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Unimodular transformations for DAE initial trajectory problems

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We consider linear time-invariant differential-algebraic equations (DAEs). For high-index DAEs, it is often the first step to perform an index reduction, which can be realized with a unimodular matrix. In this contribution, we illustrate the effect of unimodular transformations on initial trajectory problems associated with DAEs.

1 Introduction

We consider linear time-invariant differential-algebraic equations (DAEs) of the form

$$E\dot{x} = Ax + f \tag{1}$$

with $E, A \in \mathbb{R}^{\ell\times n}$ and inhomogeneity $f \in \mathbb{D}^{\ell}_{\mathrm{pw}\mathcal{C}^\infty}$, where $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ denotes the space of piecewise-smooth distributions [1] and the more general higher order form

$$P(\tfrac{\mathrm{d}}{\mathrm{d}t})x = f \tag{2}$$

where $P(s) \in \mathbb{R}[s]^{\ell\times n}$. The first order formulation (1) is included in (2) as the special case $P(s) = sE - A$. It is easily seen (cf. the proof of [2, Thm. 7]) that these two descriptions are isomorphic, in particular, any higher order model can be written as a first order model. We also consider initial trajectory problems (ITPs) of the following form

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)}, \\ (P(\tfrac{\mathrm{d}}{\mathrm{d}t})x)_{[0,\infty)} &= f_{[0,\infty)}, \end{aligned} \tag{3}$$

with given initial trajectory x^0. We want to study how unimodular transformation effect a solution of the ITP (3). More precisely, we are interested how a solution (x, f) of (3) is related to a solution $(\widetilde{x}, \widetilde{f})$ for the ITP

$$\begin{aligned} \widetilde{x}_{(-\infty,0)} &= \widetilde{x}^0_{(-\infty,0)}, \\ (\widetilde{P}(\tfrac{\mathrm{d}}{\mathrm{d}t})\widetilde{x})_{[0,\infty)} &= \widetilde{f}_{[0,\infty)}, \end{aligned} \tag{4}$$

with $\widetilde{P}(s) := U(s)P(s)V(s)$ where $U(s) \in \mathbb{R}[s]^{\ell\times\ell}$ ad $V(s) \in \mathbb{R}[s]^{n\times n}$ are unimodular. Our motivation for such a transformation is twofold: First, index reduction for a DAE can be performed via a suitable unimodular matrix $U(s)$, and second, transformations with unimodular matrices can be used to decide whether a delay differential-algebraic equation is delay-regular [3].

2 Unimodular transformations and initial trajectory problems

In order to compare solutions of (3) and (4) we make the following definition.

Definition 2.1 The DAE (2), respectively the ITP (3), is called *regular* if, and only if, for all initial trajectories $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ and all inhomogeneities $f \in \mathbb{D}^{\ell}_{\mathrm{pw}\mathcal{C}^\infty}$ a unique solution $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ for the ITP (3) exists.

The ITP (3) can be interpreted as a switched DAE with a switch at time $t = 0$. Following [1, Thm. 3.5.2] and the proof of [2, Thm. 7] we conclude that the ITP (3) is regular if, and only if, $\ell = n$ and $\det(P(s)) \not\equiv 0$. Unimodularity of $U(s)$ and $V(s)$ implies that the ITP (3) is regular if, and only if, the transformed ITP (4) is regular.

Before we present our main result, we need the following technical observation.

Lemma 2.2 *Consider $F \in \mathbb{D}^{\ell}_{\mathrm{pw}\mathcal{C}^\infty}$ and $U(s) = \sum_{k=0}^{d} U_k s^k \in \mathbb{R}[s]^{\ell\times\ell}$. Then*

$$U(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(F_{[0,\infty)}\right) = \left(U(\tfrac{\mathrm{d}}{\mathrm{d}t})F\right)_{[0,\infty)} + \sum_{k=0}^{d} U_k \sum_{i=0}^{k-1} F^{(k-1-i)}(0^-)\delta^{(i)},$$

where δ denotes the Dirac impulse.

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P r o o f. The result is a direct consequence of [4, Prop. 12]. Indeed, applying [4, Prop. 12] yields

$$\begin{aligned} U(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(F_{[0,\infty)}\right) &= \sum_{k=0}^{d} U_k \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k F_{[0,\infty)} = \sum_{k=0}^{d} U_k \left(\left(F^{(k)}\right)_{[0,\infty)} + \sum_{i=0}^{k-1} F^{(k-1-i)}(0^-)\delta^{(i)}\right) \\ &= \left(U(\tfrac{\mathrm{d}}{\mathrm{d}t})F\right)_{[0,\infty)} + \sum_{k=0}^{d} U_k \sum_{i=0}^{k-1} F^{(k-1-i)}(0^-)\delta^{(i)}. \end{aligned}$$ □

Theorem 2.3 *Let $P(s) \in \mathbb{R}[s]^{\ell\times n}$ and define $\widetilde{P}(s) := U(s)P(s)V(s)$ with unimodular matrices $V(s) \in \mathbb{R}[s]^{n\times n}$ and $U(s) = \sum_{k=0}^{d} U_k s^k \in \mathbb{R}[s]^{\ell\times\ell}$. For any $\widetilde{x}^0, \widetilde{x} \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ and $\widetilde{f} \in \mathbb{D}^\ell_{\mathrm{pw}\mathcal{C}^\infty}$ define*

$$x := V(\tfrac{\mathrm{d}}{\mathrm{d}t})\widetilde{x}, \quad x^0 := V(\tfrac{\mathrm{d}}{\mathrm{d}t})\widetilde{x}^0, \quad \text{and} \quad f := U^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(\widetilde{f}_{[0,\infty)} + \sum_{k=0}^{d} U_k \sum_{i=0}^{k-1} (P(\tfrac{\mathrm{d}}{\mathrm{d}t})x^0)^{(k-1-i)}(0^-)\delta^{(i)}\right).$$

Then $\widetilde{x}$ is a solution of the ITP (4) *with initial trajectory $\widetilde{x}^0$ and inhomogeneity $\widetilde{f}$ if, and only if, x is a solution of the ITP* (3) *with initial trajectory x^0 and inhomogeneity f.*

P r o o f. First assume that x is a solution of the ITP (3) with initial trajectory x^0 and inhomogeneity f. Then

$$\begin{aligned} \widetilde{x}_{(-\infty,0)} &= (V^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})x)_{(-\infty,0)} = (V^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})x_{(-\infty,0)})_{(-\infty,0)} \\ &= (V^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})x^0_{(-\infty,0)})_{(-\infty,0)} = (V^{-1}(\tfrac{\mathrm{d}}{\mathrm{d}t})x^0)_{(-\infty,0)} = \widetilde{x}^0_{(-\infty,0)}. \end{aligned}$$

Lemma 2.2 implies

$$\begin{aligned} (\widetilde{P}(\tfrac{\mathrm{d}}{\mathrm{d}t})\widetilde{x})_{[0,\infty)} &= (U(\tfrac{\mathrm{d}}{\mathrm{d}t})P(\tfrac{\mathrm{d}}{\mathrm{d}t})x)_{[0,\infty)} \\ &= U(\tfrac{\mathrm{d}}{\mathrm{d}t})(P(\tfrac{\mathrm{d}}{\mathrm{d}t})x)_{[0,\infty)} - \sum_{k=0}^{d} U_k \sum_{i=0}^{k-1} (P(\tfrac{\mathrm{d}}{\mathrm{d}t})x)^{(k-1-i)}(0^-)\delta^{(i)} \\ &= U(\tfrac{\mathrm{d}}{\mathrm{d}t})\left(f_{[0,\infty)}\right) - \sum_{k=0}^{d} U_k \sum_{i=0}^{k-1} (P(\tfrac{\mathrm{d}}{\mathrm{d}t})x^0)^{(k-1-i)}(0^-)\delta^{(i)} \\ &= \widetilde{f}_{[0,\infty)}, \end{aligned}$$

i.e., $\widetilde{x}$ is a solution of the ITP (4) with initial trajectory $\widetilde{x}^0$ and inhomogeneity $\widetilde{f}$.

The converse direction follows analogously. In fact, from $\widetilde{x}$ being a solution of the ITP (4) with initial trajectory $\widetilde{x}^0$ and inhomogeneity $\widetilde{f}$, it follows that $x_{(-\infty,0)} = x^0_{(-\infty,0)}$ and

$$U(\tfrac{\mathrm{d}}{\mathrm{d}t})((P(\tfrac{\mathrm{d}}{\mathrm{d}t})x)_{[0,\infty)}) = U(\tfrac{\mathrm{d}}{\mathrm{d}t})f = U(\tfrac{\mathrm{d}}{\mathrm{d}t})f_{[0,\infty)}.$$ □

Remark 2.4 In an index-reduction procedure the inhomogeneity f in (2) is replaced with a linear combination of f and its derivatives. If however the solution of (2) is not smooth at $t = 0$, then additional Dirac impulses and derivatives of Dirac impulses may occur. In Theorem 2.3 this is accounted for in the definition of f.

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