Optimal control of DAEs with unconstrained terminal costs

Paul Wijnbergen and Stephan Trenn

Abstract— **This paper is concerned with the linear quadratic optimal control problem for impulse controllable differential algebraic equations on a bounded half open interval. With respect to the cost functional, a general positive semi-definite weight matrix is considered in the terminal cost. It is shown that for this problem, there generally does not exist an input that minimizes the cost functional. First it is shown that the problem can be reduced to finding an input to an index-1 DAE that minimizes a different quadratic cost functional. Second, necessary and sufficient conditions in terms of matrix equations are given for the existence of an optimal control are stated.**

I. INTRODUCTION

In this paper, we consider the problem of finding an input u that minimizes the cost functional

$$J^-(x,u)=\int_{t_0}^{t_f}\left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}\right) + x(t_f^-)^\top P x(t_f^-) \quad (1)$$

on a half open interval $[t_0, t_f)$ for some positive definite $R = R$ and positive semi-definite $Q = Q$ and $P = P^\top$, subject to differential algebraic equation (DAE)

$$E\dot{x} = Ax + Bu, \qquad x(0^-) = x_0, \quad (2)$$

which is assumed to be impulse controllable and where, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, for $n, m \in \mathbb{N}$, $x : [t_0, t_f) \to \mathbb{R}^n$ is the state and $u : [t_0, t_f) \to\in \mathbb{R}^m$ is the input. Note that as the weight matrix in (1) is symmetric and positive semi definite we can decompose it as

$$\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} C^\top \\ D^\top \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix},$$

such that the running cost in (1) can be regarded as the integral over the norm of the output $y(t) = Cx(t) + Du(t)$. Besides positive semi-definiteness of the weight matrix P, no further assumptions are made on the terminal cost. This is in contrast to the assumption commonly made in the literature that the weight matrix only penalizes differential states, *i.e.,* is of the form $x(t_f)E^\top PEx(t_f)$ [1]–[3] or no terminal cost is considered with an infinite time horizon [4]–[6]. Also note that whereas commonly a closed interval is of interest, in this paper a half open interval is considered. Consequently, the terminal cost penalizes $x(t_f^-)$. However the algebraic states of (2) can possibly be controlled instantaneously and as a result $x(t_f^-)$ is not necessarily equal to $x(t_f)$ or even well defined such that an optimal solution might fail to exist as is shown by the following example.

Example 1: Consider the DAE given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} 0 \\ -1 \end{bmatrix}u, \quad x(t_0^-) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad (3)$$

for some $x_0 \in \mathbb{R}$ and the cost functional

$$J^-(x,u)=\int_{t_0}^{t_f}(x_1^2 + x_2^2 + u^2) + (x_1(t_f^-) + x_2(t_f^-))^2. \quad (4)$$

Using the structure of the DAE and considering $t_0 = 0$ and $t_f = 1$ this simplifies to

$$J^-(x,u) = x_0^2 + 2\int_0^1 u(\tau)^2 d\tau + (x_0 + u(1^-))^2 \geqslant x_0^2.$$

By choosing $u(t) = 0$ on $[0, 1-\varepsilon)$ for some $\varepsilon > 0$ and $u(t) = -x_0$ on $[1-\varepsilon, 1)$, we obtain

$$J(x,u) = x_0^2 + 2\int_{1-\varepsilon}^1 u(t)^2 = (1+2\varepsilon)x_0^2.$$

This shows that $\inf_u J(x,u) = x_0^2$. However for any input[1] for which $u(1^-) = x_0 \neq 0$ we have that $\inf_u J(x,u) < J(x,u)$, because $\int_0^1 u^2 > 0$. Hence there does not exists an optimal control.

However, in the case that the terminal cost only penalizes the algebraic states, *i.e.,* the cost functional is given by

$$J^-(x,u) = \int_0^1 (x_1^2 + x_2^2 + u^2) + x_2(1^-)^2. \quad (5)$$

it is obvious that the optimal input is given by $u(t) = 0$ for all $t \in [0, 1)$. This shows that a terminal cost of the form $x(t_f)^T E^\top PEx(t_f)$ is only sufficient for existence of an optimal control, but not necessary. ◇

The problem described above is motivated by the study of linear quadratic optimal control of *switched DAEs*. By assuming that (2) is the first mode of *e.g.,* a single switched DAE defined on the interval $[t_0, \infty)$ and regarding t_f as the switching time, the terminal cost represents the cost originating from the system on the interval $[t_f, \infty)$. In that case, the weight matrix P has no direct structural relation to the DAE (2) and can only be assumed to be positive semi-definite.

The literature on optimal control of non-switched DAEs is quite mature, (besides the already mentioned literature) see *e.g.,* [7], [8], and several structural properties of switched DAEs have been investigated recently [9], [10]. Also results on optimal control of switched ordinary differential equations have appeared *e.g.,* [11]–[14]. However, to the best of the authors knowledge, optimal control of switched DAEs has

The authors are with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, Nijenborgh 9, 9747 AG, Groningen, The Netherlands.
Email: p.wijnbergen@rug.nl; s.trenn@rug.nl
This work was supported by the NWO Vidi-grant 639.032.733.

[1] By writing $u(1^-)$ we implicitely assume that $t = 1$ is a left-Lebesgue point of u, see Section II-B

not been studied yet. This paper can thus be regarded as a step towards the optimal control of switched DAEs. The main contributions can be summarized as follows.

The aim of this paper is to state conditions on the existence of inputs that minimize (1). First, by applying a preliminary feedback, we will show that minimizing (1) subject to (2) is equivalent to minimizing a different quadratic cost functional subject to an index-1 DAE. Second, we will show that if there exists an optimal control, it is a feedback. Then, using a completion of the squares formula we show that the cost functional can be expressed in terms of a solution to a Riccati differential equation and a quadratic form. Finally we state results on the existence of a solution that minimizes (1) in terms of matrix equations.

The remainder of this paper is structured as follows. The mathematical preliminaries are given in Section II. The main results are stated in Section III. Conclusions and a discussion on future work are given in Section IV.

II. Mathematical preliminaries

In this section we recall some notation and properties related to the non-switched DAE (2).

A. Properties and definitions for regular matrix pairs

In the following, we call a matrix pair (E, A) and the associated DAE (2) *regular* iff the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form* [15].

Proposition 2: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{6}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.
The matrices S and T can be calculated by using the so-called *Wong sequences* [15], [16]:

$$\begin{array}{lll} \mathcal{V}_0 := \mathbb{R}^n, & \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), & i = 0, 1, ... \\ \mathcal{W}_0 := \{0\}, & \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), & i = 0, 1, ... \end{array} \tag{7}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup_i \mathcal{W}_i. \tag{8}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (6) holds.

Based on the Wong sequences we define the following projector and selectors.

Definition 3: Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (6). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Furthermore, the *differential selector* and *impulse selector* are respectively given by

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \quad \Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$\begin{array}{ll} A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, & E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E, \\ B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B, & B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B. \end{array}$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). Furthermore if a classical solution to (2) exists, it satisfies

$$\begin{aligned} x(t) &= e^{A^{\text{diff}}t}\Pi x_0 + \int_0^t e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau)\, d\tau \\ &\quad - \sum_{i=0}^{\nu} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t), \\ &:= x^{\text{diff}}(t) - \sum_{i=0}^{\nu} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t) \\ &:= x^{\text{diff}}(t) + x^{\text{imp}}(t), \end{aligned} \tag{9}$$

where x^{diff} iand x^{imp} are referred to as the differential and algebraic states, respectively.

B. Distributional solutions and left-, right- and impulsive time evaluation

For studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [17] as the solution space, that is, we seek a solution $(x, u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+m}$ to the following initial-trajectory problem (ITP):

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{10a}$$

$$(E\dot{x})_{[0,\infty)} = (Ax + Bu)_{[0,\infty)}, \tag{10b}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_\mathcal{I}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In [17] it is shown that the ITP (10) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. Furthermore, it can be shown that the solution of the ITP (10) considered on $[0, \infty)$ is uniquely determined by $x^0(0^-)$ and does not depend on the whole past trajectory. Because of this, (2) with inconsistent initial condition $x(0^-) = x_0$ can be interpreted as a short hand notation for the ITP (10) for some x^0 with $x^0(0^-) = x_0$.

When considering distributional solutions of the DAE (2), or its corresponding ITP (10), it is not possible to simply evaluate x at some time $t \in \mathbb{R}$, because x operates on the space of test-functions. However, when restricting the distributional solution space to piecewise-smooth distribution the left-sided evaluation $x(t^-)$, right-sided evaluation $x(t^+)$ and impulsive evaluation $x[t]$ are well-defined for any $t \in \mathbb{R}$.

Furthermore, after an index reduction it is not necessary to consider distributional solutions anymore, instead any locally integrable function pair (x,u) is considered a solution if Ex is absolutely continuous and (2), or (10) is satisfied almost everywhere. In that case, when we write $x(t_f^-)$ (or $u(t_f^-)$) we make the implicite assumption that t_f is a left-Lebesgue point of x (or u), *i.e.,*

$$x(t_f^-) := \lim_{h\searrow 0} \frac{1}{h}\int_{t_f-h}^{t_f} x(\tau)d\tau \quad \text{is well defined.}$$

C. Properties of DAEs

Definition 4 ([18]): The DAE (2) is impulse controllable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x,u) of the ITP (10) such that $x(0^-) = x_0$ and $(x,u)[0] = 0$, *i.e.,* the state and the input are impulse free at $t = 0$.

Lemma 5 ([19]): The DAE (2) is impulse controllable if and only if there exists a feedback $u = Kx + v$ such that the closed loop DAE

$$E\dot{x} = (A + BK)x + Bv \tag{11}$$

is index-1.

D. Optimal control of non-switched DAEs

Consider the following cost functional

$$J_E(x,u) = \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) + x(t_f)^\top E^\top P E x(t_f) \tag{12}$$

where $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$ and P are positive semidefinite and $R = R^\top$ is positive definite.

Applying Pontryagin's maximum principle leads to the following necessary conditions.

Theorem 6 ([2]): Consider the impulse controllable DAE (2). Let (x^*, u^*) be a solution minimizing the cost functional (12). Then there exits a costate function $\mu^*(t)$ such that (x^*, u^*, μ^*) satisfies the boundary value problem

$$\begin{bmatrix} A & 0 & B \\ Q & A^\top & S \\ S^\top & B^\top & R \end{bmatrix} \begin{bmatrix} x \\ \mu \\ u \end{bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & -E^\top & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{bmatrix}, \tag{13}$$

$$Ex(t_0) = Ex_0, \quad E^\top \mu(t_f) = E^\top P E x(t_f). \tag{14}$$

Given these results, we can state the following result.

Theorem 7 ([2]): Let $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix}$ and let U and V be matrices such that $U^\top E V = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $V^\top Q V = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$. If (2) is impulse-controllable and $\begin{bmatrix} C_2 & D \end{bmatrix}$ has full rank, then there exists a unique solution to (13) and (14) and hence a unique optimal control.

Finally we conclude this recapitulation by recalling the fact that the optimal control can be expressed in the form of a state-feedback.

Theorem 8 ([3]): Consider the DAE (2) with cost functional (12). If (2) is impulse controllable and stabilizable, and (13) is regular, impulse-free and no finite eigenvalue lies on the imaginary axis, then there exists a solution $(X(t), Y(t))$ to the Generalized Riccati Differential Equation (GRDE) given by

$$\begin{aligned} &E^\top \dot{X}(t) + Y(t)A + A^\top X(t) + Q \\ &\qquad - (S + Y(t)B)R^{-1}(B^\top X(t) + S^\top) = 0, \\ &Y(t)E = E^\top X(t), \end{aligned} \tag{15}$$

with terminal condition $E^\top X(t_f) = E^\top P E$. The optimal cost is quadratic and given by

$$J_E(x^*, u^*) = \frac{1}{2} x_0^\top E^\top X(0) x_0, \tag{16}$$

and the optimal input is given by

$$u^*(t) = K(t)x(t) = -R^{-1}(B^\top X(t) + S^\top)x^*(t).$$

III. Optimal control with unconstrained terminal cost

Equipped with the mathematical preliminaries, we can return the focus on the existence of an input that minimizes (1) subject to (2). As (2) is assumed to be impulse controllable, it follows from Lemma 5 that there exists a preliminary feedback $u = Kx + v$ such that the resulting closed loop system is index-1. The form of cost functional (1) is invariant under this feedback as

$$\begin{aligned} J^-(x,u) &= \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ Kx+v \end{bmatrix} \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} x \\ Kx+v \end{bmatrix} \right) \\ &\qquad + x(t_f^-)^\top P x(t_f^-) \\ &= \int_0^{t_f} \left(\begin{bmatrix} x \\ v \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \right) + x(t_f^-)^\top P x(t_f^-) \\ &=: \bar{J}^-(x,v), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \bar{Q} &= Q + SK + K^\top S^\top + K^\top R K, \\ \bar{S} &= S + K^\top R, \\ \bar{R} &= R \end{aligned}$$

Furthermore, we observe the following.

Lemma 9: Consider the DAE (2) and assume it is impulse controllable. There exists an input $u(\cdot)$ that minimizes (1) if and only if there exists an optimal input $v(\cdot)$ that minimizes $\bar{J}^-(\bar{x}, v)$ given by (17) subject to

$$E\dot{\bar{x}} = \bar{A}\bar{x} + Bv, \qquad \bar{x}(0^-) = x_0, \tag{18}$$

where $\bar{A} := A + BK$.

Proof: Applying a feedback to (2) can be regarded as a change of coordinates

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}. \tag{19}$$

Writing (2) as $\begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$ reveals

$$\begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{x}} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} = \begin{bmatrix} \bar{A} & B \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}.$$

Hence (x,u) solves (2) if and only if, $(\bar{x}, v)$ satisfying (19) solves (18). Furthermore, it follows naturally from (17) that $J^-(x,u) = \bar{J}^-(\bar{x}, v)$. ■

This shows that the existence of an input that minimizes (1) subject to the DAE (2) (which is of arbitrary index), is equivalent to the existence of an input that minimizes (17) subject to the index-1 DAE (18). Its solution $\bar{x}(t)$ can be decomposed as $\bar{x}(t) = \bar{x}^{\text{diff}}(t) + \bar{x}^{\text{imp}}(t) = \bar{x}^{\text{diff}}(t) - \bar{B}^{\text{imp}}v(t)$ and the cost functional can be rewritten further as

$$\begin{aligned}\bar{J}^-(\bar{x}, v) &= \int_0^{t_f} \left(\begin{bmatrix} \bar{x} \\ v \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} \right) + \bar{x}(t_f^-)^\top P \bar{x}(t_f^-) \\ &= \int_0^{t_f} \left(\begin{bmatrix} \bar{x}^{\text{diff}} - \bar{B}^{\text{imp}}v \\ v \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} \bar{x}^{\text{diff}} - \bar{B}^{\text{imp}}v \\ v \end{bmatrix} \right) \\ &\quad + \bar{x}(t_f^-)^\top P \bar{x}(t_f^-), \\ &= \int_0^{t_f} \left(\begin{bmatrix} \bar{x}^{\text{diff}} \\ v \end{bmatrix}^\top \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^\top & \tilde{R} \end{bmatrix} \begin{bmatrix} \bar{x}^{\text{diff}} \\ v \end{bmatrix} \right) + \bar{x}(t_f^-)^\top P \bar{x}(t_f^-), \end{aligned} \tag{20}$$

where

$$\begin{aligned} &\tilde{Q} = \bar{Q}, \quad \tilde{S} = \bar{S} - \bar{Q}\bar{B}^{\text{imp}}, \\ &\tilde{R} = \bar{R} - \bar{B}^{\text{imp}\top}\bar{S} - \bar{S}^\top \bar{B}^{\text{imp}} + \bar{B}^{\text{imp}\top}\bar{Q}\bar{B}^{\text{imp}}. \end{aligned}$$

For the sake of our analaysis, we will assume invertibility of $\tilde{R}$ throughout the remainder of the paper. Note that this is not necessarily the case, as it might depend on the feedback matrix K that is used to reduce the index of (2). However, under an assumption which is similar to one of the conditions of Theorem 7, positive definiteness of $\tilde{R}$ is guaranteed regardless of the choice of K.

Lemma 10: Let $\bar{W}$ be any matrix of full rank with $\operatorname{im} \bar{W} = \ker E$. If $\begin{bmatrix} C\bar{W} & D \end{bmatrix}$ has full rank, then $\tilde{R}$ is positive definite.

Proof: Note that

$$\begin{aligned} \tilde{R} &= \begin{bmatrix} \bar{B}^{\text{imp}} \\ I \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & -\bar{S} \\ -\bar{S}^\top & \bar{R} \end{bmatrix} \begin{bmatrix} \bar{B}^{\text{imp}} \\ I \end{bmatrix} \\ &= \begin{bmatrix} \bar{B}^{\text{imp}} \\ I \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ K & I \end{bmatrix}^\top \begin{bmatrix} Q & -S \\ -S^\top & R \end{bmatrix} \begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} \bar{B}^{\text{imp}} \\ I \end{bmatrix} \\ &= \begin{bmatrix} I \\ I \end{bmatrix}^\top \begin{bmatrix} \bar{B}^{\text{imp}} & 0 \\ K\bar{B}^{\text{imp}} & I \end{bmatrix}^\top \begin{bmatrix} Q & -S \\ -S^\top & R \end{bmatrix} \begin{bmatrix} \bar{B}^{\text{imp}} & 0 \\ K\bar{B}^{\text{imp}} & I \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \\ &= \begin{bmatrix} \bar{B}^{\text{imp}} \\ I + K\bar{B}^{\text{imp}} \end{bmatrix}^\top \begin{bmatrix} Q & -S \\ -S^\top & R \end{bmatrix} \begin{bmatrix} \bar{B}^{\text{imp}} \\ I + K\bar{B}^{\text{imp}} \end{bmatrix} \\ &= \begin{bmatrix} \bar{B}^{\text{imp}} \\ I + K\bar{B}^{\text{imp}} \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \bar{B}^{\text{imp}} \\ I + K\bar{B}^{\text{imp}} \end{bmatrix}. \end{aligned}$$

However, as $E\bar{B}^{\text{imp}} = \bar{S}^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \bar{T}^{-1}\bar{T} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \bar{S}B = 0$, it follows that $\operatorname{im} \bar{B}^{\text{imp}} \subseteq \ker E = \operatorname{im} \bar{W}$. Furthermore, since $\begin{bmatrix} C\bar{W} & D \end{bmatrix}$ has full rank it follows that

$$\begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \bar{B}^{\text{imp}} \\ I + K\bar{B}^{\text{imp}} \end{bmatrix} v \neq 0,$$

for all $v \in \mathbb{R}^m$ and thus $v^\top \tilde{R} v > 0$ for all v, which proves that $\tilde{R}$ is positive definite. ■

Remark 11: The assumption that $\begin{bmatrix} C\bar{W} & D \end{bmatrix}$ has full rank and (2) is impulse controllable is in fact the same as Assumption 2 in [2], in which impulse controllability and full rank of $\begin{bmatrix} C_2 & D \end{bmatrix}$ was assumed, where $V^\top Q V = \begin{bmatrix} C_1 & C_2 \end{bmatrix}^\top \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ for some matrices V and U such that $U^\top E V = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Furthermore, note that $\bar{x}^{\text{diff}} = \bar{\Pi}\bar{x}$, hence we can rewrite (20) again directly in terms of $\bar{x}$ by replacing $\tilde{Q}$ by $\bar{\Pi}^\top \tilde{Q} \bar{\Pi}$ and $\tilde{S}$ by $\tilde{S}\bar{\Pi}$. Altogether, we have shown that the existence of an input that minimizes (1) subject to (2) is equivalent to the following simpler problem.[2]

Problem 1: Consider the DAE (2) and assume it is of index-1 and with consistency projector Π. Furthermore, consider the cost function

$$\begin{aligned} J_\Pi^-(x, u) = \int_{t_0}^{t_f} &\left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) \\ &+ x(t_f^-)^\top P x(t_f^-), \end{aligned} \tag{21}$$

where $R = R^\top$ is positive definite, $Q = Q^\top$ and $P = P^\top$ are positive semidefinite. Find an input $u(\cdot)$ that minimizes (21) subject to (2). Any such input we will call *optimal input* with corresponding *optimal solution* x.

Due to the quadratic nature of the cost functional (21) we can prove that if there exists an input that minimizes (21), it is linear in the optimal state, *i.e.,* the optimal input is a feedback.

Lemma 12: If there exists a solution $u(\cdot)$ to Problem 1, then $u(t) = K(t)x(t)$ for some $K(t) \in \mathbb{R}^{n \times n}$.

Proof: First we will show that the map $x_0 \mapsto u$ is linear, where u is minimizing (21) subject to (2); in particular, we will show that λu is the optimal control for the initial value λx_0 and that for any optimal inputs u_x, u_z corresponding to any initial values $x_0, z_0 \in \mathbb{R}^n$ the input $u_x + u_z$ is optimal for the initial value $x_0 + z_0$. To that extent, let $V(x_0, t_0)$ be the value function defined by

$$\begin{aligned} V(x_0, t_0) &= \min_u J_\Pi^-(x, u) \\ &= \min_u \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) \\ &\quad + x(t_f^-)^\top P x(t_f^-), \end{aligned}$$

i.e., the cost for the optimal input input u and the corresponding trajectory x with initial condition $x(t_0^-) = x_0$. Applying the input λu to an initial condition λx_0 results in a trajectory λx, due to the linearity of solutions of (2). This means that $J_\Pi^-(\lambda x, \lambda u) = \lambda^2 J_\Pi^-(x, u)$ for any $\lambda \in \mathbb{R}$ and we can conclude that

$$\lambda^2 V(x_0, t_0) = \lambda^2 J_\Pi^-(x, u) = J_\Pi^-(\lambda x, \lambda u) = V(\lambda x_0, t_0),$$

which shows that λu is optimal for λx. Furthermore, since $J_\Pi^-(x+z, u+v) + J_\Pi^-(x-z, u-v) = 2J_\Pi^-(x, u) + 2J_\Pi^-(z, v)$ we have

$$\begin{aligned} V(x_0 + z_0, t_0) &+ V(x_0 - z_0, t_0) \\ &\leqslant J_\Pi^-(x+z, u+v) + J_\Pi^-(x-z, u-v) \\ &= 2J_\Pi^-(x, u) + 2J_\Pi^-(z, v). \end{aligned}$$

[2]Note that the matrices Q, S, R appearing in Problem 1 are not equal to the original matrices Q, S, R appearing in (1), but need to be adjusted according to the derivation of (17) and (20), furthermore, Π is the consistence projector *after* the application of the possible index-reducing feedback. However, the terminal cost matrix P is *not* effected by all these preliminary transformations.

Which means that $V(x_0 + z_0, t_0) + V(x - z, t_0) \leqslant 2V(x_0, t_0) + 2V(z_0, t_0)$. Conversely

$$\begin{aligned} 2V(x_0,t_0) &+ 2V(z_0,t_0) \\ &\leqslant 2J_\Pi^-(x,u) + 2J_\Pi^-(z,v) \\ &= J_\Pi^-(x+z,u+v) + J_\Pi^-(x-z,u-v) \end{aligned}$$

and hence we can conclude $V(x_0+z_0,t_0)+V(x_0-z_0,t_0) = 2V(x_0,t_0) + 2V(z_0,t_0)$. Furthermore, if u_x is the optimal input for x and u_z is the optimal input for z then

$$\begin{aligned} V(x_0-z_0,t_0)+V(x_0+z_0,t_0) &= 2V(x_0,t_0)+2V(z_0,t_0) \\ &= 2J_\Pi^-(x,u_x)+2J_\Pi^-(z,u_z) \\ &= J_\Pi^-(x+z,u_x+u_z) \\ &\quad + J_\Pi^-(x-z,u_x-u_z). \end{aligned}$$

Since $V(x_0 + z_0, t_0) \leqslant J_\Pi^-(x + z, u_x + u_z)$ and similarly $V(x_0 - z_0, t_0) \leqslant J_\Pi^-(x - z, u_x - u_z)$, it follows that

$$\begin{aligned} 0 &\leqslant J_\Pi^-(x+z,u_x+u_z) - V(x_0+z_0,t_0) \\ &= V(x_0-z_0,t_0) - J_\Pi^-(x-z,u_x-u_z) \leqslant 0, \end{aligned}$$

and thus

$$V(x_0+z_0,t_0) = J_\Pi^-(x+z,u_x+u_z)$$

which also shows that $u_x + u_z$ is optimal for $x + z$. Hence there exists a linear map between the optimal trajectory and the optimal input. In particular, the map $x(t_0^-) = x_0 \mapsto u(t_0)$ is linear, *i.e.,* there exists a matrix $K(t_0) \in \mathbb{R}^{m\times n}$ such that $u(t_0) = K(t_0)x(t_0^-)$.

From the dynamic programming principle [20] it follows that $u_{[\tau,t_f)}$ is the optimal control for the cost function (20) considered on the interval $[\tau, t_f)$ for any $\tau \in [t_0, t_f)$, hence by replacing the initial time t_0 in the above argumentation by $\tau \in [t_0, t_f)$ we can conclude that for every $\tau \in [t_0, t_f)$ a matrix $K(\tau) \in \mathbb{R}^{m\times n}$ exists such that the optimal control satisfies $u(\tau) = K(\tau)x(\tau^-)$. ■

Corollary 13: If Problem 1 has a solution u then $u(t) = K(t)x^{\text{diff}}(t)$ for some $K(t)$.

Proof: Note that as $\Pi x = x^{\text{diff}}$ and $x^{\text{imp}} = -B^{\text{imp}}u$ the cost functional (21) written as

$$\begin{aligned} J_\Pi^-(x,u) = \int_{t_0}^{t_f} & \left(\begin{bmatrix} x^{\text{diff}} \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q\Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x^{\text{diff}} \\ u \end{bmatrix} \right) \\ & + x^{\text{diff}}(t_f^-)^\top P x^{\text{diff}}(t_f^-) \\ & \quad - 2x^{\text{diff}\top}(t_f^-) P B^{\text{imp}} u(t_f^-) \\ & \quad\; + u(t_f^-)^\top B^{\text{imp}\top} P B^{\text{imp}} u(t_f^-) \\ = \bar{J}_\Pi^-(x^{\text{diff}}, u), & \end{aligned} \tag{22}$$

Hence it follows analogously as in the proof of Lemma 12 that $u(t) = K(t)x^{\text{diff}}(t)$. ■

Given the fact that if there exists an optimal control it is linear in x^{diff}, leads to the following auxiliary results.

Corollary 14: Assume that there exists an input u that minimizes (21) subject to the index-1 DAE (2). Let $x = x^{\text{diff}} + x^{\text{imp}}$ be the corresponding trajectory. If $x^{\text{diff}}(\tau^-) = 0$ for some $\tau \in [t_0, t_f)$ then $x^{\text{diff}}(t^-) \equiv 0$ on $[t_0, t_f)$. Consequently, $x^{\text{diff}}(\tau^-) = 0$ for some $\tau \in [t_0, t_f)$ if and only if $x^{\text{diff}}(t_0^-) = 0$.

Proof: Since $x(t)$ is a solution to the DAE (2), it follows that for the optimal input $u = K(t)x^{\text{diff}}(t)$, the optimal x^{diff} is a solution to the ODE

$$\dot{x}^{\text{diff}}(t) = \left(A^{\text{diff}} + B^{\text{diff}}K(t)\right)x^{\text{diff}}(t),\ x^{\text{diff}}(0^-) = \Pi x_0.$$

Consequently, if $x^{\text{diff}}(\tau^-) = 0$ for some $\tau \in [t_0, t_f)$, $x^{\text{diff}}(t) = 0$ for all $t \in [\tau, t_f)$. Solving the ODE backwards in time yields that $x^{\text{diff}}(t_0^-) = 0$ and hence $x^{\text{diff}} \equiv 0$. ■

Finally, we can show that for any element $p \in \operatorname{im}\Pi$ there exists an initial condition and an input such that $J_\Pi^-(x,u)$ is minimal and $x(t_f^-) = p$.

Corollary 15: Assume that Problem 1 has a solution for all initial values. Then for any $p \in \operatorname{im}\Pi$, there exists an initial value x_0 such that the optimal trajectory satisfies $x(t_0^-) = x_0$ and $x^{\text{diff}}(t_f^-) = p$.

Proof: As the optimal input $u(t) = K(t)x^{\text{diff}}(t)$ the optimal x^{diff} is a solution to an ODE. Time reversing this ODE given the final state $x^{\text{diff}}(t_f^-) = p$ yields the initial $x^{\text{diff}}(t_0^-) = q$. Then it follows that any initial condition $x(t_0^-) = x^{\text{diff}}(t_0^-) + x^{\text{imp}}(t_0^-)$ for which $x^{\text{diff}}(t_0^-) = q$ yields $x^{\text{diff}}(t_f^-) = p$. ■

Next we will derive some necessary conditions on the input u. To do so, we will first show that the extension of u to $[t_0, t_f]$ by defining $u(t_f) = u(t_f^-)$ has to minimize another optimization problem.

Lemma 16: An input u solves Problem 1 if, and only if, $\bar{u}$ defined on $[t_0, t_f]$ as $\bar{u}_{[t_0,t_f)} = u_{[t_0,t_f)}$ and $\bar{u}(t_f) = u(t_f^-)$ minimizes

$$\begin{aligned} J_\Pi(\bar{x},\bar{u}) = \int_{t_0}^{t_f} & \left(\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q\Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \right) \\ & + \bar{x}(t_f)^\top P \bar{x}(t_f), \end{aligned} \tag{23}$$

where $\bar{x}$ is the corresponding solution of DAE (2) on $[t_0, t_f]$ with input $\bar{u}$.

Proof: ($\Leftarrow$) Assume there exists input $\bar{u}(t)$ that minimizes (23) for which $\bar{u}(t_f) = \bar{u}(t_f^-)$. Let $\bar{x}$ be the corresponding optimal trajectory. Since $\bar{x}^{\text{diff}}$ is continuous on $[t_0, t_f]$, it follows that

$$\begin{aligned} \bar{x}(t_f) &= \bar{x}^{\text{diff}}(t_f) - B^{\text{imp}}\bar{u}(t_f) \\ &= \bar{x}^{\text{diff}}(t_f^-) - B^{\text{imp}}\bar{u}(t_f^-) = \bar{x}(t_f^-). \end{aligned}$$

Consequently, we have that for the input $u = \bar{u}_{[t_0,t_f)}$ that the corresponding trajectory $x = \bar{x}_{[t_0,t_f)}$. Hence $J_\Pi^-(x,u) = J_\Pi(\bar{x},\bar{u})$.

Seeking a contradiction, suppose that there exists an input v and a corresponding trajectory y for which $J_\Pi^-(y,v) < J_\Pi^-(x,u)$ then the input $\bar{v}(t)$ with $\bar{v}(t_f^-) = \bar{v}(t_f)$ and $\bar{v}_{[t_0,t_f)} = v_{[t_0,t_f)}$ yields $J_\Pi(\bar{y},\bar{v}) = J_\Pi^-(y,v) < J_\Pi^-(x,u) = J_\Pi(\bar{x},\bar{u})$, which contradicts the optimality of $\bar{u}$. Hence for all inputs v and corresponding trajectories y we have $J_\Pi^-(x,u) \leqslant J_\Pi^-(y,v)$ and thus u minimizes (21), which proves the desired result.

($\Rightarrow$) Assume that u solves Problem 1 and x is the corresponding optimal trajectory on $[t_0, t_f)$. Then for the input $\bar{u}$

defined as $\bar{u} = u$ on $[t_0, t_f)$ and $\bar{u}(t_f) = u(t_f^-)$ we obtain that $J_\Pi^-(x,u) = J_\Pi(\bar{x}, \bar{u})$. Seeking a contradiction, suppose that there exists an input $\bar{w}$ and a corresponding trajectory $\bar{y}$ defined on $[t_0, t_f]$ for which $J_\Pi(\bar{y}, \bar{w}) < J_\Pi(\bar{x}, \bar{u})$. For $\delta > 0$ define $\bar{w}_\delta(t)$ as $\bar{w}_\delta(t) = \bar{w}(t)$ on $[t_0, t_f - \delta)$ and $\bar{w}_\delta(t) = \bar{w}(t_f)$ on $[t_f - \delta, t_f]$, then $J_\Pi^-(y_\delta, w_\delta) := J_\Pi^-((\bar{y}_\delta)_{[t_0,t_f)}, (\bar{w}_\delta)_{[t_0,t_f)}) = J_\Pi(\bar{y}_\delta, \bar{w}_\delta)$, where $\bar{y}_\delta$ is the solution corresponding to $\bar{w}_\delta$ on $[t_0, t_f]$. Furthermore, for every $\varepsilon > 0$ we find $\delta > 0$ such that $J_\Pi(\bar{y}_\delta, \bar{w}_\delta) < J_\Pi(\bar{y}, \bar{w}) + \varepsilon$. Hence for sufficiently small $\varepsilon > 0$ and corresponding $\delta > 0$, we have $J_\Pi^-(y_\delta, w_\delta) = J_\Pi(\bar{y}_\delta, \bar{w}_\delta) < J_\Pi(\bar{y}, \bar{w}) + \varepsilon < J_\Pi(\bar{x}, \bar{u}) = J_\Pi^-(x, u)$; a contradiction to optimality of (x, u). ∎

Corollary 17: If there exists an input u that solves Problem 1, then the optimal trajectory $x = x^{\text{diff}} + x^{\text{imp}}$ and optimal input u satisfy

$$B^{\text{imp}\top} P x^{\text{diff}}(t_f^-) = B^{\text{imp}\top} P B^{\text{imp}} u(t_f^-).$$

Proof: Since u minimizes (21), by Lemma 16 we have that the extension $\bar{u}$ of u to $[t_0, t_f]$ defined by $\bar{u}(t) = u(t)$ on $[t_0, t_f)$ and $\bar{u}(t_f) = u(t_f^-)$ minimizes

$$J_\Pi(x,u) = \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) + x(t_f)^\top P x(t_f).$$

However, as we can control $x^{\text{imp}}(t_f) = -B^{\text{imp}} \bar{u}(t_f)$ instantly to any state without changing the running cost or $x^{\text{diff}}(t_f)$, it follows that $\bar{u}(t_f)$ is such that it minimizes

$$\begin{aligned} x(t_f)^\top P x(t_f) = {} & x^{\text{diff}\top}(t_f) P x^{\text{diff}}(t_f) \\ & - 2 x^{\text{diff}\top}(t_f) P B^{\text{imp}} \bar{u}(t_f) \\ & + \bar{u}(t_f)^\top B^{\text{imp}\top} P B^{\text{imp}} \bar{u}(t_f) \end{aligned} \tag{24}$$

However, as (24) is a convex function of $\bar{u}(t_f)$ it follows that the global minimum for a given $x^{\text{diff}}(t_f)$ satisfies

$$\begin{aligned} \frac{\partial x(t_f)^\top P x(t_f)}{\partial \bar{u}(t_f)} = {} & - 2 B^{\text{imp}\top} P x^{\text{diff}}(t_f) \\ & + 2 B^{\text{imp}\top} P B^{\text{imp}} \bar{u}(t_f) = 0, \end{aligned}$$

from which it follows that $B^{\text{imp}\top} P x^{\text{diff}}(t_f) = B^{\text{imp}\top} P B^{\text{imp}} \bar{u}(t_f)$. By assumption $\bar{u}(t_f) = \bar{u}(t_f^-) = u(t_f^-)$ and as $\bar{x}^{\text{diff}}$ is continuous it follows that $\bar{x}^{\text{diff}}(t_f) = \bar{x}^{\text{diff}}(t_f^-) = x^{\text{diff}}(t_f^-)$ and thus the result follows. ∎

In the following we will show that we can express the cost for a given trajectory in terms of a quadratic function and the input. This is done by using the completion of the squares formula.

Lemma 18: Consider Problem 1. Then for any input $u(\cdot)$ the cost (21) is given by

$$\begin{aligned} J_\Pi^-(x,u) = {} & x^{\text{diff}\top}(t_0^-) X(t_0^-) x^{\text{diff}}(t_0^-) \\ & + \int_{t_0}^{t_f} \left(\left\| R^{\frac{1}{2}} u + R^{-\frac{1}{2}} (B^{\text{diff}\top} X + S^\top) x^{\text{diff}} \right\|_2^2 \right) \\ & + x(t_f^-)^\top P x(t_f^-) - x^{\text{diff}\top}(t_f^-) X(t_f^-) x^{\text{diff}}(t_f^-), \end{aligned} \tag{25}$$

where $X(\cdot)$ is a solution to the Riccati equation

$$\begin{aligned} \dot{X} = {} & -A^{\text{diff}\top} X - X^\top A^{\text{diff}} - \Pi^\top Q \Pi \\ & + (\Pi^\top S + X^\top B^{\text{diff}}) R^{-1} (B^{\text{diff}\top} X + S^\top \Pi), \end{aligned} \tag{26}$$

on $[t_0, t_f]$.

Proof: For any differentiable and symmetric matrix function $X(\cdot)$ we have that the cost functional can be expressed as

$$\begin{aligned} J_\Pi^-(x,u) & - x^{\text{diff}\top}(t_0) X(t_0) x^{\text{diff}}(t_0) \\ = {} & \int_{t_0}^{t_f} \left(\begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) \\ & + \int_{t_0}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t} \left(x^{\text{diff}\top} X(\cdot) x^{\text{diff}} \right) + x(t_f^-)^\top P x(t_f^-) \\ & - x^{\text{diff}\top}(t_f^-) X(t_f^-) x^{\text{diff}}(t_f^-). \end{aligned}$$

Then, invoking $x^{\text{diff}} = \Pi x$ and $\Pi^2 = \Pi$,

$$\begin{aligned} & \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \Pi^\top Q \Pi & \Pi^\top S \\ S^\top \Pi & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \tfrac{\mathrm{d}}{\mathrm{d}t} (x^{\text{diff}\top} X(\cdot) x^{\text{diff}}) \\ & = x^{\text{diff}\top} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} X + A^{\text{diff}\top} X + X^\top A^{\text{diff}} + \Pi^\top Q \Pi \right) x^{\text{diff}} \\ & \quad + 2 x^{\text{diff}\top} (X B^{\text{diff}} + \Pi^\top S) u + u^\top R u \\ & = \left\| R^{\frac{1}{2}} u + R^{-\frac{1}{2}} (B^{\text{diff}\top} X + S^\top) x^{\text{diff}} \right\|_2^2 + x^{\text{diff}\top} W x^{\text{diff}}, \end{aligned}$$

where

$$\begin{aligned} W := {} & \dot{X} + A^{\text{diff}\top} X + X^\top A^{\text{diff}} + \Pi^\top Q \Pi \\ & - (\Pi^\top S + X^\top B^{\text{diff}}) R^{-1} (B^{\text{diff}\top} X + S^\top \Pi). \end{aligned}$$

This means that if $X(\cdot)$ is a solution to the Riccati equation (26) then $W(t) = 0$ for all $t \in [t_0, t_f]$ and the cost can be expressed as (25). ∎

Proposition 19: Let $X(t)$ be a solution to the Riccati differential equation (26). Then $Y(t) = \Pi^{\text{diff}\top} X(t) \Pi$ is a solution to the generalized Riccati differential equation

$$E \dot{Y} = -A^\top Y - Y^\top A - \Pi^\top Q \Pi + Y^\top B R^{-1} B^\top Y \tag{27}$$

Proof: Let $Y(t) = \Pi^{\text{diff}\top} X(t) \Pi$, then $Y(t)$ is a solution to the generalized Riccati equation

$$\begin{aligned} E^\top \dot{Y} & = E^\top \Pi^{\text{diff}\top} \dot{X}(t) \Pi \\ & = \Pi^\top \dot{X}(t) \Pi \\ & = \Pi^\top (-A^{\text{diff}\top} X - X^\top A^{\text{diff}} - \Pi^\top Q \Pi \\ & \qquad + X B^{\text{diff}} R^{-1} B^{\text{diff}\top} X) \Pi \\ & = -A^{\text{diff}\top} X \Pi - \Pi^\top X^\top A^{\text{diff}} - \Pi^\top Q \Pi \\ & \qquad + \Pi^\top X B^{\text{diff}} R^{-1} B^{\text{diff}\top} X \Pi \\ & = -A^\top Y - Y^\top A - \bar{Q} + Y^\top B R^{-1} B^\top Y, \end{aligned}$$

which proves the desired result. ∎

Combining all auxiliary results leads to the following main result.

Theorem 20: Problem 1 is solvable if, and only if, there exist $X \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{m \times n}$ such that

$$K = -R^{-1} (B^{\text{diff}\top} X + S^\top) \tag{28a}$$

$$B^{\text{imp}\top} P \Pi = B^{\text{imp}\top} P B^{\text{imp}} K \Pi \tag{28b}$$

$$X = \Pi^\top (I - K^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} K) \Pi. \tag{28c}$$

Proof: $(\Rightarrow)$ Suppose that $u(t)$ minimizes (21). By Corollary 13 $u(t) = K(t) x^{\text{diff}}(t)$ and thus by Corollary 17 we have

$$B^{\text{imp}\top} P \Pi x^{\text{diff}}(t_f^-) = B^{\text{imp}} P B^{\text{imp}} K(t_f^-) x^{\text{diff}}(t_f^-). \tag{29}$$

By Corollary 15 equation (29) must hold for any $x^{\text{diff}}(t_f) \in \operatorname{im}\Pi$. Consequently

$$B^{\text{imp}\top} P\Pi = B^{\text{imp}} P B^{\text{imp}} K(t_f^-)\Pi,$$

and thus $K = K(t_f^-)$ solves (28b). It follows from Lemma 18 that if $X(t)$ is a solution to the Riccati equation (26) with terminal condition

$$X(t_f^-) = \Pi^\top (I - K(t_f^-)^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} K(t_f^-))\Pi,$$

and substituting $x(t_f^-) = (I - B^{\text{imp}} K(t_f^-))\Pi x^{\text{diff}}(t_f^-)$, that the optimal cost is given by

$$J_\Pi^-(x,u) = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} + \int_{t_0}^{t_f} \left(\left\| \left(R^{\frac{1}{2}} K(\cdot) + R^{-\frac{1}{2}} \left(B^{\text{diff}\top} X(\cdot) + S^\top \right)\right) x^{\text{diff}} \right\|_2^2 \right).$$

Now we will show that this implies $K(t) = -R^{-1}(B^{\text{diff}\top} X(t) + S^\top)$ and that $K(t_f^-) = K$, $X(t_f^-) = X$ solve (28a) and (28c). To seek a contradiction, assume that $K(t) \neq -R^{-1}(B^{\text{diff}\top} X(t) + S^\top)$. Then for some $M > 0$ the optimal cost is given by

$$J_\Pi^-(x,u) = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}}(t_0^-) + M.$$

Then we will show that there exists an input $\bar{u}$ that results in $J(\bar{x},\bar{u}) < J(x,u)$.

Consider $\varepsilon > 0$ arbitrarily small. Then the input $\bar{u}$, defined by $\bar{u}(t) = \bar{K}(t) x^{\text{diff}}(t)$, where

$$\bar{K}(t) = \begin{cases} -R^{-1}(B^{\text{diff}\top} X(\cdot) + S^\top) & t \in [t_0, t_f - \varepsilon) \\ K(t) & t \in [t_f - \varepsilon, t_f) \end{cases}$$

results in the following cost

$$\begin{aligned} J_\Pi^-(\bar{x},\bar{u}) &= x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} \\ &+ \int_{t_f-\varepsilon}^{t_f} \left(\left\| \left(R^{\frac{1}{2}} K(\cdot) + R^{-\frac{1}{2}} (B^{\text{diff}\top} X(\cdot) + S^\top \right) \bar{x}^{\text{diff}} \right\|_2^2 \right) \\ &< x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} + M \\ &= J_\Pi^-(x,u). \end{aligned}$$

This contradicts the optimality of the input u. Hence $u(t) = -R^{-1}(B^{\text{diff}\top} X(t) + S^\top) x^{\text{diff}}(t)$. Consequently it must hold that

$$-R^{-1}(B^{\text{diff}\top} X(t_f^-) + S^\top) x^{\text{diff}}(t_f^-) = K(t_f^-) x^{\text{diff}}(t_f^-),$$

which proves that if $K = K(t_f^-)$ and $X = X(t_f^-)$ that X, Y solve (28).

($\Leftarrow$) Conversely, assume that there exists a solution to (28). For any matrix K that satisfies

$$B^{\text{imp}\top} P\Pi = B^{\text{imp}\top} P B^{\text{imp}\top} K \Pi, \tag{30}$$

we have that for all $\bar{K}$

$$\begin{aligned} &x^{\text{diff}}(t_f^-)(I - B^{\text{imp}} K)^\top P (I - B^{\text{imp}} K) x^{\text{diff}}(t_f^-) \\ &\quad \leqslant x^{\text{diff}}(t_f^-)(I - B^{\text{imp}} \bar{K})^\top P (I - B^{\text{imp}} \bar{K}) x^{\text{diff}}(t_f^-), \end{aligned}$$

and that equality holds for any K that satisfies (30). Consequently, for any feedback matrix $\bar{K}$ we have for all $x^{\text{diff}}(t_f^-) \in \operatorname{im}\Pi$

$$\begin{aligned} 0 \leqslant x^{\text{diff}}(t_f^-)^\top (I - \bar{K}^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} \bar{K}) x^{\text{diff}}(t_f^-) \\ - x^{\text{diff}}(t_f^-)^\top X x^{\text{diff}}(t_f^-). \end{aligned}$$

For the feedback matrix $K(t) = R^{-1}(B^{\text{diff}\top} X(t) + S^\top) x^{\text{diff}}(t)$ where $X(t)$ is a solution to (26) with terminal condition $X(t_f) = X$, we have that $K(t_f) = K$ and thus satisfies (28b). Furthermore, as X satisfies (28c), the cost for the feedback $u(t) = K(t)x(t)$ is given by

$$J_\Pi^-(x,u) = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}}.$$

Note that for any other input $\bar{u}(t) = \bar{K}(t) x^{\text{diff}}(t)$ the cost is given by

$$\begin{aligned} &J_\Pi^-(\bar{x},\bar{u}) \\ &\quad = x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}}. \\ &\qquad + \int_{t_0}^{t_f} \left(\| R^{\frac{1}{2}} \bar{u} + R^{-\frac{1}{2}} (B^{\text{diff}\top} X(t) + S^\top) \bar{x}^{\text{diff}} \|_2^2 \right) \\ &\qquad + \bar{x}(t_f^-)^\top P \bar{x}(t_f^-) - \bar{x}^{\text{diff}}(t_f^-)^\top X(t_f^-) \bar{x}^{\text{diff}}(t_f^-), \\ &\quad \geqslant x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} - \bar{x}^{\text{diff}}(t_f^-)^\top X(t_f^-) \bar{x}^{\text{diff}}(t_f^-), \\ &\qquad + x^{\text{diff}}(t_f^-)^\top ((I - \bar{K}^\top B^{\text{imp}\top}) P (I - B^{\text{imp}} \bar{K}) x^{\text{diff}}(t_f^-) \\ &\quad \geqslant x_0^{\text{diff}\top} X(t_0^-) x_0^{\text{diff}} \\ &\quad \geqslant J_\Pi^-(x,u), \end{aligned}$$

and hence u minimizes $J_\Pi^-(x,u)$. ∎

Corollary 21: Consider Problem 1 and assume it has a solution. Let X, K solve (28). Then the u that minimizes (21) is given by

$$u(t) = -R^{-1} \left(B^{\text{diff}\top} X(t) + S^\top \right) x^{\text{diff}}(t),$$

where $X(t)$ is a solution to (26) with terminal condition $X(t_f^-) = X$

Corollary 22: Consider Problem 1 and assume $\Pi^\top P B^{\text{imp}} = 0$. Then Problem 1 is solvable if and only if

$$P B^{\text{imp}} R^{-1} (B^{\text{diff}\top} P + \bar{S}^\top) \Pi = 0. \tag{31}$$

Proof: Assume that there exists an input that minimizes (21) that is continuous at t_f. It follows from Corollary 17.

$$B^{\text{imp}} P \Pi x^{\text{diff}}(t_f^-) = -B^{\text{imp}} P B^{\text{imp}} u(t_f^-) = 0.$$

Furthermore, the optimal control on $[t_0, t_f)$ is given by

$$u(t) = -R^{-1}(B^{\text{diff}\top} X(t) + S^\top) \Pi x^{\text{diff}}(t),$$

where $X(t)$ is the solution to the Riccati equation (26) with terminal constraint $X(t) = \Pi^\top P \Pi$. Since $u(t)$ is continuous it follows that

$$B^{\text{imp}\top} P B^{\text{imp}} R^{-1} (B^{\text{diff}\top} P + S^\top) \Pi x^{\text{diff}}(t_f^-) = 0.$$

Since this holds for any $x^{\text{diff}}(t_f)$ the result follows.

Conversely, if (31) holds, it follows that $X = \Pi^\top P \Pi$ solves (28). ∎

Remark 23: In the case $P = E^\top \bar{P} E$ for some positive semi-definite $\bar{P}$ it follows that $\Pi^\top P B^{\text{imp}} = \Pi^\top E^\top \bar{P} E B^{\text{imp}} = 0$ as $EB^{\text{imp}} = 0$. Condition (31) is clearly satisfied and thus there exist an optimal input u. Furthermore,, the optimal input is given by

$$u(t) = -R^{-1}(B^{\text{diff}} X(t) + S^\top)\Pi x^{\text{diff}}(t),$$

where $X(t)$ is the solution of (26) with terminal constraint $X(t_f) = \Pi^\top P \Pi$. Note that the input can also be written in terms of a solution to (27) as

$$u(t) = -R^{-1}(BY(t) + S^\top)x(t),$$

where $Y(t)$ is a solution of (27) with terminal condition $EY(t_f^-) = P$ and $Y(t) = \Pi^{\text{diff}\top} X(t)\Pi$.

Example 24 (Example 1 revisted): Returning to the example in the introduction, we have for (4) that $R = 1$, $S = 0$, $P = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $\Pi = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $B^{\text{imp}} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $B^{\text{diff}} = 0$. It follows that $\Pi^\top P B^{\text{imp}} = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \neq 0$. This means that there exist an optimal input if and only if the conditions given in Theorem 20 are satisfied. However, according to (28a) $K = 0$, whereas (28b) states that K also satisfies

$$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] = B^{\text{imp}\top} P \Pi = B^{\text{imp}\top} P B^{\text{imp}} K \Pi$$

Hence the conditions are not satisfied and there indeed does not exist an optimal input u.

For the cost functional (5) we have $R = 1$, $S = 0$, $P = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $\Pi = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $B^{\text{imp}} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $B^{\text{diff}} = 0$.This means that $\Pi^\top P B^{\text{imp}} = 0$. However,

$$B^{\text{imp}\top} P B^{\text{imp}} R^{-1}(B^{\text{diff}\top} P + S^\top)\Pi = 0,$$

which shows that there exists an optimal input u. ◇

IV. Conclusion

In this paper, the linear quadratic optimal control problem for DAEs with unconstrained terminal cost has been studied. It was shown that for a general weight matrix in the terminal cost, optimal input might fail to exist. Necessary and sufficient condition that guarantee the existence of an optimal solution in terms of matrix equations were formulated.

References

[1] Frank L Lewis. Optimal control for singular systems. In *1985 24th IEEE Conference on Decision and Control*, pages 266–272. IEEE, 1985.

[2] Douglas J. Bender and Alan J. Laub. The linear-quadratic optimal regulator for descriptor systems. In *Proc. 24th IEEE Conf. Decis. Control, Ft. Lauderdale, FL*, pages 957–962, 1985.

[3] Tohru Katayama and Katsuki Minamino. Linear quadratic regulator and spectral factorization for continous-time descriptor systems. In *[1992] Proceedings of the 31st IEEE Conference on Decision and Control*, pages 967–972. IEEE, 1992.

[4] J. Daniel Cobb. Descriptor variable systems and optimal state regulation. *IEEE Trans. Autom. Control*, 28:601–611, 1983.

[5] Volker Mehrmann. Existence, uniqueness and stability of solutions to singular, linear-quadratic control problems. *Linear Algebra Appl.*, pages 291–331, 1989.

[6] Timo Reis and Matthias Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems: Existence of nonpositive solutions. *Systems & Control Letters*, 86:1–8, 2015.

[7] Peter Kunkel and Volker Mehrmann. Optimal control for unstructured nonlinear differential-algebraic equations of arbitrary index. *Math. Control Signals Syst.*, 20:227–269, 2008.

[8] Peter Kunkel and Volker Mehrmann. The linear quadratic control problem for linear descriptor systems with variable coefficients. *Math. Control Signals Syst.*, 10:247–264, 1997.

[9] Paul Wijnbergen and Stephan Trenn. Impulse controllability of switched differential-algebraic equations. In *2020 European Control Conference (ECC)*, pages 1561–1566. IEEE, 2020.

[10] Paul Wijnbergen, Mark Jeeninga, and Stephan Trenn. On stabilizability of switched differential algebraic equations. In *Proc. IFAC World Congress*, 2020.

[11] Alessandro Giua, Carla Seatzu, and Cornelis Van Der Mee. Optimal control of switched autonomous linear systems. In *Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228)*, volume 3, pages 2472–2477. IEEE, 2001.

[12] Sorin C Bengea and Raymond A DeCarlo. Optimal control of switching systems. *automatica*, 41(1):11–27, 2005.

[13] Xuping Xu and Panos J Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE transactions on automatic control*, 49(1):2–16, 2004.

[14] Guangyu Wu, Jian Sun, and Jie Chen. Optimal linear quadratic regulator of switched systems. *IEEE transactions on automatic control*, 64(7):2898–2904, 2018.

[15] Thomas Berger, Achim Ilchmann, and Stephan Trenn. The quasi-Weierstraß form for regular matrix pencils. *Linear Algebra Appl.*, 436(10):4052–4069, 2012.

[16] Kai-Tak Wong. The eigenvalue problem $\lambda Tx + Sx$. *J. Diff. Eqns.*, 16:270–280, 1974.

[17] Stephan Trenn. *Distributional differential algebraic equations*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany, 2009.

[18] J. Daniel Cobb. Controllability, observability and duality in singular systems. *IEEE Trans. Autom. Control*, 29:1076–1082, 1984.

[19] J. Daniel Cobb. Feedback and pole placement in descriptor variable systems. *Int. J. Control*, 33(6):1135–1146, 1981.

[20] Dimitri P Bertsekas. *Dynamic programming and optimal control*, volume 1. Athena scientific Belmont, MA, 1995.