# A forward approach to controllability of switched DAEs

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### Introduction

In this note we consider *switched differential algebraic equations* (switched DAEs) of the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p,n,m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in [2]. We study controllability of (1) on the interval $[t_0,t_f)$ where (1) is controllable if for all initial values there exists an input such that $x_u(t_f^-,x_0) = 0$. Furthermore, we assume that the switching signal only has finitely many switches in that interval and assume that the switching signal is of the form $\sigma(t) = p$ if $t \in [t_p,t_{p+1})$.

### Problem setting

Necessary and sufficient conditions for controllability of a switched DAE are given in [1]. However, these conditions involve a backward approach, where all calculations redone again if the length of the interval increases. This abstract introduces necessary and sufficient conditions for controllability of switched DAEs that only depend on calculations that run forward in time.

Since controllability implies the existence of an input such that the state is zero at $t_f^-$ for any initial condition, it means that $\min_u x_u(t_f^-,x_0) = 0$ for all $x_0$. Hence in order to verify whether the system is controllable, we compute the minimum norm of the state at the end of the interval $[t_0,t_f)$. To that extent, we first introduce certain projectors for a given switched DAE. Let $\mathcal{C}_i$ be the controllable space of the $i^{\text{th}}$ mode. Then $\Pi_{\mathcal{C}_i^\perp}$ is the projector onto $\mathcal{C}_i^\perp$ along $\mathcal{C}_i$. These projectors project solutions on the interval $(t_i,t_{i+1})$ to elements of the augmented consistency space of the $i^{\text{th}}$ mode. This allows us to compute the $x_0$-uncontrollable orthogonal subspace, defined as follows.

**Definition 1** *Consider the system* (1). *The* $x_0$-*uncontrollable orthogonal subspace* $\Psi_i$ *is defined by the following sequence*

$$\Psi_0 = \operatorname{im} \Pi_{\mathcal{C}_0^\perp} e^{A_0^{\mathit{diff}}(t_1-t_0)} \Pi_0,$$
$$\Psi_{i+1} = \operatorname{im} \Pi_{\mathcal{C}_{i+1}^\perp} e^{A_{i+1}^{\mathit{diff}}(t_{i+1}-t_i)} \Pi_{i+1} \Psi_i.$$

In addition to the $x_0$-uncontrollable orthogonal subspace we also introduce the reachable space, which can be computed by the algorithm given by Proposition 1.

**Definition 2** *Consider the system* (1). *The time* $t_i$ *reachable space is defined by*

$$\mathcal{R}^{[0,t_i]} = \{x \in \mathbb{R}^n \mid \exists u \text{ s.t. } x_u(t_i,0) = x\}.$$

**Proposition 1** *Consider the following sequence of sets:*

$$\mathcal{S}_0 = \mathcal{C}_0,$$
$$\mathcal{S}_{i+1} = e^{A_{i+1}^{\mathit{diff}}(t_{i+2}-t_{i+1})} \Pi_{i+1} \mathcal{S}_i + \mathcal{C}_{i+1},$$

*then we have* $\mathcal{S}_i = \mathcal{R}^{[0,t_{i+1}]}$ *for* $i \in \{0,1,\dots,\mathtt{n}\}$.

It turns out that we can decompose a solution, evaluated before the switching instance, as follows.

**Lemma 1** *Consider the system* (1). *If* $x_u(t,x_0)$ *is a solution of* (1)*, then we can compose the solution befor each switching instance as follows.*

$$x_u(t_i^-,x_0) = \psi + \eta, \qquad \psi \in \Psi_{i-1}, \eta \in \mathcal{R}^{[t_0,t_i)}$$

### Main result

If the system is controllable, there must exist an input such that $x_u(t_f^-,x_0) = 0$. In terms of the decomposition of Lemma 1 this yields the following theorem.

**Theorem 1** *Consider the switched DAE* (1). *The initial condition* $x_0 \in \mathcal{R}$ *of the switched system is controllable if and only if* $\Psi_{\mathtt{n}} \subseteq \mathcal{R}^{[t_0,t_f)}$.

Note that if the conditions in Theorem 1 are satisfied for all basis vectors of the state-space, then the condition is satisfied for all elements of the state-space. Furthermore, both the computation of the $\Psi_{\mathtt{n}}$ and $\mathcal{R}^{[t_0,t_f)}$ run forward in time, allowing for an extension of $[t_0,t_f)$ without much computational effort.

### References

[1] Ferdinand Küsters, Markus G.-M. Ruppert, and Stephan Trenn. Controllability of switched differential-algebraic equations. *Syst. Control Lett.*, 78(0):32 – 39, 2015.

[2] Stephan Trenn. *Distributional differential algebraic equations*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany, 2009.