Minimal realization for linear switched systems with a single switch

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***Abstract*— We discuss the problem of minimal realization for linear switched systems with a given switching signal and present some preliminary results for the single switch case. The key idea is to extend the reachable subspace of the second mode to include nonzero initial values (resulting from the first mode) and also extend the observable subspace of the first mode by taking information from the second mode into account. We provide some simple examples to illustrate the approach.**

I. Introduction

Realization theory is a classical topic in the area of systems and control. In general, the aim of realization theory is to construct a state-space model from a given input-output behavior of the system [1], [2]. Realization theory provides a theoretical foundation for model reduction, system identification and filtering/observer design. Indeed, finding a minimal realization could be seen as the first step towards model reduction. Realization theory of switched systems has been discussed e.g. in [3]–[10]. The author in [6] combines the theory of rational formal power series with the classical automata theory to discuss the realization theory of hybrid systems. In particular, the cases of arbitrary and constrained switching are discussed where switching signal are viewed as input to the switched systems. Moreover, some other works have been done on realization theory, observability and reachability of linear switched systems, e.g., [11]–[15].

In contrast to the existing literature, we view a switched system as a piecewise-constant time-varying linear system, in particular, a (minimal) realization in general depends on the specifically given switching signal. Several minimal realization approaches have been developed in the context of time-varying systems, and they are classified by constant rank systems, piecewise constant rank systems, global and intervalwise Kalman decomposition, e.g., [2], [16]–[19]. Most available methods are considered by Gramian based *differential* Kalman decomposition and some drawbacks are summarized in [19]. Moreover, the authors in [18] have been discussed the detection of temporal properties for piecewise constant rank systems (PCR), indeed a PCR system differs from the switched linear systems because the time-instants are *a priori fixed* and the linear system over each time interval is *time-varying*.

To be more specific, we try to find a minimal realization of linear switched systems (LSSs) with a given switching signal of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_q(t) = A_{\sigma(t)}x_q(t) + B_{\sigma(t)}u(t),\ t \in (t_q, t_{q+1}) \\ x_q(t_q^+) = J_{\sigma(t_q^+),\sigma(t_q^-)}x_{q-1}(t_q^-), \\ y(t) = C_{\sigma(t)}x_q(t), \quad t \in \mathbb{R}, \end{cases} \tag{1}$$

where $x_q : (t_q, t_{q+1}) \to \mathbb{R}^{n_q}$ is the absolutely continuous q-th piece of the state, $u : \mathbb{R} \to \mathbb{R}^u$ is the input and y is the measured output. The switching signal $\sigma : \mathbb{R} \to \mathscr{Q} = \{1, 2, \cdots, f\} \subset \mathbb{N}$ is a given piecewise constant function with finitely many switching times: $\{\ t_q \mid q \in \mathscr{Q}, t_1 < t_2 < \cdots < t_f\ \}$ in the bounded interval (t_1, t_{f+1}) of interest. For each $q \in \mathscr{Q}$, the system matrices A_q, B_q, C_q, are of appropriate size, describing the dynamics correspond to the linear system active in mode $q \in \mathscr{Q}$. Note that we do allow different state-space dimension n_q in each mode, in particular, it is necessary to provide a jump-map $J_{q^+,q^-} : \mathbb{R}^{n_{q^-}} \to \mathbb{R}^{n_{q^+}}$ relating the state-value $x_{q^-}(t_q^-)$ immediately before the switch with the state value $x_{q^+}(t_q^+)$ after the switch at t_q.

We assume that the switching signal is known and fixed, hence without restricting generality we assume (unless stated otherwise) that $\sigma(t) = q$ on (t_q, t_{q+1}). In particular, we can simplify the notation for the jump matrix $J_{\sigma(t_q^+),\sigma(t_q^-)} = J_{q,q-1} =: J_q$. Furthermore, in some slight abuse of notation we will simply speak of the solution $x(\cdot)$ instead of the different solution pieces $x_q(\cdot)$.

As mentioned above, finding a minimal realization of (1) is a first step towards model reduction. Recently, we have presented in [20] a time-varying model reduction approach for linear switched system (with identical state-dimensions and without jumps). However, the resulting reduced systems was not a switched system anymore, instead it was fully time-varying and is therefore difficult to handle numerically. Therefore, our aim is to gain insight into a more suitable model-reduction approach by studying the minimal realization problem for switched systems of the form (1) *within* this system class. In particular, the process of going from a non-minimal representation (with initial value zero) to a minimal one can be seen as removing "unreachable" and "unobservable" states; understanding what the notions "unreachable" and "unobservable" exactly means in this context allows to generalize these ideas to "difficult to reach" and "difficult to observe" which then allows to perform model reduction.

This note is just the first step in this research field and provides some preliminary results for the single switch case.

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II. Problem Setting and Preliminaries

A. Minimal realization: definition

In this section, we introduce some notions and properties related to minimality of linear switched systems (1) and we begin with the formal definition of minimality.

Definition 1 (cf. [6]): For Σ_σ as in (1) we define its *total dimension* as follows

$$\dim \Sigma_\sigma := \sum_{q\in\mathscr{Q}} n_q.$$

Furthermore, we define its input-output behaviour as follows

$$\mathfrak{B}^{io}_\sigma := \left\{ (u,y) \;\middle|\; \begin{array}{l} \exists x_q : (t_q,t_{q+1}) \to \mathbb{R}^{n_q} \text{ satisfying (1)} \\ \text{and } x(t_1^-) = 0 \end{array} \right\}.$$

A linear switched system $\widehat{\Sigma}_\sigma$ with corresponding input-output behavior $\widehat{\mathfrak{B}}^{io}_\sigma$ is said to be a minimal realization of switched system Σ_σ if 1) $\mathfrak{B}^{io}_\sigma = \widehat{\mathfrak{B}}^{io}_\sigma$ and 2) for any $\widetilde{\Sigma}_\sigma$ with $\mathfrak{B}^{io}_\sigma = \widetilde{\mathfrak{B}}^{io}_\sigma$ we have

$$\dim \widehat{\Sigma}_\sigma \le \dim \widetilde{\Sigma}_\sigma.$$

For non-switched linear systems it is well known that a realization is minimal if, and only if, it is minimal, however for switched systems of the form (1) this is not the case in general as the following example shows:

Example 2: Consider a switched system with two modes

$$(A_1,B_1,C_1) = \left(\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right),$$
$$(A_2,B_2,C_2,J_2) = \left(\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right).$$

Assume the single switching signal defined by

$$\sigma(t) = \begin{cases} 1, & \text{on } (t_1,t_2), \\ 2, & \text{on } (t_2,t_f). \end{cases} \tag{2}$$

It is easily seen, that each modes is unreachable and unobservable. However, the switched system is reachable in the sense that each value $x(t_f^-) \in \mathbb{R}^3$ can be reached from zero by a suitable input and it is also observable in the sense that (for a vanishing input) only a zero initial value leads to a zero output.

On the other hand, the second state is unreachable in the 1st mode and unobservable in the 2nd mode. In particular, when starting with a zero initial value, for any input the value of the second state does not effect the output (because in the first mode it is identically zero and in the second mode the corresponding coefficient in the C-matrix is zero). Therefore, we can remove the second state without altering the input-output behavior.

Remark 3: The above definition of minimality is not specifying any *method* how to obtain a minimal realization from a given switched system. In particular, it does not take into account constraints like the requirement that the reduced state is obtained via a uniform projection map (cf. [21], [22] in the context of model reduction). In general, a minimal realization can only be obtained by considering each mode individually (and by properly taking the effect on the other modes into account).

It is important to note that a naive approach to reduce each individual mode by removing unreachable and unobservable states will not work in general, this is illustrated with the following example:

Example 4: Consider the switched system Σ_σ with

$$(A_1,B_1,C_1) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right),$$
$$(A_2,B_2,C_2) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right),$$

with switching signal (2) and without jumps. The second state is not observable in the first mode and not reachable in the second mode, hence one may be tempted to remove this state to obtain an (unswitched) minimal realization given by

$$\dot{x}_1 = x_1 + u, \quad y = x_1. \tag{3}$$

However, it is easily seen, that a non-zero input leads to a non-zero second state during the first mode, which then will effect the output of the second mode, i.e. system (3) obtained by simply reducing each mode individually does *not* have the same input-output behavior as the original switched system.

In Section III we will propose a method which takes into account the effect of the different modes have on each other. Although this method will not simply consider a minimal realization of each mode individually, the method of reducing a given (unswitched) linear system to a minimal one, will play an important role and we will recalled first.

B. The Kalman decomposition and minimal realization

Assume a linear system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t), \end{cases} \tag{4}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. The well known Kalman decomposition (KD), [2], is a coordinate transformation $x = \mathbb{V}z$ which leads to the following block triangular form

$$(\mathbb{V}^{-1}A\mathbb{V}, \mathbb{V}^{-1}B, C\mathbb{V}) = \left(\begin{bmatrix} A_{r\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{ro} & 0 & A_{24} \\ 0 & 0 & A_{\bar{r}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C_{ro} & 0 & C_{\bar{r}o} \end{bmatrix} \right),$$

where $\left(\begin{bmatrix} A_{r\bar{o}} & A_{12} \\ 0 & A_{ro} \end{bmatrix}, \begin{bmatrix} B_{r\bar{o}} \\ B_{ro} \end{bmatrix} \right)$ is reachable and $\left(\begin{bmatrix} A_{ro} & A_{24} \\ 0 & A_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} C_{ro} & C_{\bar{r}o} \end{bmatrix} \right)$ is observable. In fact $\mathbb{V} = [\mathbb{V}_{r\bar{o}}\ \mathbb{V}_{ro}\ \mathbb{V}_{\bar{r}\bar{o}}\ \mathbb{V}_{\bar{r}o}]$, where $\operatorname{im}\mathbb{V}_{r\bar{o}}$ is the intersection of the reachable and unobservable space, $\operatorname{im}[\mathbb{V}_{r\bar{o}}, \mathbb{V}_{ro}]$ is the reachable space and $\operatorname{im}[\mathbb{V}_{r\bar{o}}, \mathbb{V}_{\bar{r}\bar{o}}]$ is the unobservable space. A minimal realization of (4) is now given by (A_{ro}, B_{ro}, C_{ro}).

This method is based on the assumption that the initial value is zero. If arbitrary initial values are considered, it is easily seen that only the unobservable states can be removed.

In the context of switched systems, the initial values for the second mode are neither zero nor completely arbitrary, but are constraint to the reachable space of the first mode, this motivates to find a minimal realization for the linear system

$$\Sigma^{\mathscr{X}_0} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in \mathscr{X}_0, \\ y(t) = Cx(t), \end{cases} \tag{5}$$

where $\mathscr{X}_0 \subseteq \mathbb{R}^n$ is the subspace of relevant initial values. Inspired by the fact, that the ODE $\dot{x} = Ax$ with initial condition $x(0) = x_0$ has the same solution as the impulsive ODE $\dot{x} = Ax + x_0\delta$, $x(0^-) = 0$ we propose the following *input-extended* system corresponding to (5) (cf. [23] in the context of model-reduction)

$$\Sigma_e^{\mathscr{X}_0} : \begin{cases} \dot{x}_e(t) = Ax_e(t) + \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix}, \quad x_e(0) = 0, \\ y_e(t) = Cx_e(t), \end{cases} \tag{6}$$

where $\operatorname{im} X_0 = \mathscr{X}_0$. We can now apply the KD on the extended system and obtain a minimal extended realization

$$\widehat{\Sigma}_e^{\mathscr{X}_0} : \begin{cases} \dot{\widehat{x}}_e(t) = \widehat{A}_e\widehat{x}_e(t) + \begin{bmatrix} \widehat{B}_e & \widehat{X}_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix}, \quad \widehat{x}_e(0) = 0, \\ \widehat{y}_e(t) = \widehat{C}_e\widehat{x}_e(t), \end{cases} \tag{7}$$

and the corresponding minimal realization

$$\widehat{\Sigma}^{\mathscr{X}_0} : \begin{cases} \dot{\widehat{x}}(t) = \widehat{A}_e\widehat{x}(t) + \widehat{B}_e u(t), \quad \widehat{x}(0) = \widehat{x}_0 \in \operatorname{im}\widehat{X}_0, \\ \widehat{y}(t) = \widehat{C}_e\widehat{x}(t). \end{cases} \tag{8}$$

The properties of the reduced system $\widehat{\Sigma}^{\mathscr{X}_0}$ can formalized in the following lemma.

Lemma 5: Consider $\Sigma^{\mathscr{X}_0}$ as in (5) and $\widehat{\Sigma}^{\mathscr{X}_0}$ as in (8) obtained by first extending (5) to (6), then reducing it via the KD to a minimal extended realization (7) and finally removing the extension. Then $\Sigma^{\mathscr{X}_0}$ and $\widehat{\Sigma}^{\mathscr{X}_0}$ are input-output equivalent in the sense that for all trajectories (x,u,y) satisfying (5) there exists $\widehat{x}$ such that $(\widehat{x},u,y)$ satisfies (8). Furthermore, $\widehat{\Sigma}^{\mathscr{X}_0}$ has the minimal state-dimension under all systems which are input-output equivalent to $\Sigma^{\mathscr{X}_0}$.

Proof: The output equation of $\Sigma^{\mathscr{X}_0}$ with initial value $x(0) = x_0 \in \mathscr{X}_0$ is given by

$$y^{\mathscr{X}_0}(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau. \tag{9}$$

Construct an input-extended system $\Sigma_e^{\mathscr{X}_0}$ as in (6), and let $x_0 = X_0 z_0$ for some z_0, where the columns of X_0 form the basis for the subspace of relevant initial values. Then system (5) is input-output equivalent to system (6) with the input $u_0 = z_0\delta$, where δ denotes the Dirac delta distribution and $Ce^{At}x_0 = Ce^{At}X_0 z_0$.

The output equation of $\Sigma_e^{\mathscr{X}_0}$ as in (6), can be written by

$$\begin{aligned} y^{\mathscr{X}_0}(t) &= \int_0^t Ce^{A(t-\tau)} \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix} d\tau \\ &= \int_0^t C\mathbb{V}e^{\mathbb{V}^{-1}A\mathbb{V}(t-\tau)}\mathbb{V}^{-1} \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix} d\tau, \end{aligned}$$

where $\mathbb{V}$ is the KD transformation matrix which transforms $(A,[B,X_0],C)$ into a KD, i.e.

$$\mathbb{V}^{-1}A\mathbb{V} = \begin{bmatrix} * & * & * & * \\ 0 & \widehat{A}_e & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \mathbb{V}^{-1}[B\, X_0] = \begin{bmatrix} * & * \\ \widehat{B}_e & \widehat{X}_0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C\mathbb{V} = \begin{bmatrix} 0 & \widehat{C}_e & 0 & * \end{bmatrix}.$$

So we have, utilizing $\mathbb{V}^{-1}e^{A}\mathbb{V} = e^{\mathbb{V}^{-1}A\mathbb{V}}$,

$$y^{\mathscr{X}_0}(t) = \int_0^t \widehat{C}_e e^{\widehat{A}_e(t-\tau)} \begin{bmatrix} \widehat{B}_e & \widehat{X}_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix} d\tau, \tag{10}$$

which is the output of minimal extended realization, as in (7). Again, for $u_0 = z_0\delta$, equation (10) can be written as

$$y^{\mathscr{X}_0}(t) = \widehat{C}_e e^{\widehat{A}_e t}\widehat{x}_0 + \int_0^t \widehat{C}_e e^{\widehat{A}_e(t-\tau)}\widehat{C}_e u(\tau)d\tau, \tag{11}$$

where $\widehat{x}_0 = \widehat{X}_0 z_0$ and it represents the output equation for (8). Therefore, for any arbitrary trajectories (x,u,y) satisfying (5) there exists $\widehat{x}$ such that $(\widehat{x},u,y)$ satisfies (8).

2nd part: Assume $\widehat{\Sigma}_e^{\mathscr{X}_0}$ is the minimal realization of $\Sigma_e^{\mathscr{X}_0}$. Consider a system $\widetilde{\Sigma}^{\mathscr{X}_0}$ which is input-output equivalent to $\Sigma^{\mathscr{X}_0}$. Then for all $x_0 \in \mathscr{X}_0$, there exists $\widetilde{x}_0 \in \widetilde{\mathscr{X}_0}$ such that for all input $u(\cdot)$, satisfies

$$y(\cdot,u,x_0) = \widetilde{y}(\cdot,u,\widetilde{x}_0).$$

Construct an input-extended system $\widetilde{\Sigma}_e^{\mathscr{X}_0}$ as in (6) for $\widetilde{\Sigma}^{\mathscr{X}_0}$ such that

$$\widetilde{y}_e(\cdot,\left[\begin{smallmatrix} u \\ u_0 \end{smallmatrix}\right],0) = y_e(\cdot,\left[\begin{smallmatrix} u \\ u_0 \end{smallmatrix}\right],0),\ \forall u,u_0. \tag{12}$$

Then we have that

$$\dim\widetilde{\Sigma}^{\mathscr{X}_0} = \dim\widetilde{\Sigma}_e^{\mathscr{X}_0} \geq \dim\widehat{\Sigma}_e^{\mathscr{X}_0} = \dim\widehat{\Sigma}^{\mathscr{X}_0},$$

because $\widehat{\Sigma}_e^{\mathscr{X}_0}$ has by construction the minimal state dimension under all (extended) systems satisfying (12), i.e., $\widehat{\Sigma}^{\mathscr{X}_0}$ has indeed the minimal state-dimension under all systems which are input-output equivalent to $\Sigma^{\mathscr{X}_0}$. ■

III. Minimal realization of switched system

In this section, we proposed a method to find the minimal realization of linear switched systems

$$\begin{aligned} (A_1,B_1,C_1), &\quad \text{on} \quad t \in (t_1,t_2), \\ (A_2,B_2,C_2,J_2), &\quad \text{on} \quad t \in (t_2,t_f), \end{aligned} \tag{13}$$

with the single switch switching signal given by (2). We propose a technique consisting of three main steps. First, we construct a minimal realization of second mode by taking into account the reachable subspace of the first mode. As a second step, we find a minimal realization of first mode by taking into account the observable states of the second system. The final step consists of defining the reduced jump map from mode 1 to mode 2.

A. Step 1: Reduction of second mode

As discussed above the second mode will in general not start with initial value zero, instead the values of $x(t_2^-)$ cover the whole reachable space $\mathscr{R}_1 = \operatorname{im}[B_1, A_1B_1, \ldots, A_1^{n_1-1}B_1]$, consequently $x(t_2^+) \in J_2\mathscr{R}_1$. Using the method described above we extend the second mode to $(A_2, B_{2,e}, C_2)$ with $B_{2,e} := [B_2, J_2R_1]$ where $\operatorname{im} R_1 = \mathscr{R}_1$ and apply to reduction method described above to obtain the reduced mode $(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2)$ which is input-output equivalent with the second mode (under the assumption that the initial values for the second mode are determined by the reachable states of mode 1).

B. Step 2: Reduction of first mode

Restricted to the first interval, all unobservable and unreachable states can be removed from the first mode without changing the input-output behavior. However, some unobservable, but reachable, state from the first mode may become observable in the second mode and hence the input of the first mode may therefore indirectly influence the output of the second mode (via the initial value for the second mode). We arrive therefore at the following general problem: Given a linear system (4) and a subspace $\mathscr{L} \subseteq \mathbb{R}^n$ of indirectly-observable states, find a minimal realization of (4) which does not "remove" indirectly-observable states. To be more precise, we assume that the reduction of (4) is achieved via a left projection W and a right projection V, i.e. $WV = I$ and the reduced system of (A,B,C) is given by $(\widehat{A},\widehat{B},\widehat{C}) = (WAV, WB, CV)$, in particular, $\ker W$ corresponds to the removed states. The condition that the space of indirectly-observed states $\mathscr{L}$ is not "removed" by the reduction procedure, can now be formalized by the condition

$$\ker W \cap \mathscr{L} = \{0\}.$$

Similar as in Step 1 where we extended the input matrix to enlarge the reachability space, we now propose to extend the output matrix to extend the observability space, i.e. we consider

$$\Sigma_e^{\mathscr{L}} : \begin{cases} \dot{x}_e(t) = Ax_e(t) + Bu(t), \quad x_e(0) = 0, \\ y_e(t) = \begin{bmatrix} C \\ L^\top \end{bmatrix} x_e(t), \end{cases} \tag{14}$$

where $\operatorname{im} L = \mathscr{L}$. Very similar as before, we utilize the KD to obtain a minimal realization of (14) given by

$$\widehat{\Sigma}_e^{\mathscr{L}} : \begin{cases} \dot{\widehat{x}}_e(t) = \widehat{A}_e\widehat{x}_e(t) + \widehat{B}_e u(t), \quad \widehat{x}_e(0) = 0, \\ y_e(t) = \begin{bmatrix} \widehat{C}_e \\ \widehat{L}^\top \end{bmatrix} \widehat{x}_e(t). \end{cases} \tag{15}$$

Removing the additional rows in the output-matrix we then arrive at the proposed minimal realization of (4) which does not "remove" states from $\mathscr{L}$:

$$\widehat{\Sigma}^{\mathscr{L}} : \begin{cases} \dot{x}(t) = \widehat{A}_e\widehat{x}(t) + \widehat{B}_e u(t), \quad \widehat{x}(0) = 0, \\ \widehat{y}(t) = \widehat{C}_e\widehat{x}(t). \end{cases} \tag{16}$$

Let us investigate the properties of the reduced system $\widehat{\Sigma}^{\mathscr{L}}$.

Lemma 6: The two systems (4) and (16) have the same input output behavior. Furthermore, let W and V be the two projectors transforming (4) into (16), i.e. $WV = I$, $\widehat{A}_e = WAV$, $\widehat{B}_e = WB$ and $\widehat{C}_e = CV$, then $\ker W \cap \mathscr{L} = \{0\}$.

Proof: The output equation of system (4) is

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \\ &= \int_0^t \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} C \\ L^\top \end{bmatrix} \mathbb{V} e^{\mathbb{V}^{-1}A\mathbb{V}(t-\tau)}\mathbb{V}^{-1}Bu(\tau)d\tau, \end{aligned}$$

where $\mathbb{V}$ is the transformation for obtaining a KD for $(A, B, [C^\top, L]^\top)$, i.e.

$$\mathbb{V}^{-1}A\mathbb{V} = \begin{bmatrix} * & * & * & * \\ 0 & \widehat{A}_e & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \mathbb{V}^{-1}B = \begin{bmatrix} * \\ \widehat{B}_e \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} C \\ L^\top \end{bmatrix} \mathbb{V} = \begin{bmatrix} 0 & \widehat{C}_e & 0 & * \\ 0 & \widehat{L}^\top & 0 & * \end{bmatrix}.$$

Now, together with

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} C \\ L^\top \end{bmatrix} \mathbb{V} = \begin{bmatrix} 0 & \widehat{C}_e & 0 & * \end{bmatrix}, \mathbb{V}^{-1}e^A\mathbb{V} = e^{\mathbb{V}^{-1}A\mathbb{V}},$$

we have

$$y(t) = \int_0^t \widehat{C}_e e^{\widehat{A}_e(t-\tau)}\widehat{B}_e u(\tau)d\tau = \widehat{y}(t),$$

where $\widehat{y}(t)$ is the output of minimal realization as in (16).

2nd part: Let the transformation matrices to obtain a KD of system (14) be $\widehat{\mathbb{V}} = [\widehat{V}_{r\bar{o}}\ \widehat{V}_{ro}\ \widehat{V}_{\bar{r}\bar{o}}\ \widehat{V}_{\bar{r}o}]$, $\widehat{\mathbb{V}}^{-1} = \begin{bmatrix} \widehat{W}_{r\bar{o}} \\ \widehat{W}_{ro} \\ \widehat{W}_{\bar{r}\bar{o}} \\ \widehat{W}_{\bar{r}o} \end{bmatrix}$.

Clearly, $W = \widehat{W}_{ro}$, $V = \widehat{V}_{ro}$ and $\operatorname{im} L \subseteq \operatorname{im} \begin{bmatrix} C \\ L^\top \end{bmatrix}^\top =: \operatorname{im} C_e^\top$, it follows that

$$\operatorname{im} L \subseteq \left(\ker \begin{bmatrix} C_e \\ C_eA \\ \vdots \\ C_eA^{n-1} \end{bmatrix} \right)^\perp = \left(\operatorname{im}[\widehat{V}_{r\bar{o}}, \widehat{V}_{\bar{r}\bar{o}}] \right)^\perp = \operatorname{im}[\widehat{W}_{ro}^\top, \widehat{W}_{\bar{r}o}^\top].$$

Furthermore, by assumption $\operatorname{im} L \subseteq \operatorname{im}[\widehat{V}_{r\bar{o}}, \widehat{V}_{ro}]$ (the reachable subspace). Hence

$$\operatorname{im} L \subseteq \operatorname{im}[\widehat{V}_{r\bar{o}}, \widehat{V}_{ro}] \cap \operatorname{im}[\widehat{W}_{ro}^\top, \widehat{W}_{\bar{r}o}^\top].$$

Consider an arbitrary $z \in \operatorname{im} L \cap \ker W$. From $Wz = 0$ and $z \in \operatorname{im} L \subseteq \operatorname{im}[\widehat{V}_{r\bar{o}}, \widehat{V}_{ro}]$, we can conclude that there exists z_1, z_2 such that $z = \widehat{V}_{r\bar{o}}z_1 + \widehat{V}_{ro}z_2$, so $0 = Wz = W\widehat{V}_{r\bar{o}}z_1 + W\widehat{V}_{ro}z_2 = z_2$, where $W\hat{V}_{r\bar{o}} = 0$, $W\hat{V}_{ro} = I$. Therefore, $z = \widehat{V}_{r\bar{o}}z_1$ and

$$z \in \operatorname{im} \widehat{V}_{r\bar{o}} = \ker \begin{bmatrix} \widehat{W}_{ro} \\ \widehat{W}_{\bar{r}\bar{o}} \\ \widehat{W}_{\bar{r}o} \end{bmatrix} \subseteq \ker \begin{bmatrix} \widehat{W}_{ro} \\ \widehat{W}_{\bar{r}o} \end{bmatrix} = \left(\operatorname{im}[\widehat{W}_{ro}^\top, \widehat{W}_{\bar{r}o}^\top] \right)^\perp.$$

Altogether, we have that

$$z \in \operatorname{im}[\widehat{W}_{ro}^\top, \widehat{W}_{\bar{r}o}^\top] \cap \left(\operatorname{im}[\widehat{W}_{ro}^\top, \widehat{W}_{\bar{r}o}^\top] \right)^\perp = \{0\}.$$

■

We believe that the above reduction method results in a minimal realization, however, a complete proof is still work in progress and we therefore formulate here the minimality result as a conjecture.

Conjecture 7: Let $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ be a system which is input-output equivalent to (4) and which is obtained via a projection method with left-projector $\widetilde{W}$. If $\ker \widetilde{W} \cap \mathscr{L} = \{0\}$ then the state-dimension $\widetilde{n}$ of $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is at least the state dimension of (16).

C. Step 3: Reduced jump map

Assume the mode before the jump $J_2 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ was reduced by a projection method with left- and right-projectors W_1, V_1 and the mode after the jump was reduced by left- and right-projectors W_2, V_2. Then the reduced jump $\widehat{J}_2 : \mathbb{R}^{\widehat{n}_1} \to \mathbb{R}^{\widehat{n}_2}$ is defined as

$$\widehat{J}_2 := W_2 J_2 V_1. \tag{17}$$

D. Summary minimal realization algorithm: single switch case

Overall, the algorithm is summarized as follows.

Step 1a. Compute the reachable subspace $\mathscr{R}_1 = \operatorname{im} R_1$ of first subsystem (A_1, B_1, C_1) and extend the input matrix of the second mode to

$$B_{2,e} := \operatorname{im}[B_2, J_2 R_1].$$

Step 1b. Calculate the KD of $(A_2, B_{2,e}, C_2)$ with corresponding transformation matrix $\mathbb{V}_2$ and left- and right-projectors W_2, V_2 (i.e. the corresponding rows and columns of $\mathbb{V}_2^{-1}$ and $\mathbb{V}_2$) and let

$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = (W_2 A_2 V_2, W_2 B_2, C_2 V_2).$$

Step 2a. Calculate the space $\mathscr{L}_{12} = \mathscr{R}_1 \cap \mathscr{K}_{12} =: \operatorname{im} L_{12}$ of additional observable states, where $\mathscr{K}_{12} = \operatorname{im} K_{12}$ for some full column rank matrix $K_{12} \in \mathbb{R}^{n_1 \times n_2^J}$ such that $J_2 K_{12} = V_2^J$ for a full column rank matrix $V_2^J \in \mathbb{R}^{n_2 \times n_2^J}$ with $\operatorname{im} V_2^J := \operatorname{im} V_2 \cap \operatorname{im} J_2$. Then extend the output matrix of the first mode as

$$C_{1,e} := \operatorname{im} \begin{bmatrix} C_1 \\ L_{12}^\top \end{bmatrix}.$$

Step 2b. Calculate the KD of $(A_1, B_1, C_{1,e})$ with corresponding transformation matrix $\mathbb{V}_1$ and left- and right-projectors W_1, V_1 (i.e. the corresponding rows and columns of $\mathbb{V}_1^{-1}$ and $\mathbb{V}_1$) and let

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (W_1 A_1 V_1, W_1 B_1, C_1 V_1).$$

Step 3. Calculate the reduced jump $\widehat{J}_2$ according to (17).

The overall reduced switched system is then given by

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_1 = \widehat{A}_1 \widehat{x}_1 + \widehat{B}_1 u, & \text{on } (t_1, t_2), \quad \widehat{x}_1(t_1^+) = 0, \\ \dot{\widehat{x}}_2 = \widehat{A}_2 \widehat{x}_2 + \widehat{B}_2 u, & \text{on } (t_2, t_f), \quad \widehat{x}_2(t_2^+) = \widehat{J}_2 \widehat{x}_1(t_2^-). \end{cases}$$

The steps of the algorithm is illustrated by the following example.

Example 8 (Example 2 revisited): We now apply the reduction method to Example 2.

Step 1a. $R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ (reachable space of 1st mode).

Step 1b. Via the KD of the extended 2nd mode $(A_2, [B_2, J_2 R_1], C_2]$ we obtain the left- and right-projectors $W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the corresponding reduced 2nd mode

$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = (W_2 A_2 V_2, W_2 B_2, C_2 V_2) = \left(\begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right).$$

Step 2a. $\operatorname{im} R_1 \cap \operatorname{im} K_{12} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} =: \operatorname{im} L_{12}$; $\operatorname{im} K_{12} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\operatorname{im}(J_2 K_{12}) = \operatorname{im} V_2^J := \operatorname{im} J_2 \cap \operatorname{im} V_2 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Step 2b. Via the KD of the extended 1st mode $(A_1, B_1, [C_1^\top, L_{12}]^\top)$ we obtain the left- and right-projectors $W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the corresponding reduced 1st mode

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (W_1 A_1 V_1, W_1 B_1, C_1 V_1) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right).$$

Step 3. The reduced jump map is $\widehat{J}_2 = W_2 J_2 V_1 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$.

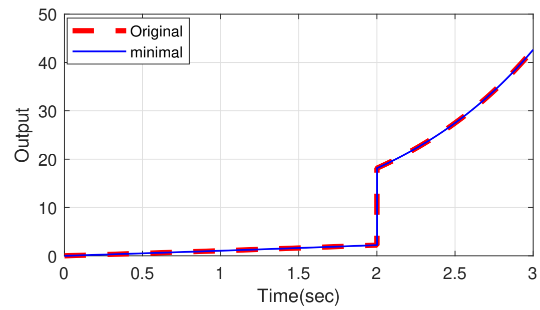

Fig. 1. Outputs of original system and its minimal system.

Figure 1 shows the output of the original and reduced switched system for input $u(t) = 1$ and switching signal (2) with switching time $t_2 = 2$ and clearly both outputs coincide.

Consider now Example 2 with the reversed switching signal

$$\sigma_2(t) = \begin{cases} 2, & \text{on } (t_1, t_2), \\ 1, & \text{on } (t_2, t_f), \end{cases} \tag{18}$$

and jump matrix $J_1 = J_2$. Then the minimal realizations is obtained as follows.

Step 1a. $R_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (reachable space of 1st mode).

Step 1b. Via the KD of the extended 2nd mode $(A_1, [B_1, J_1 R_1], C_1]$ we obtain the left- and right-projectors $W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and the corresponding reduced 2nd mode

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (W_2 A_1 V_2, W_2 B_1, C_1 V_2) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right).$$

Step 2a. $\operatorname{im} R_1 \cap \operatorname{im} K_{12} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =: \operatorname{im} L_{12}$; $\operatorname{im} K_{12} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\operatorname{im}(J_2 K_{12}) = \operatorname{im} V_2^J := \operatorname{im} J_2 \cap \operatorname{im} V_2 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Step 2b. Via the KD of the extended 1st mode $(A_2, B_2, [C_2^\top, L_{12}]^\top)$ we obtain the left- and right-projectors $W_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and the corresponding reduced 1st mode

$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = (W_1 A_2 V_1, W_1 B_2, C_2 V_1) = \begin{bmatrix} 0.1 & 1 & 0 \end{bmatrix}.$$

Step 3. The reduced jump map is $\widehat{J}_1 = W_2 J_1 V_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

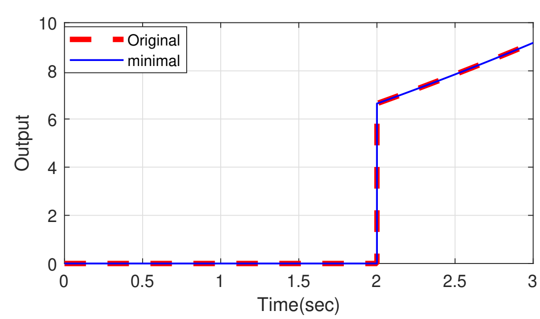

Fig. 2. Outputs of original system and its minimal system.

Figure 2 shows the output of the original and reduced switched system for input $u(t) = 1$ and switching signal (18) with switching time $t_2 = 2$ and clearly both outputs coincide.

E. Correctness of algorithm

Theorem 9: Consider the switched system Σ_σ with the single-switch switching signal (2) and the reduced system $\widehat{\Sigma}_\sigma$ obtained via the above algorithm. Then both systems are input-output equivalent in the sense of Definition 1.

Proof: Consider the switched system Σ_σ as in (13) then the output equation is given by

$$y_\sigma(t) = \begin{cases} \int_{t_1}^{t} C_1 e^{A_1(t-\tau)} B_1 u(\tau) d\tau, \; x_1(t_1^+) = 0, \\ \underbrace{C_2 e^{A_2(t-t_2)} J_2 x_1(t_2^-)}_{y_J} + \int_{t_2}^{t} C_2 e^{A_2(t-\tau)} B_2 u(\tau) d\tau. \end{cases}$$

From Lemma 6, we have that

$$\int_{t_1}^{t} C_1 e^{A_1(t-\tau)} B_1 u(\tau) d\tau = \int_{t_1}^{t} \widehat{C}_1 e^{\widehat{A}_1(t-\tau)} \widehat{B}_1 u(\tau) d\tau.$$

For jump map, original and its minimal system is given by

$$\begin{aligned} y_J = C_2 e^{A_2(t-t_2)} J_2 x_1(t_2^-) &= C_2 V_2 e^{W_2 A_2 V_2 (t-t_2)} W_2 J_2 V_1 \widehat{x}_1(t_2^-) \\ &= \widehat{C}_2 e^{\widehat{A}_2(t-t_2)} \widehat{J}_2 \widehat{x}_1(t_2^-) = \widehat{y}_J. \end{aligned}$$

Again from Lemma 5, we have that

$$y_J + \int_{t_2}^{t} C_2 e^{A_2(t-\tau)} B_2 u(\tau) d\tau = \widehat{y}_J + \int_{t_2}^{t} \widehat{C}_2 e^{\widehat{A}_2(t-\tau)} \widehat{B}_2 u(\tau) d\tau.$$

The above results conclude that $y_\sigma(t) = \widehat{y}_\sigma(t)$, $\forall t$, where $\widehat{y}_\sigma(t)$ is the output of reduced system $\widehat{\Sigma}_\sigma$. ■

Theorem 10: The reduced system $\widehat{\Sigma}_\sigma$ has minimal total dimension under all possible input-output equivalent system of Σ_σ, provided Conjecture 7 is true.

Proof: Because of space limitations we only give a sketch of the proof. Consider a system $\widetilde{\Sigma}_\sigma$ with modes $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1)$, $(\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2, \widetilde{J}_2)$ which is input-output equivalent to Σ_σ.

First of all, this implies that the second mode $(\widetilde{A}_2, \widetilde{B}_2, \widetilde{C}_2)$ with initial values $\widetilde{x}_0 \in \widetilde{J}_2 \widetilde{\mathscr{R}}_1$ ($\widetilde{\mathscr{R}}_1$ is the reachable space of 1st mode) is input-output equivalent with the original second mode (A_2, B_2, C_2) with initials value $x_0 \in J_2 \mathscr{R}_1$ (where $\mathscr{R}_1$ is the reachable space of the first mode). Hence Lemma 5 yields $\widetilde{n}_2 \geq \widehat{n}_2$.

Furthermore, input-output equivalence implies that for $\mathscr{L}_{12}$ as calculated in Step 2a, the left projector $\widetilde{W}_1$ used to obtain $\widetilde{A}_1$, has to satisfy $\ker \widetilde{W}_1 \cap \mathscr{L}_{12} = \{0\}$. Hence Conjecture 7 implies $\widetilde{n}_1 \geq \widehat{n}_1$.

Altogether we therefore have

$$\dim \widehat{\Sigma}_\sigma := (\widehat{n}_1 + \widehat{n}_2) \leq (\widetilde{n}_1 + \widetilde{n}_2) := \widetilde{\Sigma}_\sigma.$$

Therefore, $\widehat{\Sigma}_\sigma$ has minimal total dimension under all possible input-output equivalent system of Σ_σ. ■

IV. Conclusion

In this paper, we have presented preliminary results concerning the minimal realization of switched linear systems. The key novelty is the viewpoint of the switched systems as a piecewise-constant time-varying system (with known time-variance). We have focused here on the single-switch case and obtained a minimal realization by first reducing an input-extended version of the second mode, then reducing an output-extended version of the first mode and finally reduced the jump-map between mode 1 and 2. The proposed method seems also suitable for more than one switch; the details are currently under investigation. It also seems possible to extend the underlying ideas to the discrete time setup which is another topic of ongoing investigations.

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