An approximation for nonlinear differential-algebraic equations via singular perturbation theory

Yahao Chen * **Stephan Trenn** *

* *Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, The Netherlands.*

Abstract: In this paper, we study the jumps of nonlinear DAEs caused by inconsistent initial values. First, we propose a simple normal form called the index-1 nonlinear Weierstrass form **(INWF)** for nonlinear DAEs. Then we generalize the notion of consistency projector introduced in Liberzon and Trenn (2009) for linear DAEs to the nonlinear case. By an example, we compare our proposed nonlinear consistency projectors with two existing consistent initialization methods (one is from the paper Liberzon and Trenn (2012) and the other is given by a MATLAB function) to show that the two existing methods are not coordinate-free, i.e., the consistent points calculated by the two methods are not invariant under nonlinear coordinates transformations. Next we propose a singular perturbed system approximation for nonlinear DAEs, which is an ordinary differential equation (ODE) with a small perturbation parameter and we show that the solutions of the proposed perturbation system approximate both the jumps resulting from the nonlinear consistency projectors and the $\mathcal{C}^1$-solutions of the DAE. At last, we use a numerical simulation of a nonlinear DAE model arising from an electric circuit to illustrate the effectiveness of the proposed singular perturbed system approximation of DAEs.

Keywords: differential-algebraic equations, singular perturbation, jumps, index-1, nonlinear Weierstrass form, inconsistent initial values

1. INTRODUCTION

We consider a nonlinear differential-algebraic equation (DAE),

$$\Xi : E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is the vector of generalized states and X is an open subset of $\mathbb{R}^n$, and where $E : X \to \mathbb{R}^{n\times n}$ and $F : X \to \mathbb{R}^n$ are $\mathcal{C}^\infty$-smooth maps. A DAE of the form (1) will be denoted by $\Xi = (E, F)$ or Ξ. The matrix-valued function $E(x)$ is not necessarily invertible, which implies that there may exist some algebraic constraints and some algebraic variables in the DAE Ξ. A particular case of Ξ is a semi-explicit DAE

$$\Xi^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \ \ 0 = f_2(x_1, x_2), \end{cases} \tag{2}$$

with $E = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$ being constant. The DAE Ξ^{SE} has the algebraic variables x_2 (since the derivatives of x_2 are not present) and the algebraic constraints $0 = f_2(x_1, x_2)$. We will study also linear DAEs of the form

$$\Delta : \ E\dot{x} = Hx, \tag{3}$$

where $E \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times n}$. A linear DAE of the form (3) will be denoted by $\Delta = (E, H)$ or, shortly, Δ. A linear DAE Δ is called *regular* if $sE - H \in \mathbb{R}^{n\times n}[s]\backslash 0$.

A $\mathcal{C}^1$-solution of a DAE $\Xi = (E, F)$ is a differentiable function $x : I \to X$ defined on an open interval I such that for all $t \in I$, the curve $x(t)$ satisfies $E(x(t))\,\dot{x}(t) = F(x(t))$, where $\dot{x}$ denotes the classical time-derivative defined everywhere on I. A point x_0^+ is called *consistent* if there exists at least one $\mathcal{C}^1$-solution $x : I \to X$ with $t_0 \in I$ such that $x_0^+ = x(t_0)$. The set of all consistent points will be called consistency space and denoted by S_c.

It is known that the $\mathcal{C}^1$-solutions of a nonlinear DAE Ξ exist on its consistency space S_c only (see Section 2). For a given inconsistent initial point $x_0^- \in X\backslash S_c$, there does not exist any $\mathcal{C}^1$-solution starting from x_0^-. Then it is natural to search for the consistent point $x_0^+ \in S_c$ such that we can get the $\mathcal{C}^1$-solutions of Ξ starting from x_0^+. The instant change from the inconsistent point x_0^- to a consistent one x_0^+ is called a jump of the DAE at $t = 0$. Note that the jumps which we study in the paper are called external or exogenous jumps, which are different from the jumps at the impasse (or singular) points as discussed in Takens (1976); Chua and Deng (1989); Sastry and Desoer (1981). We assume throughout that once starting from the point x_0^+, there will not exist any jump and we will study only the $\mathcal{C}^1$-solutions of Ξ. In conclusion, we consider the following initial value problem:

$$\begin{cases} \text{Jumps}: \lim\limits_{t\to 0^-} x(t) = x_0^- \notin S_c \to \lim\limits_{t\to 0^+} x(t) = x_0^+ \in S_c, \\ \mathcal{C}^1\text{-solutions: } (E(x)\dot{x})_{(0,T)} = F(x)_{(0,T)}, \end{cases}$$

for some function $x : I \to \mathbb{R}^n$ differentiable on $(0, T) \subseteq (-\varepsilon, T) \subseteq I$ for some $\varepsilon > 0$. The problem of finding the consistent point x_0^+ for a DAE with an inconsistent initial value x_0^- is called consistent initialization, which is a significant problem for hybrid DAE systems involving

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with jump behaviors. Some examples of such systems are the electric circuits with instant connections or switching devices (see e.g., Zuhao (1991); Vlach et al. (1995); Trenn (2012)), the power systems with DC transmissions in Susuki et al. (2008), the multi-body dynamics in Hamann and Mehrmann (2008) and the battery model of Methekar et al. (2011).

For a regular linear DAE $\Delta = (E, H)$, given by (3), the consistent initialization can be solved by the linear consistency projector introduced by Liberzon and Trenn (2009, 2012), which is a linear map constructed with the help of the well-known Weierstrass form **(WF)**. For a semi-explicit DAE Ξ^{SE} of the form (2), the singular perturbation theory (see e.g., Kokotović et al. (1999); Khalil (2001)) was frequently used to study system approximations of the discontinues solutions of Ξ^{SE} (see e.g., Sastry and Desoer (1981); Rabier and Rheinboldt (2002); Susuki et al. (2008) and section 4 of the present paper). Two existing methods of solving the consistent initialization problem for nonlinear DAEs are, the jump rule of Liberzon and Trenn (2012), which determines the consistent initial value x_0^+ through the formula $x_0^+ - x_0^- \in \ker E(x_0^+)$, and the function *decic* of MATLAB (see MathWorks (2006)), which calculates the consistent initial values via a numerical searching method, we will show in Example 9 below that both of those two methods are not coordinate-free, i.e., the calculated consistent values depends on which local coordinates are chosen for the DAE.

The aims of this paper are, on one hand, to give a nonlinear generalization of the linear consistency projector in order to calculate consistent initial points for nonlinear DAEs, on the other hand, to extend the singular perturbed system approximation method to nonlinear DAEs of the form (1) to study the jump behaviors. This paper is organized as follows: We introduce the notations of the paper and some notions as invariant submanifolds, external equivalence and linear consistency projectors in Section 2. We propose a normal form called the index-1 nonlinear Weierstrass form **(INWF)** and extend the linear consistency projector to nonlinear DAEs in Section 3. A singular perturbed system approximation of nonlinear DAEs is proposed in Section 4 and we show the simulation results of an electric circuit by the singular perturbation method in Section 5. Conclusions of the paper are given in Section 6.

2. NOTATIONS AND SOME PRELIMINARIES OF LINEAR AND NONLINEAR DAES

We use the following notations: The symbol $\mathcal{C}^k$ denotes the class of functions which are k-times differentiable. For a map $A : X \to \mathbb{R}^{n\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively. The general linear group over $\mathbb{R}$ of degree n (in other words, the set of invertible matrices of size $n \times n$) is denoted by $GL(n, \mathbb{R})$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$. Let $f_i : X \to \mathbb{R}$ for $i = 1, \ldots, m$, in coordinates $x = (x_1, \ldots, x_n)$, the differential of f_i is $\mathrm{d}f_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\mathrm{d}x_j = [\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n}]$, the differentials of a vector-valued function $f = (f_1, \ldots, f_m)$ are $\mathrm{D}f = \begin{bmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{bmatrix}$. We assume that the reader is familiar with some basic notions as smooth embedded submanifolds, tangent spaces, involutive distributions from differential geometry, the reader can also consult the book by Lee (2001) for the definitions of such notions.

The existence and uniqueness of $\mathcal{C}^1$-solutions for nonlinear DAEs of the form (1) have been discussed using geometric methods in e.g., Reich (1991); Rabier and Rheinboldt (2002); Chen and Trenn (2020); Chen et al. (2020). An important notion in the geometric solutions theory of DAEs is the invariant submanifold defined as follows:

Definition 1. For a DAE $\Xi = (E, F)$, a smooth connected embedded submanifold M is called *invariant* if for any $x_0^+ \in M$, there exists a $\mathcal{C}^1$-solution $x : I \to X$ such that $x(0) = x_0^+$ and $x(t) \in M$, $\forall\, t \in I$. Fix a point $x_p \in X$, a smooth embedded submanifold M containing x_p is called *locally invariant*, if there exists a neighborhood U of x_p such that $M \cap U$ is invariant.

A locally invariant submanifold M^*, around a point x_p, is called locally *maximal*, if there exists a neighborhood U of x_p such that for any other locally invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$. It is shown in Chen and Trenn (2020); Chen et al. (2020) that the maximal invariant submanifold M^* around a nominal point x_p locally coincides with the consistency space S_c, i.e., there exists a neighborhood U^* of x_p such that

$$M^* \cap U^* = S_c \cap U^*.$$

Hence in the present paper, we make no difference between the notion of maximal invariant submanifold M^* and that of consistency space S_c when considering a DAE Ξ around a point x_p. Note that there is an iterative way of calculating the locally maximal invariant submanifold M^* of DAEs, called the geometric reduction method (see e.g., Rabier and Rheinboldt (2002); Chen and Trenn (2020); Chen et al. (2020)), the number of steps for the geometric reduction method to produce M^* and to get the solutions of a DAE is called the *geometric* index (see Chen and Trenn (2020)) of the DAE.

We now recall a definition of equivalence for linear DAEs: two linear DAEs $\Delta = (E, H)$ and $\tilde{\Delta} = (\tilde{E}, \tilde{H})$ are called externally equivalent (see Chen and Respondek (2021)) or strictly equivalent if there exist constant and invertible matrices Q and P such that $\tilde{E} = QEP^{-1}$ and $\tilde{H} = QHP^{-1}$. The same concept can be generalized to nonlinear DAEs of form (1) as follows.

Definition 2. (external equivalence). Consider two DAEs $\Xi = (E, F)$ and $\tilde{\Xi} = (\tilde{E}, \tilde{F})$ defined on X and $\tilde{X}$, respectively. Then Ξ and $\tilde{\Xi}$ are called externally equivalent, shortly ex-equivalent, if there exist a diffeomorphism $\psi : X \to \tilde{X}$ and $Q : X \to GL(n, \mathbb{R})$ such that

$$\tilde{E}(\psi(x)) = Q(x)E(x)\left(\frac{\partial \psi(x)}{\partial x}\right)^{-1}, \quad \tilde{F}(\psi(x)) = Q(x)F(x).$$

The ex-equivalence of two DAEs will be denoted by $\Xi \overset{ex}{\sim} \tilde{\Xi}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x_p and $\tilde{U}$ of $\tilde{x}_p$, and $Q(x)$ is defined on U, we will speak about local ex-equivalence.

Remark 3. It is easily seen, that for two externally equivalent systems Ξ and $\tilde{\Xi}$ a $\mathcal{C}^1$-curve $x : I \to X$ is a solution of Ξ if and only if $\psi \circ x$ is a solution of $\tilde{\Xi}$.

To illustrate the notions of maximal invariant submanifold and external equivalence, we use the following example.

Example 4. Consider a DAE $\Xi = (E, F)$, given by

$$\Xi : \begin{bmatrix} 1 & 3x_2^2 - 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}. \quad (4)$$

Fix a point $x_p = (x_{1p}, x_{2p}) = (0, 1)$, the locally maximal invariant submanifold of Ξ around x_p is $M^* = \left\{ x \in \mathbb{R}^2 \,|\, x_1 = 0, x_2 > \frac{\sqrt{3}}{3} \right\}$ (note that M^* is connected). We have that Ξ is locally ex-equivalent to the following form (i.e., the **(INWF)**, see Definition 5)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -f(\xi_1, 0) \\ \xi_2 \end{bmatrix}, \quad (5)$$

on the neighborhood $V = \left\{ x \in \mathbb{R}^2 \,|\, x_2 > \frac{\sqrt{3}}{3} \right\}$ of x_p, via $\psi = \xi = (\xi_1, \xi_2) = (x_1 + x_2^3 - x_2, x_1)$ and $Q = \left[\begin{smallmatrix} 1 & -f' \\ 0 & 1 \end{smallmatrix}\right]$, where $f(\xi) = f(\xi_1, 0) + f'(\xi)\xi_2$, $f = \frac{1}{3}\left(a + (a^2 - \frac{1}{27})^{\frac{1}{2}}\right)^{-\frac{1}{3}} + \left(a + (a^2 - \frac{1}{27})^{\frac{1}{2}}\right)^{\frac{1}{3}}$, $a(\xi_1, \xi_2) = \frac{\xi_1 - \xi_2}{2}$.

3. INDEX-1 NONLINEAR WEIERSTRASS FORM AND NONLINEAR CONSISTENCY PROJECTOR

Consider a nonlinear DAE $\Xi = (E, F)$, let $H(x, \dot{x}) = E(x)\dot{x} - F(x)$, define the k-th order differential array of $H(x, \dot{x}) = 0$ by

$$H_k(x, x', w) = \begin{bmatrix} H \\ \mathrm{D}_x H x' + \mathrm{D}_{x'} H x'' \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k} H \end{bmatrix} (x, x', w) = 0, \quad (6)$$

where $w = \left(x^{(2)}, \ldots, x^{(k+1)}\right)$, the differentiation index or shortly, the index, of the DAE Ξ is the least integer k such that equation (6) uniquely determines x' as a function of x, i.e., $x' = v(x)$. In Chen and Trenn (2020), we have shown that under some constant rank assumptions, the differential index coincides with the geometric index, we will use a simplification of those constant rank assumptions in the present paper: For a DAE $\Xi = (E, F)$, fix a point x_p, define $F_2 = F \backslash \mathrm{Im}\, E$, assume that $F_2(x_p) = 0$ and introduce the following constant rank condition, there exists a neighborhood U of x_p such that

(CR) $\mathrm{rank}\, E(x) = \textit{const.}$, $\forall x \in U$; $\mathrm{rank}\, \mathrm{D}F_2(x) = \textit{const.}$ and $\mathrm{rank}\, (E \ker \mathrm{D}F_2(x)) = \textit{const.}$, $\forall x \in U$ such that $F_2(x) = 0$.

The assumption $\mathrm{rank}\, E(x) = \textit{const.}$ ensures that there exists $Q : U \to GL(n, \mathbb{R})$ such that E_1 of $QE = \left[\begin{smallmatrix} E_1 \\ 0 \end{smallmatrix}\right]$ is of full row rank. Denote $QF = \left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right]$, then the map $F \backslash \mathrm{Im}\, E$ is given by F_2. The assumption $\mathrm{rank}\, \mathrm{D}F_2(x) = \textit{const.}$ guarantees that the zero-level set $\{x \in U \,|\, F_2(x) = 0\}$ is a smooth embedded submanifold and the condition $\mathrm{rank}\, (E \ker \mathrm{D}F_2(x)) = \textit{const.}$ excludes singular/impasses points (see Chua and Deng (1989); Chen (2019)) and helps to view the DAE as an ODE defined on a submanifold. Note that under the condition **(CR)**, a DAE Ξ is of differentiation index-1 if and only if it is of geometric index-1 (Chen and Trenn (2020)). Now we define a normal form, which is a semi-explicit DAE of index-1 with the algebraic equations fully decoupled from its differential equations.

Definition 5. (index-1 nonlinear Weirstrass form). We say that a DAE Ξ is represented in the index-1 nonlinear Weirstrass form **(INWF)** if Ξ is of the form

$$\begin{cases} \dot{\xi}_1 = F^*(\xi_1), \\ 0 = \xi_2. \end{cases} \quad (7)$$

where $\xi_1 \in X_1 \subseteq \mathbb{R}^r$, $\xi_2 \in X_2 \subseteq \mathbb{R}^{n-r}$ and $F^* : X_1 \to \mathbb{R}^r$.

Remark 6. For any DAE in **(INWF)** with an inconsistent initial point $(\xi_{10}^-, \xi_{20}^-) \notin M^*$, i.e., $\xi_{20}^- \neq 0$ (it is clear that the maximal invariant submanifold of (7) is $M^* = \{(\xi_1, \xi_2) \in X_1 \times X_2 \,|\, \xi_2 = 0\}$), we could easily deduce that $(\xi_{10}^+, \xi_{20}^+) = (\xi_{10}^-, 0)$ is the only possible jumping point from (ξ_{10}^-, ξ_{20}^-). Indeed, for the DAE (7), only ξ_2-variables are allowed to jump because any jump of ξ_1-variables will produce a Dirac impulse on the left-hand side of $\dot{\xi}_1 = F^*(\xi_1)$ (see the distributional solution theory of DAEs in Trenn (2009)), which is not possible since $F^*(\xi_1)$ is not able to produce a same impulsive term on the right-hand side in order to equalize the differential equations.

Theorem 7. Consider a DAE $\Xi = (E, F)$ and fix a point $x_p \in X$. Assume that Ξ satisfies the constant rank condition **(CR)** in a neighborhood $U \subseteq X$ of x_p. Then there exists a neighborhood $V \subseteq U$ of x_p such that Ξ is locally ex-equivalent to the **(INWF)**, given by (7), if and only if Ξ is of index-1 and the distribution $\mathcal{E} = \ker E$ is involutive.

Proof. *Only if.* Assume that Ξ is locally ex-equivalent to the **(INWF)**, denoted by $\tilde{\Xi} = (\tilde{E}, \tilde{F})$. It is clear that $\tilde{\Xi}$ is index-1 and that $\ker \tilde{E}$ is involutive (since $\tilde{E}$ is constant). Notice that the Q-transformation preserves the kernels and $\ker \tilde{E}(\psi(x)) = \frac{\partial \psi}{\partial x} \ker E(x)$; let $\ker E = \mathrm{span}\, \{g_1, \ldots, g_{n-r}\}$ for some vector fields g_i, we have $\ker \tilde{E} = \mathrm{span}\left\{ \frac{\partial \psi}{\partial x} g_1, \ldots, \frac{\partial \psi}{\partial x} g_m \right\}$, so the Lie bracket $[g_i, g_j] \in \ker E$ (i.e., $\ker E$ is involutive) if and only if $[\frac{\partial \psi}{\partial x} g_i, \frac{\partial \psi}{\partial x} g_j] = \frac{\partial \psi}{\partial x} [g_i, g_j] = \frac{\partial \psi}{\partial x} \ker E = \ker \tilde{E}$ (i.e., $\ker \tilde{E}$ is involutive). We conclude that Ξ is index-1 and $\mathcal{E} = \ker E$ is involutive as well.

If. Suppose that Ξ is of index-1 and the distribution $\mathcal{E} = \ker E$ is involutive. Then by $\mathrm{rank}\, E(x) = \textit{const.}$ (denote this rank by r) of **(CR)**, there exists $Q : U \to GL(n, \mathbb{R})$ such that $\mathrm{rank}\, E_1(x) = r$ in

$$Q(x)E(x)\dot{x} = Q(x)F(x) \Rightarrow \left[\begin{smallmatrix} E_1(x) \\ 0 \end{smallmatrix}\right] \dot{x} = \left[\begin{smallmatrix} F_1(x) \\ F_2(x) \end{smallmatrix}\right]. \quad (8)$$

Notice that the condition **(CR)** implies that there exists a neighborhood $U_1 \subseteq U$ of x_p such that $\mathrm{rank}\, A(x) = \mathrm{rank} \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right] = \textit{const.}$, $\forall x \in U_1 : F_2(x) = 0$. Since the DAE is of differentiation index-1, we have that $A(x)$ has to be invertible, i.e., $\mathrm{rank}\, A(x) = n$, because only if $A(x)$ is invertible, we can uniquely solve $\dot{x} = v(x) = A^{-1}(x) \left[\begin{smallmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{smallmatrix}\right]$ with only a first order differentiation of (8) (note that we only need to differentiate the algebraic equation $0 = F_2(x)$). Since the distribution $\Xi = \ker E$ is involutive, by Frobenius theorem (see e.g., Lee (2001)), there exist a neighborhood $U_2 \subseteq U_1$ and a smooth map $\xi_1 : U_2 \to \mathbb{R}^r$ such that $\mathrm{span}\left\{ \mathrm{d}\xi_1^1, \ldots, \mathrm{d}\xi_1^r \right\} = \mathcal{E}^{\perp}$, where $\mathrm{d}\xi_1^i$

are independent rows of $\mathrm{D}\xi_1$ and $\mathcal{E} = \ker E = \ker E_1$, i.e., $\mathrm{D}\xi_1(x) \ker E_1(x) = 0$, $\forall x \in U_2$. It follows that there exists $Q_1 : U_2 \to GL(r, \mathbb{R})$ such that $\mathrm{D}\xi_1(x) = Q_1(x)E_1(x)$. Set $\xi_2 = F_2$, then we have $\psi(x) = (\xi_1(x), \xi_2(x))$ is a local diffeomorphism on U_2 since

$$\frac{\partial\psi(x)}{\partial x} = \begin{bmatrix} \mathrm{D}\xi_1(x) \\ \mathrm{D}F_2(x) \end{bmatrix} = \begin{bmatrix} Q_1(x) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E_1(x) \\ \mathrm{D}F_2(x) \end{bmatrix} = \begin{bmatrix} Q_1(x) & 0 \\ 0 & I \end{bmatrix} A(x)$$

is invertible for all $x \in U_2$. Define the new local coordinates $\xi = \psi(x) = (\xi_1(x), \xi_2(x))$ on U_2, the DAE (8) under the new ξ-coordinates is represented by

$$\begin{bmatrix} E_1(x) \\ 0 \end{bmatrix} \left(\frac{\partial\psi(x)}{\partial x}\right)^{-1} \frac{\partial\psi(x)}{\partial x}\dot{x} = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} E_1^1(\xi_1,\xi_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(\xi_1,\xi_2) \\ \xi_2 \end{bmatrix}, \tag{9}$$

where $E_1^1 : U_2 \to \mathbb{R}^{r\times r}$, $[E_1^1 \circ \psi, E_1^2 \circ \psi] = E_1(\frac{\partial\psi}{\partial x})^{-1}$ with $E_1^2 \equiv 0$, $\tilde{F}_1 \circ \psi = F_1$. Notice that $E_1^2 \equiv 0$ because $\operatorname{Im} E_1^2(x) = E_1(x) \ker \mathrm{D}\xi_1(x) = 0$ and that $E_1^1(x)$ is invertible for $x \in U_2$ since $\operatorname{rank} E(x) = \mathit{const.} = r$, $\forall x \in U_2$. Let $\bar{F}_1 = (E_1^1)^{-1}\tilde{F}_1$, we can always find $\bar{F}_1' : U_2 \to \mathbb{R}^{r\times m}$ such that $\bar{F}_1(\xi_1, \xi_2) = \bar{F}_1(\xi_1, 0) + \bar{F}_1'(\xi_1, \xi_2)\xi_2$. Then via $\tilde{Q} = \begin{bmatrix} (E_1^1)^{-1} & -\bar{F}_1' \\ 0 & I \end{bmatrix}$, the DAE (9) is ex-equivalent to the **(INWF)** with $F^*(\xi_1) = \bar{F}_1(\xi_1, 0)$. Finally, it is seen that Ξ is locally (on $V = U_2$) ex-equivalent to the **(INWF)** via the $\tilde{Q}Q$-transformation and the diffeomorphism ψ.

With the help of the **(INWF)**, we can generalize the notion of consistency projector to nonlinear DAEs:

Definition 8. (nonlinear consistency projector). For a nonlinear DAE $\Xi = (E, F)$, fix a point x_p and assume that there exists a neighborhood V of x_p such that Ξ is locally (on V) ex-equivalent to the **(INWF)** of (7) via a Q-transformation and a local diffeomorphism ψ. The (local) *nonlinear consistency projector* $\Omega_{E,F} : V\backslash M^* \to V \cap M^*$ of Ξ is then defined by

$$\Omega_{E,F} := \psi^{-1} \circ \pi \circ \psi,$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is the canonical projection $(\xi_1, \xi_2) \mapsto (\xi_1, 0)$.

For a DAE Ξ being locally (on V) ex-equivalent to the **(INWF)** with an inconsistent initial value $x_0^- \in V\backslash M^*$, we can get a unique consistent point $x_0^+ = \Omega_{E,F}(x_0^-) \in V \cap M^*$ since in the ξ-coordinates of the **(INWF)**, the inconsistent point $(\xi_{10}^-, \xi_{20}^-) = \psi(x_0^-)$ has to jump into $(\xi_{10}^+, \xi_{20}^+) = (\xi_{10}^-, 0)$ (see Remark 6), hence $x_0^+ = \psi^{-1}(\xi_{10}^+, \xi_{20}^+) = \psi^{-1} \circ \pi \circ \psi(x_0^-) = \Omega_{E,F}(x_0^-)$. Then we compare the consistent initial values calculated by the nonlinear consistency projector with that from the jump rule in Liberzon and Trenn (2012) and MATLAB *decic* function (see MathWorks (2006)).

Example 9. (continuation of *Example 4*). The DAE (4) satisfies the condition **(CR)** in the neighborhood $U = \{x \in \mathbb{R}^2 \,|\, x \neq \pm\sqrt{3}/3\}$ of x_p. We have shown in Example 4 that (4) is ex-equivalent (on $V \subseteq U$) to the **(INWF)**, given by (5), via Q and ψ. Thus the nonlinear (local) consistency projector of Ξ is

$$\Omega_{E,F} = \psi^{-1} \circ \pi \circ \psi = \begin{bmatrix} 0 \\ f(x_1 + x_2^2 - x_2, 0) \end{bmatrix}.$$

Take an inconsistent initial value $x_0^- = (1, 0.7) \in V\backslash M^*$, the consistent point calculated by the nonlinear consistency projector is $x_0^+ = \Omega(x_0^-) = (0, 1.233) \in M^*$. Note that the inconsistent initial point of (5) is $\xi_0^- = \psi(x_0^-) = (0.643, 1)$ and the consistent point is $\xi_0^+ = (0.643, 0)$ since only ξ_2-variables are allowed to jump (see Remark 6). Then we use the jump rule $x_0^+ - x_0^- = \ker E(x_0^+)$ in Liberzon and Trenn (2012) to calculate the consistent value $\tilde{x}_0^+$ for (4) with x_0^- and $\tilde{\xi}_0^+$ for (5) with ξ_0^-, we get

$$\tilde{x}_0^+ = (0, 0.109) \text{ and } \tilde{\xi}_0^+ = (0.643, 0).$$

Similarly, we use MATLAB *decic* function to determine the consistent value $\bar{x}_0^+$ for (4) and $\bar{\xi}_0^+$ for (5) to get

$$\bar{x}_0^+ = (0, 0.7) \text{ and } \tilde{\xi}_0^+ = (0.643, 0).$$

Since $\tilde{\xi}_0^+ \neq \psi(\tilde{x}_0^+)$ and $\bar{\xi}_0^+ \neq \psi(\bar{x}_0^+)$, we conclude that the two consistent initialization methods in Liberzon and Trenn (2012) and MathWorks (2006) do *not* preserve the calculated consistent points when changing the coordinates of the given DAE. On the other hand, the jump $x_0^- \to x_0^+$ of (4), given by the nonlinear consistency projector, and the jump $\xi_0^- \to \xi_0^+$ of (5) are clearly the same jump in different coordinates since $\xi_0^- = \psi(x_0^-)$, $\xi_0^+ = \psi(x_0^+)$, which proves that the consistent initialization calculated by the consistency projector is coordinate-free.

4. SINGULAR PERTURBED SYSTEM APPROXIMATION OF NONLINEAR DAES

We first recall a singular perturbed system for a semi-explicit DAE Ξ^{SE} of the form (2). Replacing the algebraic constraint $0 = f_2(x_1, x_2)$ by $\epsilon\dot{x}_2 = f_2(x_1, x_2)$, where ϵ represents some ignorable small modeling parameters (e.g, the small inductance of an inductor in electrical circuits, see page 367 of Rabier and Rheinboldt (2002)), we get a perturbed ODE system Ξ_ϵ^{SE} on the left-hand side of the following, then by rescaling time t to τ by $\frac{d\tau}{dt} = \frac{1}{\epsilon}$, we get a perturbed system in the time-scale τ on the right-hand side.

$$\Xi_\epsilon^{SE} : \begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \epsilon\dot{x}_2 = f_2(x_1, x_2). \end{cases} \overset{\epsilon = \frac{dt}{d\tau}}{\Leftrightarrow} \begin{cases} \dfrac{dx_1}{d\tau} = \epsilon f_1(x_1, x_2), \\ \dfrac{dx_2}{d\tau} = f_2(x_1, x_2). \end{cases}$$

There are, in general, two assumptions in the singular perturbed approximation method of semi-explicit DAEs: (a) $\frac{df_2}{dx_2}$ is invertible (which is actually equivalent to that Ξ^{SE} is of index-1); (b) the so-called boundary layer model $\frac{dx_2}{d\tau} = f_2(x_{10}^-, x_2)$ is asymptotically stable uniformly in x_2. Then under assumptions (a),(b), the well-known Tihkonov's theorem (see e.g., Khalil (2001) and a similar result in Theorem III.1 of Sastry and Desoer (1981)) states that if a unique solution (x_1, x_2) of Ξ^{SE} starting from a consistent initial point (x_{10}^+, x_{20}^+) exists on the interval $I = (0, \alpha)$, then there exists $\delta \geq 0$ such that a solution $(\bar{x}_1(\cdot, \epsilon), \bar{x}_2(\cdot, \epsilon))$ of Ξ_ϵ^{SE} starting from any point (x_{10}^-, x_{20}^-) with $||x_{10}^+ - x_{10}^-|| + ||x_{20}^+ - x_{20}^-|| < \delta$ satisfies

$$\begin{aligned} &\lim_{\epsilon\to 0} ||x_1(t) - \bar{x}_1(t, \epsilon)|| = 0, \\ &\lim_{\epsilon\to 0} ||x_2(t) - \bar{x}_2(t, \epsilon)|| = 0, \end{aligned} \tag{10}$$

for all $t \in I$ (uniformly on each compact subinterval of I). In this section, we will propose a singular perturbed system approximation for nonlinear DAEs of the form (1) with the help of the proposed normal form **(INWF)**.

Definition 10. (singular perturbed system). For a nonlinear DAE $\Xi = (E, F)$, fix a point x_p, assume that there exists a neighborhood V of x_p such that Ξ is locally (on V) ex-equivalent to the **(INWF)** of (7) via a Q-transformation and a local diffeomorphism ψ. Define the following singular perturbed system on V:

$$\Xi_\epsilon : \dot{x} = E_\epsilon^{-1}(x,\epsilon)F(x), \tag{11}$$

where $E_\epsilon(x,\epsilon) = E(x) + Q^{-1}(x)\begin{bmatrix}0 & 0\\ 0 & -\epsilon I_{n-r}\end{bmatrix}\frac{\partial\psi(x)}{\partial x}$.

Remark 11. Any linear index-1 regular DAE $\Delta = (E, H)$ of the form (3) is always ex-equivalent to a decoupled DAE given by $\left(\left[\begin{smallmatrix}I_{n_1} & 0\\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}A_1 & 0\\ 0 & I_{n_2}\end{smallmatrix}\right]\right)$. Applying the construction of (11) to Δ, we get the following singular perturbed system:

$$\Delta_\epsilon : \dot{x} = E_\epsilon^{-1}Hx = P^{-1}\left[\begin{smallmatrix}A_1 & 0\\ 0 & -\frac{1}{\epsilon}I_{n_2}\end{smallmatrix}\right]Px,$$

where $E_\epsilon = Q^{-1}\left[\begin{smallmatrix}I_{n_1} & 0\\ 0 & -\epsilon I_{n_2}\end{smallmatrix}\right]P$. The above perturbed linear system Δ_ϵ is proposed in section IV of Mironchenko et al. (2015) as an ODE approximation of linear DAEs.

The following theorem shows that the solution $\bar{x}(\cdot,\epsilon)$ of the proposed perturbed system Ξ_ϵ of (11) with an inconsistent initial value x_0^- converges to the $\mathcal{C}^1$-solution $x(\cdot)$ of Ξ starting from a consistent point x_0^+ calculated via the nonlinear consistency projector.

Theorem 12. Consider a DAE $\Xi = (E, F)$ and fix a point $x_p \in X$. Assume that the condition **(CR)** is satisfied in a neighborhood U of x_p. Suppose that Ξ is of geometric index-1 and that $\mathcal{E} = \ker E$ is involutive, implying that there exists a neighborhood $V \subseteq U$ of x_p such that Ξ is locally (on V) ex-equivalent to the **(INWF)** of (7) via Q and ψ. Let $x_0^- \in V\backslash M^*$ be an inconsistent initial point of Ξ and $x_0^+ = \Omega_{E,F}(x_0^-) \in M^*$ be the consistent point calculated via the nonlinear consistency projector $\Omega_{E,F}$. If $\bar{x}(\cdot,\epsilon) : I \to V$ is the solution of the perturbed system Ξ_ϵ of (11) starting from x_0^- and $x : I \to V$ is the $\mathcal{C}^1$-solution of Ξ starting from x_0^+, then we have

$$\lim_{\epsilon\to 0}||\bar{x}(t,\epsilon) - x(t)|| = 0, \quad \forall t \in I. \tag{12}$$

Proof. Suppose that Ξ is locally (on V) ex-equivalent to the **(INWF)** of (7) via Q and ψ. Consider the following disturbed system for (7):

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}I_r & 0\\ 0 & -\epsilon I_{n-r}\end{bmatrix}^{-1}\begin{bmatrix}F^*(\xi_1)\\ \xi_2\end{bmatrix} = \begin{bmatrix}F^*(\xi_1)\\ -\frac{1}{\epsilon}\xi_2\end{bmatrix}, \tag{13}$$

Let $\bar{\xi}(t,\epsilon) = (\bar{\xi}_1(t,\epsilon), \bar{\xi}_2(t,\epsilon))$ be the solution of (13) starting from $\xi_0^- = (\xi_{10}^-, \xi_{20}^-) = \psi(x_0^-)$. It is plain that $\bar{\xi}_2(t,\epsilon) = e^{-\frac{1}{\epsilon}t}\xi_{20}^-$. Then consider the following ODE

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}F^*(\xi_1)\\ 0\end{bmatrix}, \tag{14}$$

and let $\xi(t) = (\xi_1(t), \xi_2(t))$ be its solution of (13) with the initial point $\xi_0^+ = (\xi_{10}^+, \xi_{20}^+) = \psi(x_0^+) = \psi\circ\Omega_{E,F}(x_0^-) = \pi\circ\psi(x_0^-) = (\xi_{10}^-, 0)$. Define $\gamma(t,\epsilon) = \bar{\xi}(t,\epsilon) - \xi(t)$, we have

$$\dot{\gamma}(t,\epsilon) = \begin{bmatrix}0\\ -\frac{1}{\epsilon}\bar{\xi}_2(t,\epsilon)\end{bmatrix} = \begin{bmatrix}0\\ -\frac{1}{\epsilon}e^{-\frac{1}{\epsilon}t}\xi_{20}^-\end{bmatrix}$$

and $\gamma(0,\epsilon) = \xi_0^- - \xi_0^+ = (0, \xi_{20}^-)$. It follows that $\gamma(t,\epsilon) = (0, e^{-\frac{1}{\epsilon}t}\xi_{20}^-)$. Moreover, it is not hard to deduce that $\bar{x}(t,\epsilon) = \psi^{-1}\circ\xi(t,\epsilon)$ and that $x(t) = \psi^{-1}\circ\xi(t)$. Therefore we have

$$\lim_{\epsilon\to 0}||\bar{x}(t,\epsilon) - x(t)|| = \lim_{\epsilon\to 0}||\psi^{-1}\circ\bar{\xi}(t,\epsilon) - \psi^{-1}\circ\xi(t)||$$
$$\le \lim_{\epsilon\to 0}K||\bar{\xi}(t,\epsilon) - \xi(t)|| = \lim_{\epsilon\to 0}K||\gamma(t,\epsilon)|| = 0.$$

Note that the inequality "$\le$" holds in the above results since ψ^{-1} is a diffeomorphism and thus satisfies the Lipschitz condition for a Lipschitz constant K.

5. SIMULATION EXAMPLE

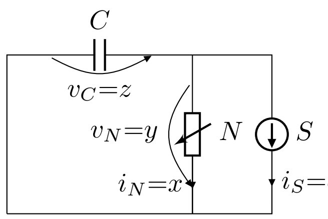


Fig. 1. An electrical circuit with a nonlinear resistor and a controlled current source

Consider the electrical circuit shown in Figure 1, which consists of a capacitor C and a nonlinear resistor N as the simple circuit discussed in Sastry and Desoer (1981); Chua and Deng (1989); Rabier and Rheinboldt (2002). A controlled current source S is additionally connected in parallel with N in order to generate nonlinear terms in $E(x)$ of the DAE model.

The relations between the current $i_N = x$ and the voltage $v_n = y$ of the nonlinear resistor N is characterized by the following algebraic equation

$$0 = a(x, y),$$

and the current i_S of S is equal to $b(x,y)\dot{y}$, where $a : \mathbb{R}^2 \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$ are smooth maps . Using Kirchoff's law, we model the circuit as a DAE $\Xi = (E, F)$:

$$\left[\begin{smallmatrix}0 & -b(x,y) & C\\ 0 & 0 & 0\\ 0 & 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\ y+z\\ a(x,y)\end{smallmatrix}\right].$$

We consider the following case: $C = 1$, $a(x,y) = x - y^2 - 2y$, $b(x,y) = y$. Let $\eta = (x,y,z)$ and $\eta_p = (0,0,0)$, then the condition **(CR)** is satisfied on $U = \{(x,y,z) \in \mathbb{R}^3 \,|\, y \neq 1\}$. The locally maximal invariant submanifold M^* (around η_p) is $M^* = \left\{\eta \in \mathbb{R}^3 \,|\, y + z = x - y^2 - 2y = 0, y < 1\right\}$. Since $\mathcal{E} = \ker E = \mathrm{span}\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\}$ is involutive and Ξ is of index-1. Since $\mathcal{E}$ is involutive, it is possible to find $\psi_1 : V \to \mathbb{R}$, where $V = \left\{\eta \in \mathbb{R}^3 \,|\, y < 1\right\}$, such that $\mathrm{span}\{\mathrm{d}\psi_1\} = \mathcal{E}^\perp$; by solving some first order PDE, we get $\psi_1(\eta) = -\frac{1}{2}y^2 + z$. Let $\psi_2(\eta) = y + z$ and $\psi_3 = a$, then the DAE Ξ is locally (on V) ex-equivalent to the following DAE represented in the **(INWF)**:

$$\left[\begin{smallmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\dot{\tilde{z}}\\ \dot{\tilde{y}}\\ \dot{\tilde{x}}\end{smallmatrix}\right] = \left[\begin{smallmatrix}-2\tilde{z}\\ \tilde{y}\\ \tilde{x}\end{smallmatrix}\right]. \tag{15}$$

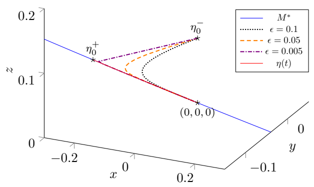
via $Q = \left[\begin{smallmatrix}1 & -2 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right]$ and $\psi(x,y,z) = (\tilde{z}, \tilde{y}, \tilde{x}) = (\psi_1, \psi_2, \psi_3)$. Following (11) of Definition 10, we construct a singular perturbed system Ξ_ϵ:

$$Q^{-1}\left[\begin{smallmatrix}1 & 0 & 0\\ 0 & -\epsilon & 0\\ 0 & 0 & -\epsilon\end{smallmatrix}\right]\frac{\partial\psi}{\partial\eta}\left[\begin{smallmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\ y+z\\ x-y^2-2y\end{smallmatrix}\right] \Rightarrow \Xi_\epsilon : \left[\begin{smallmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{smallmatrix}\right] = \left[\begin{smallmatrix}f_1(\eta,\epsilon)\\ f_2(\eta,\epsilon)\\ f_3(\eta,\epsilon)\end{smallmatrix}\right],$$

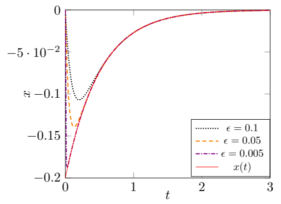
where $f_1(\eta,\epsilon) = -\frac{-x+y(2+y)-2\epsilon(y^2-2z)-2(y+z)}{\epsilon}$, $f_2(\eta,\epsilon) = -\frac{y+\epsilon y^2-2\epsilon z+z}{\epsilon+\epsilon y}$, $f_3(\eta,\epsilon) = \frac{\epsilon(y^2-2z)-y(y+z)}{\epsilon(1+y)}$. Consider an inconsistent initial point $\eta_0^- = (0,0,0.1) \in V\backslash M^*$, then we use the nonlinear consistency projector $\Omega_{E,F}$ to get

$$\eta_0^+ = \Omega_{E,F}(\eta_0^-) = \psi^{-1}\circ\pi\circ\psi(\eta_0^-) = (-0.2, -0.1056, 0.1056),$$

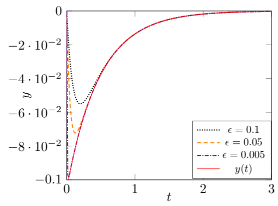
which defines the jump $\eta_0^- \to \eta_0^+$ of Ξ. Now we use MATLAB ode45 solver to simulate the solution $\bar{\eta}(t,\epsilon) = (\bar{x}(t,\epsilon), \bar{y}(t,\epsilon), \bar{z}(t,\epsilon))$ starting from η_0^- of the perturbed system Ξ_ϵ for different values of the perturbation parameter ϵ and the $\mathcal{C}^1$-solution $\eta(t) = (x(t), y(t), z(t))$ of Ξ starting from η_0^+, see Figure 2. Clearly, the proposed perturbed system approximates the DAE both for the jump $\eta_0^- \to \eta_0^+$ and for the $\mathcal{C}^1$-solution $\eta(t)$ starting from η_0^+ and evolving on M^*.



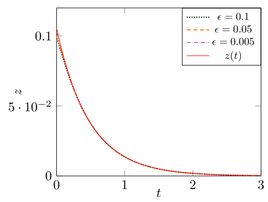
(a) Trajectories $\bar{\eta}(t,\epsilon)$ for different ϵ and $\eta(t)$ on the (x,y,z)-coordinates.



(b) Trajectories $\bar{x}(t,\epsilon)$ for different ϵ and $x(t)$ on the (t,x)-coordinates.



(c) Trajectories $\bar{y}(t,\epsilon)$ for different ϵ and $y(t)$ on the (t,y)-coordinates.



(d) Trajectories $\bar{z}(t,\epsilon)$ for different ϵ and $z(t)$ on the (t,z)-coordinates.

Fig. 2. The solutions $\bar{\eta}(t,\epsilon)$ of Ξ_ϵ for different ϵ and the solution $\eta(t)$ of Ξ

6. CONCLUSIONS

In this paper, we discuss the $\mathcal{C}^1$-solutions and the jumps from inconsistent initial points for nonlinear DAEs. First, we propose a normal form called the index-1 nonlinear Weierstrass form **(INWF)**, which has a simple and decoupled system structure. We show that a nonlinear DAE is locally externally equivalent to the **(INWF)** if and only if the DAE is index-1 and the distribution defined by ker E is involutive. Then we use the **(INWF)** to generalize the consistency projector of linear DAEs to the nonlinear case. The generalized nonlinear consistency projector offers a way to solve the consistent initialization problem for nonlinear DAEs. Finally, we propose a system approximation for nonlinear DAEs with jumps via the singular perturbation theory. The results of this paper could be a nice tool to study hybrid DAE systems involving with switchings since the consistent initialization is a fundamental problem for the solutions of switched nonlinear DAEs.

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