^{\ker E_1}) \cap \operatorname{im}\Phi_1 = \{0\}$ and $n \leq k_s \leq K - n$. Then the observability of SLSS (1) does not depend on k_s. Moreover, the third subspace in (13) can be replaced by the simpler condition $\left[\Pi_{\mathcal{S}_2}^{\ker E_1}\Phi_1^n\right]^{-1}(\mathcal{O}_2)$.

(ii) If, for some $k \in \mathbb{N}$,

$$\mathcal{O}_1^{k-1} = \mathcal{O}_1^k \text{ and} \tag{15}$$

$$\left[\Phi_{2,1}\Phi_1^{k-1}\right]^{-1}(\mathcal{O}_2) = \left[\Phi_{2,1}\Phi_1^k\right]^{-1}(\mathcal{O}_2) \tag{16}$$

then the observability of SLSS (1) does not depend on the switching time for any k_s with $k \leq k_s \leq K - n$. Moreover, if (16) holds for some $k \geq n$, then the observability of SLSS (1) does not depend on the switching time k_s with $n \leq k_s \leq K - n$.

From part (ii) and (iii) in Lemma 3.3, we can derive some situations where the observability property would be independent of the switching time as explained in the following remark.

Remark 3.5: If for some k_s with $n \leq k_s \leq K - n$ the third subspace in (13) satisfies

$$\left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}(\mathcal{O}_2) \supseteq \left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2)$$

then

$$\left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2) \supseteq \left[\Phi_{2,1}\Phi_1^{k_s+1}\right]^{-1}(\mathcal{O}_2) \supseteq \cdots.$$

In this situation the third subspace in (13) will not grow for bigger k_s resulting that if the switched system is observable with the given switching time k_s then it remain observable for any bigger switching time (with $k_s < K - n$). Similarly, if for some k_s with $n \leq k_s \leq K - n$

$$\left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}(\mathcal{O}_2) \subseteq \left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2)$$

then

$$\left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2) \subseteq \left[\Phi_{2,1}\Phi_1^{k_s+1}\right]^{-1}(\mathcal{O}_2) \subseteq \cdots.$$

In this situation the third subspace in (13) will not shrink for bigger k_s resulting that if the switched system is not-observable for the given switching time k_s then the switched system will remain non-observable for any bigger switching time. ◇

So far, we have obtained some sufficient conditions as in Corollary 3.4 to verify the independence of the observability on the switching time. When restricting to a two-dimensional state ($n = 2$), we are actually able to prove that observability cannot depend on the switching time as long as both modes are active for at least two time steps.

Theorem 3.6: Consider the SLSS (1) of index-1 with state space dimension $n = 2$ and singular E-matrices. Assume that both modes are active for at least two time steps, i.e. $2 \leq k_s \leq K - 2$. Then the observability of the switched system does not depend on k_s. In fact, the observability condition (13) can be reduced to

$$\mathcal{S}_1 \cap \mathcal{O}_1 \cap [\Phi_{2,1}\Phi_1]^{-1}(\mathcal{O}_2) = \{0\} \tag{17}$$

where $\mathcal{O}_i := \ker[C_i^\top, (C_i\Phi_i)^\top]^\top$ for $i = 1, 2$.

Proof: The proof is based on transforming Φ_1 via a coordinate change into $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right]$ (which is possible due to the index-1 assumption) and by carefully considering the different cases for λ (equal or unequal to zero) and for the entries in the matrix C_1; the details are omitted due to space limitations. ■

At the moment it is not clear to the authors whether Theorem 3.6 remains valid for higher dimensional systems.

IV. Illustrative Example

We have seen already in the introduction that even if all modes of the switched systems are observable on a given interval $[0, K]$, the switched system considered on the same interval is not necessarily observable.

On the other hand it is also possible that the switched system is observable although the individual modes are not observable, this is illustrated in the following example.

Example 2: Consider the SLSS (1) on $[0, K]$ with $K = 10$ under single switch switching signal (11), where

$$E_1 = \begin{bmatrix}0&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, E_2 = \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$
$$A_1 = \begin{bmatrix}1&0&0&0\\1&0&0&0\\1&1&0&1\\0&0&1&0\end{bmatrix}, A_2 = \begin{bmatrix}1&1&1&1\\0&0&0&1\\1&0&0&1\\1&0&1&1\end{bmatrix},$$
$$C_1 = \begin{bmatrix}0&1&1&1\end{bmatrix}, C_2 = \begin{bmatrix}1&1&0&1\end{bmatrix}.$$

Then $\mathcal{S}_1 = \operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\\0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\-1\end{smallmatrix}\right)\right\}, \mathcal{S}_2 = \operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\\0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\0\\0\end{smallmatrix}\right)\right\}, \ker E_1 = \ker E_2 = \operatorname{span}\left\{\left(\begin{smallmatrix}0\\0\\1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\\0\\1\end{smallmatrix}\right)\right\}$. Consequently, the switched system is jointly index-1, because $\mathcal{S}_i \cap \ker E_j = \{0\},\ i, j = 1, 2$. The corresponding one-step-map matrices are

$$\Phi_1 = \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\-1&0&0&0\end{bmatrix}, \Phi_{1,2} = \begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&0&0\\1&-1&0&0\end{bmatrix},$$
$$\Phi_2 = \begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}, \Phi_{2,1} = \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Each individual system i.e. mode-1 and mode-2 are both not-observable on $[0, K]$ since $\mathcal{S}_1 \cap \mathcal{O}_1 = \operatorname{span}\left(\begin{smallmatrix}0\\1\\0\\-1\end{smallmatrix}\right) \neq \{0\}$, and $\mathcal{S}_2 \cap \mathcal{O}_2 = \operatorname{span}\left(\begin{smallmatrix}1\\0\\0\\-1\end{smallmatrix}\right) \neq \{0\}$.

We observe here the observability on $[0, K]$ for varying switching times $k_s \in [1, K]$ illustrated in Fig. 2.

Mode sequence 1-2 (i.e. the system starts from mode-1 and switches to mode-2) always produces a not-observable switched system since the observability condition (12) for $k_s < 4$ or (13) for $k_s \geq 4$ does not hold. This is not surprising because each individual system is not-observable. Furthermore, by checking the condition in part (i) of Corollary 3.4 we have that

$$\ker(O_2 \Pi_{\mathcal{S}_2}^{\ker E_1}) \cap \operatorname{im} \Phi_1 = \{0\}$$

which means that independently of the switching time k_s the mode sequence 1-2 will always give a non-observable switched system.

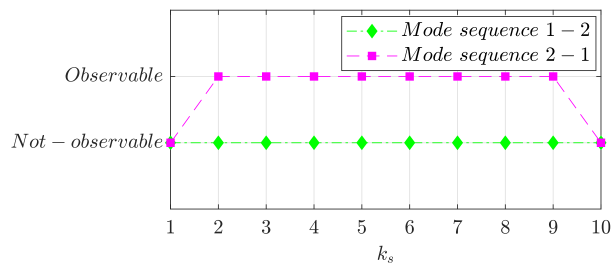

Fig. 2. Switching time vs observability Example 2

On the other hand, for mode sequence 2-1 the switched system actually becomes observable for all switching times $k_s = 2, 3, \ldots, 9$ because (12) is satisfied. This illustrates that even though each mode is not-observable, the switched system could be observable. For mode sequence 2-1, the condition in part (i) of Corollary 3.4 is not satisfied but the condition in part (ii) is satisfied i.e. we have that

$$\left[\Phi_{1,2}\Phi_2^3\right]^{-1}(\mathcal{O}_1) = \left[\Phi_{1,2}\Phi_2^4\right]^{-1}(\mathcal{O}_1).$$

This implies that we also can make sure analytically that the mode sequence 2-1 is always observable for any $3 \leq k_s \leq 7$ i.e. it doesn't depend on the switching time anymore. ◇

V. Summary and Future Works

We have presented two necessary and sufficient conditions for observability characterizations of singular linear switched systems in discrete time with single switch switching signals. The first characterization corresponds to arbitrary switching time cases whereas the second characterization corresponds to large enough switching time cases. The switching time variable appears explicitly in the condition which means that in general the observability property seems to depend on the switching time. However, we have provided some sufficient conditions when the observability is not depend on the switching time anymore. Furthermore, for systems with two-dimensional states, we have proved that it does not depend on the switching time. Finally, we have illustrated via the example that 1) even if each individual subsystem is not-observable, the switching may produce an observable switched system, and 2) swapping the mode sequence may change the observability.

The study in this paper leaves many open problems which will be studied in our future works. First, the dependence of observability on the switching time for systems with higher dimensional states. Next, the observability characterizations for multiple switches cases. Then, forward observability and observer design are also interesting to study. Last, we will also study the controllability characterization for such systems. Moreover, we expect that in some sense there is a duality between the observability and the controllability as in ordinary systems.

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