On geometric and differentiation index of nonlinear differential algebraic equations

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1 Introduction

We consider differential-algebraic equations DAEs of the following form

$$\Xi: \; E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is called the generalized state and X is an open subset of $\mathbb{R}^n$, and where $E : X \to \mathbb{R}^{l\times n}$ and $F : X \to \mathbb{R}^l$ are $\mathcal{C}^\infty$-smooth maps. A DAE of form (1) will be denoted by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. A solution of Ξ is a $\mathcal{C}^1$-curve $\gamma : I \to X$ with an open interval I such that for all $t \in I$, $\gamma(t)$ solves (1). To characterize different properties of nonlinear DAEs, various notions of index are proposed, see the survey or survey-like papers on index of DAEs: [1],[2],[3]. The most commonly used notions of index in DAEs theory seem to be the geometric index (see e.g., [4]) and the differentiation index (see [2] and [1]).

The main idea of geometric analysis of DAEs is to view the DAE as a vector field on a submanifold M^*, the later is called the maximal invariant submanifold, it is where the solutions of DAEs exist. The following locally maximal invariant submanifold algorithm is a recursive way to construct M^* [5]:
For a DAE $\Xi_{l,n} = (E, F)$, set $M_0 = X$, assume that a point $x_0 \in M_0$ satisfies $F(x_0) \in \operatorname{Im} E(x_0)$. Step 1: set

$$M_1 = \{x \in X : F(x) \in \operatorname{Im} E(x)\}\,;$$

Step k: assume that $M_{k-1} \subsetneq \cdots \subsetneq M_0$, for a certain $k \geq 1$, have been constructed and for some open neighborhood $U_{k-1} \subseteq X$ of x_0 that the intersection $M_{k-1} \cap U_{k-1}$ is a smooth submanifold and denote by M^c_{k-1} the connected component of $M_{k-1} \cap U_{k-1}$ satisfying $x_0 \in M^c_{k-1}$. Set

$$M_k = \left\{x \in M^c_{k-1} \,|\, F(x) \in E(x)T_xM^c_{k-1}\right\}.$$

Definition 1. (Geometric index) The geometric index of $\nu_g \in \mathbb{N}$ of a DAE Ξ is defined by

$$\nu_d := \min\left\{k \geq 0 \,|\, (M_k = M_{k+1}) \wedge (M_k \neq \emptyset)\right\}.$$

The differentiation index of DAEs is defined via the differential array shown in equation (2) below. Consider a nonlinear DAE $\Xi_{l,n} = (E, F)$, let $H(x, \zeta_1) = E(x)\zeta_1 - F(x)$, denote $(\frac{d^k}{dt^k}H) = H^{(k)}$ and define

$$H_k(x, \bar{\zeta}_{k+1}) = \begin{bmatrix} H^{(0)}(x, \zeta_1) \\ H^{(1)}(x, \zeta_1, \zeta_2) \\ \vdots \\ H^{(k)}(x, \bar{\zeta}_{k+1}) \end{bmatrix} = 0, \tag{2}$$

where $\bar{\zeta}_{k+1} = (\zeta_1, \ldots, \zeta_{k+1})$, set $\mathcal{M}_0 = X$, assume that for $k > 0$,

$$\begin{aligned} \mathcal{M}_k &= \left\{x \in X \,|\, H_{k-1}(x, \bar{\zeta}_k) = 0\right\} \quad \text{and} \\ \mathcal{Z}^k_1 &= \left\{\zeta_1 \in \mathbb{R}^n \,|\, H_{k-1}(x, \bar{\zeta}_k) = 0,\; x \in \mathcal{M}_k\right\} \end{aligned}$$

are smooth submanifolds of X and $\mathbb{R}^n$, respectively.
Definition 2. (Differentiation index) For a DAE $\Xi_{l,n} = (E, F)$, the differentiation index ν_d is defined by $\nu_d := 0$ if $(l = n) \wedge (E : X \to Gl(l, \mathbb{R}))$; otherwise, ν_d is the smallest integer such that $(\mathcal{M}_k \neq \emptyset) \wedge (\mathcal{Z}^k_1 = \mathcal{Z}^k_1(x)$ is a singleton$) \wedge (\mathcal{Z}^k_1(x) \in T_x\mathcal{M}_k, \forall x \in \mathcal{M}_k)$.

The aim of the present work is to have a comprehensive understanding of the two notions of DAE index. Both of the two indices serve as a measure of the difficulties of solving DAEs, but we show that they are actually related to the existence and uniqueness of solutions in a different manner. We also show that the two DAE indices have close relations with each other when some assumptions of smoothness and constant rankness are satisfied.

References

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