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Impulse-free interval-stabilization of switched differential algebraic equations☆

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ABSTRACT

In this paper stabilization of switched differential algebraic equations is considered, where Dirac impulses in both the input and the state trajectory are to be avoided during the stabilization process. First it is shown that stabilizability of a switched DAE and the existence of impulse-free solutions are merely necessary conditions for impulse-free stabilizability. Then necessary and sufficient conditions for the existence of impulse-free solutions are given, which motivate the definition of (impulse-free) interval-stabilization on a finite interval. Under a uniformity assumption, which can be verified for a broad class of switched systems, stabilizability on an infinite interval can be concluded based on interval-stabilizability. As a result a characterization of impulse-free interval stabilizability is given and as a corollary we provide a novel impulse-free null-controllability characterization. Finally, the results are compared to results on interval-stabilizability where Dirac impulses are allowed in the input and state trajectory.

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1. Introduction

In this paper we consider *switched differential algebraic equations* (switched DAEs) of the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p, n, m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in [1]. We study impulse-free stabilizability of (1) where impulse-free stabilizability means the ability to find for each initial value a control signal such the state converges towards zero and remains impulse free (see the forthcoming Definition 9).

Differential algebraic equations (DAEs) arise naturally when modeling physical systems with certain algebraic constraints on the state variables. Examples of applications of DAEs in electrical circuits (with distributional solutions) can be found in *e.g.* [2]. The algebraic constraints are often eliminated such that the system is described by ordinary differential equations (ODEs). In the case that a system undergoes abrupt structural changes a switched DAE is obtained. Examples of applications are electronic circuits containing switches or mechanical systems with component failure. Since each mode generally has different algebraic constraints, there does not exist a switched ODE description of the system with a common state variable. This problem can be overcome by studying switched DAEs directly.

Several structural properties of switched DAEs have been studied recently. Among those are null-controllability [3], stability [4], stabilizability [5] and observability [6]. All of these studies allow for Dirac impulses in the state trajectory, whereas for some applications Dirac impulses are undesirable. Examples of such applications are electrical circuits containing switches where Dirac impulses in the voltage can damage components or cause electric sparks to occur at the switch [7]. Furthermore, in the case components need to be replaced, *e.g.* for maintenance reasons, or if new components are attached to an operational electrical circuit, some state components might, for safety reasons, be required to be stabilized in a impulse-free fashion. Impulse-controllability of switched DAEs, *i.e.*, the ability to avoid Dirac impulses in the state trajectory by means on an input, has been studied in [8]. However, switched systems that are both impulse-controllable and stabilizable are not necessarily stabilizable with an impulse-free trajectory, as is shown in the following example.

Consider the electrical circuit given in Fig. 1. For maintenance reasons the capacitor and the component consisting of the operational amplifier combined with an inductor are disconnected at $t = t_1$. In order to keep the network to which this circuit is connected running, the voltage source V_0 needs to remain constant. However, there is another controllable voltage source u available.

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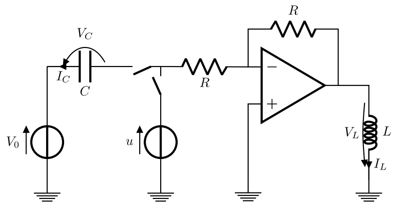

Fig. 1. An example of an electrical circuit that is stabilizable and impulse-controllable, but not stabilizable without Dirac impulses.

Since the system is operational at $t = t_0$ it is assumed that the state at t_0 is consistent. Defining the state as $x = [\, V_L\, I_L\, V_C, I_C, V_0\,]$, we obtain that for $t \in [t_0, t_1)$ the system is described by Eq. (2), whereas for $t \in [t_1, \infty)$ it is described by Eq. (3).

$$\begin{bmatrix} 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u, \tag{2}$$

$$\begin{bmatrix} 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x \tag{3}$$

The current through the resistors is given by $I_R = u/R$ and hence $RI_L = u$. After opening the switch, the voltage over the resistor is zero and thus $I_R = I_L = 0$. Hence for a non-zero input u at t_1^-, we obtain that I_L jumps to zero at t_1^+ and consequently a Dirac impulse occurs in $V_L = L\dot{I}_L$. However, if the input is brought to zero smoothly, no Dirac impulses occur and hence the system is impulse-controllable.

However, since the amount of charge stored on the capacitor is given by $q = C(V_0 - u)$, we have for a nonzero u at t_1, that the capacitor is charged and is unable to discharge, since the current $I_C = 0$. The capacitor can be discharged before t_1, but that requires a nonzero u at t_1^-, which produces a Dirac impulse yet stabilizes the state of the components. Hence we have an example of a system which is impulse controllable and stabilizable, but not stabilizable with an impulse free trajectory.

Motivated by this example, this paper considers stabilization of switched DAEs where Dirac impulses are to be avoided, so called impulse-free stabilization.

The outline of the paper is as follows: notations and results for non-switched DAEs are presented in Section 2. In Section 3 the definition of impulse-free stabilizability is given, together with the introduction of the concept of interval-stabilizability. Results on impulse controllability and (impulse-free) stabilizability are given in Section 4. Finally, the paper ends with a conclusion in Section 5.

Notation The sets of natural, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \mid \Re(\lambda) \geqslant 0\}$ denotes the closed right-half complex plane. For a vector $x \in \mathbb{R}^n$, $|x|$ is its Euclidean norm; $e_i \in \mathbb{R}^n$ is the vector of all zeros except for a one in position i. Let $\mathcal{X}$, $\mathcal{Y} \subseteq \mathbb{R}^n$ be vector spaces and consider the linear map $A : \mathcal{X} \to \mathcal{Y}$. Then, $\operatorname{im} A = \{Ax \mid x \in \mathcal{X}\}$ and $\ker A = \{x \in \mathcal{X} \mid Ax = 0\}$, respectively. The inverse image of a subspace $\mathcal{V} \subseteq \mathcal{Y}$ is given as $A^{-1}\mathcal{V} = \{x \in \mathcal{X} \mid Ax \in \mathcal{V}\}$. Finally, $\mathcal{V}^\perp = \{x \in \mathcal{X} \mid x^\top v = 0\ \forall v \in \mathcal{V}\}$ is the orthogonal complement of a subspace $\mathcal{V} \subseteq \mathcal{X}$ in the inner product space $\mathcal{X}$.

2. Mathematical preliminaries

2.1. Properties and definitions for regular matrix pairs

In the following, we consider *regular* matrix pairs (E, A), i.e. for which the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form* [9].

Proposition 1. *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{4}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.

The matrices S and T can be calculated by using the so-called *Wong sequences* [9,10]:

$$\begin{aligned} &\mathcal{V}_0 := \mathbb{R}^n, && \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), && i = 0, 1, \ldots \\ &\mathcal{W}_0 := \{0\}, && \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), && i = 0, 1, \ldots \end{aligned} \tag{5}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup_i \mathcal{W}_i. \tag{6}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (4) holds.

Based on the Wong sequences we define the following projectors and selectors.

Definition 2. Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (4). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential selector* and *impulse selector* are given by

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \quad \Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S,$$

respectively.

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \qquad B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B,$$

$$E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E = T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}, \qquad B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B.$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). An important feature for DAEs is the so called consistency space, defined as follows:

Definition 3. Consider the DAE $E\dot{x}(t) = Ax(t) + Bu(t)$, then the *consistency space* is defined as

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } x \text{ of} \\ E\dot{x} = Ax, \text{ with } x(0) = x_0 \end{array} \right\},$$

and the *augmented consistency space* is defined as

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solutions } (x, u) \text{ of} \\ E\dot{x} = Ax + Bu \text{ and } x(0) = x_0 \end{array} \right\}.$$

In order to express (augmented) consistency spaces in terms of the Wong limits we introduce the following notation for matrices A, B of conformable sizes:

$$\langle A \mid B \rangle := \operatorname{im}[B, AB, \ldots, A^{n-1}B].$$

Proposition 4 ([11]). *Consider the DAE $E\dot{x} = Ax + Bu$ and assume the matrix pair (E, A) is regular, then $\mathcal{V}_{(E,A)} = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\text{diff}}_{(E,A)}$ and $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.*

2.2. Distributional solutions

The switched DAE (1) usually will not have classical solutions, because each mode of the switched DAE given by the DAE $E_i\dot{x} = A_ix + B_iu$ might have different (augmented) consistency spaces which enforce jumps in the state-variable x. We therefore utilize the piecewise-smooth distributional framework as introduced in [1], i.e. x and u are vectors of piecewise-smooth distributions given by

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, T \subseteq \mathbb{R} \text{ is} \\ \text{discrete}, \forall t \in T : D_t \\ \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \dots\} \end{array} \right\},$$

where $\mathcal{C}^\infty_{\mathrm{pw}}$ denotes the space of piecewise-smooth functions, $f_{\mathbb{D}}$ denotes the regular distribution induced by f, δ_t denotes the Dirac impulse with support $\{t\}$ and δ_t' denotes distributional derivative of δ_t. For a piecewise smooth distribution $D = f_{\mathbb{D}} + \sum_{t\in T} D_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ three types of "evaluation at time t" are defined: left side evaluation $D(t^-) := f(t^-)$, right side evaluation $D(t^+) := f(t^+)$ and the impulsive part $D[t] := D_t$ if $t \in T$ and $D[t] = 0$ otherwise.

It can be shown (see e.g. [12]) that the space $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ can be equipped with a multiplication, in particular, the multiplication of a piecewise-constant function with a piecewise-smooth distribution is well defined and the switched DAE (1) can be interpreted as an equation within the space of piecewise-smooth distributions. Hence the following solution behavior (depending on σ) is well defined:

$$\mathfrak{B}_\sigma := \{(x, u) \in \mathbb{D}^{n+m}_{\mathrm{pw}\mathcal{C}^\infty} \mid E_\sigma\dot{x} = A_\sigma x + B_\sigma u\},$$

and restrictions of x and u to intervals, are well defined as well. Given the notation $x_{\mathcal{I}}$ for the restriction of x to the interval $\mathcal{I} \subseteq \mathbb{R}$, it is shown in [1] that the *initial trajectory problem* (ITP) (7)

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{7a}$$

$$(E\dot{x})_{[0,\infty)} = (Ax)_{[0,\infty)} + (Bu)_{[0,\infty)}, \tag{7b}$$

has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. As a direct consequence, the switched DAE (1) with regular matrix pairs is also uniquely solvable (with piecewise-smooth distributional solutions) for any switching signal with locally finitely many switches.

2.3. Properties of DAEs

For the rest of this section we are considering the DAE

$$E\dot{x} = Ax + Bu. \tag{8}$$

Recall the following definitions and characterization of (impulse) controllability [11].

Proposition 5. *The reachable space of the regular DAE (8) defined as*

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \;\exists \text{ smooth solution } (x, u) \text{ of (8)} \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\}$$

satisfies $\mathcal{R} = \langle A^{\mathrm{diff}} \mid B^{\mathrm{diff}} \rangle \oplus \langle E^{\mathrm{imp}} \mid B^{\mathrm{imp}} \rangle$.

It is easily seen that the reachable space for (8) coincides with the (null-)controllable space, i.e.

$$\mathcal{R} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \;\exists \text{ smooth solution } (x, u) \text{ of (8)} \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Note that $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R} = \mathcal{V}_{(E,A)} \oplus \langle E^{\mathrm{imp}}, B^{\mathrm{imp}} \rangle$.

Definition 6. The DAE (8) is impulse controllable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x, u) of the ITP (7) such that $x(0^-) = x_0$ and $(x, u)[0] = 0$, i.e. the state and the input are impulse free at $t = 0$. The space of impulse controllable states of the DAE (8) is given by

$$\mathcal{C}^{\mathrm{imp}}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } (x, u) \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \text{ of (7)} \\ \text{s.t. } x(0^-) = x_0 \text{ and } (x, u)[0] = 0. \end{array} \right\}.$$

In particular, the DAE (8) is impulse controllable if and only if $\mathcal{C}^{\mathrm{imp}}_{(E,A,B)} = \mathbb{R}^n$.

The impulse controllable space can be characterized as follows [13].

Proposition 7. *Consider the DAE (8) then*

$$\begin{aligned} \mathcal{C}^{\mathrm{imp}}_{(E,A,B)} &= \mathcal{V}_{(E,A,B)} + \ker E \\ &= \mathcal{V}_{(E,A)} + \mathcal{R} + \ker E \\ &= \mathcal{V}_{(E,A)} + \langle E^{\mathrm{imp}} \mid B^{\mathrm{imp}} \rangle + \ker E. \end{aligned}$$

Definition 8. The DAE (8) is stabilizable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x, u) of the ITP (7) such that $x(0^-) = x_0$ and $\lim_{t\to\infty} x(t) = 0$.

According to [7] if the input $u(\cdot)$ is sufficiently smooth, trajectories of (8) are continuous on the open interval (t_0, ∞) and given by

$$x(t) = x_u(t, t_0; x_0) = e^{A^{\mathrm{diff}}(t-t_0)} \Pi_{(E,A)} x_0 + \int_{t_0}^{t} e^{A^{\mathrm{diff}}(t-s)} B^{\mathrm{diff}} u(s)\, ds - \sum_{i=0}^{n-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t). \tag{9}$$

In particular, all trajectories can be written as the sum of an autonomous part $x_{\mathrm{aut}}(t, t_0; x_0) = e^{A^{\mathrm{diff}} t} \Pi_{(E,A)} x_0$ and a controllable part $x_u(t, t_0)$ as follows:

$$x_u(t, t_0; x_0) = x_{\mathrm{aut}}(t, t_0; x_0) + x_u(t, t_0).$$

This decomposition remains valid for switched DAEs when evaluated at the initial condition at time t_0^-; the impulsive part of x at the initial time t_0 is then given by

$$x[t_0] = -\sum_{i=0}^{n-1} (E^{\mathrm{imp}})^{i+1} \left(x_0 \delta^{(i)} + \sum_{j=0}^{i} B^{\mathrm{imp}} u^{(i-j)}(t_0^+) \delta^{(j)} \right).$$

3. Stabilizability concepts

The concepts introduced in the previous section are now utilized to investigate impulse free stabilizability of switched DAEs. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals from the following class

$$\Sigma := \left\{ \sigma : \mathbb{R} \to \mathbb{N} \;\middle|\; \begin{array}{l} \sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps} \end{array} \right\},$$

i.e. we exclude an accumulation of switching times (see [1]). By further excluding infinitely many switches in the past and by appropriately relabeling the matrices we can assume that

$$\sigma(t) = k, \qquad \text{for } t_k \leqslant t < t_{k+1}. \tag{10}$$

and that for the first switching instant t_1 it holds that $t_1 > t_0 := 0$. After some results relating interval-wise properties to global properties in the remainder of this section, we will restrict our attention to the bounded interval (t_0, t_f) for some $t_f > 0$. As a consequence there are only finitely many switches in this interval, say $\mathrm{n} \in \mathbb{N}$, and for notation convenience we let $t_{\mathrm{n}+1} = t_f$.

Roughly speaking, in classical literature on non-switched systems, a linear system is called stabilizable if every trajectory can be steered towards zero as time tends to infinity. This definition can readily be applied to switched DAEs. Hence we will define impulse free stabilizability for switched DAEs in a similar fashion as follows, based on the definition of stabilizability in [5].

Definition 9 (*Impulse-free Stabilizability*)**.** The switched DAE (1) with switching signal (10) is stabilizable if the corresponding solution behavior $\mathfrak{B}_\sigma$ is stabilizable in the behavioral sense on the interval $[0, \infty)$, i.e.

$$\forall (x, u) \in \mathfrak{B}_\sigma \ \exists (x^*, u^*) \in \mathfrak{B}_\sigma :$$
$$(x^*, u^*)_{(-\infty,0)} = (x, u)_{(-\infty,0)},$$
$$\text{and } \lim_{t\to\infty} (x^*(t^+), u^*(t^+)) = 0.$$

If in addition $(x, u)[t] = 0$ for all $t \in [0, \infty)$, then the system is called impulse-free stabilizable.

In the case of switched DAEs, it is reasonable to assume that there are an infinite amount of switching instances as time tends to infinity. This poses a problem when it comes to verifying conditions for stabilizability in a finite amount of steps. To overcome this problem, we investigate stabilizability on a bounded interval. Therefore we introduce the following definition of (impulse-free) interval stabilizability.

Definition 10 (*Interval Stabilizability*)**.** The switched DAE (1) is called (t_0, t_f)*-stabilizable* for a given switching signal σ, if there exists a class $\mathcal{KL}$ function[1] $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with

$$\beta(r, t_f - t_0) < r, \quad \forall r > 0,$$

and for any initial value $x_0 \in \mathcal{V}_{E_0,A_0,B_0}$ there exists a local solution (x, u) of (1) on (t_0, t_f) with $x(t_0^-) = x_0$ such that

$$|x(t^+)| \leqslant \beta(|x_0|, t - t_0), \quad \forall t \in (t_0, t_f).$$

If in addition $(x, u)[t] = 0$ for all $t \in (t_0, t_f)$, then the system is called impulse-free (t_0, t_f)-stabilizable.

One should note that a solution on some interval is not necessarily a part of a solution on a larger interval. Consequently, stabilizability does not always imply interval stabilizability. The switched system $0 = x$ on $[0, t_1)$ and $\dot{x} = 0$ on $[t_1, \infty)$ is obviously stabilizable, since the only global solution is the zero solution. However, on the interval $[t_1, s)$ there are nonzero solutions which do not converge towards zero.

Furthermore according to Definition 10 it is required that the norm of the state is smaller at the end of an interval. This means that (impulse-free) interval stability could depend on the length of the interval considered instead of the asymptotic behavior of the system. An unstable oscillating system is thus possibly (impulse-free) interval stable and an asymptotically stable oscillating system is not necessarily (impulse-free) interval stable, depending on the choice of interval. However, under the following uniformity assumption on the switched DAE we can conclude global stabilizability.

Assumption 11 (*Uniform Interval-stabilizability*)**.** Consider the switched system (1) with switching signal σ. Let $\tau_0 := t_0$ and assume that there exists an unbounded, strictly increasing sequence $\tau_i \in (t_0, \infty)$, $i \in \mathbb{N} \setminus \{0\}$, of non-switching times such that the system is (impulse-free) (τ_{i-1}, τ_i) -stabilizable with $\mathcal{KL}$ function β_i for which additionally it holds that

$$\beta_i(r, \tau_i - \tau_{i-1}) \leqslant \alpha r, \quad \forall r > 0, \forall i \in \mathbb{N}_{>0}$$
$$\beta_i(r, 0) \leqslant Mr, \quad \forall r > 0, \forall i \in \mathbb{N}_{>0},$$

for some uniform $\alpha \in (0, 1)$ and $M \geqslant 1$.

Proposition 12. *If the switched system (1) is uniformly (impulse-free) interval-stabilizable in the sense of Assumption 11 then (1) is (impulse-free) stabilizable.*

The proof of Proposition 12 is along the same lines as the proof of Proposition 8 in [14].

4. Impulse-free stabilization and controllability

Assumption 11 can be verified for a general class of systems such as systems with periodic switching and systems with a finite amount of modes. Therefore we turn our attention to finding necessary and sufficient conditions for interval stabilizability.

As follows from Definition 10, for any initial condition x_0, there needs to exist a solution on $[t_0, t_f)$ that is impulse-free and satisfies the stability property. Hence we will first discuss necessary and sufficient conditions for a switched DAE to have impulse free solutions for any initial condition x_0 on a bounded interval, i.e. impulse controllability for switched DAEs. Once these conditions are discussed, we will investigate under which conditions these impulse-free solutions are satisfying the stability property.

In the remainder of this section we will use Π_i, A_i^{diff}, E_i^{imp}, B_i^{imp}, B_i^{diff}, $\mathcal{R}_i$, $\mathcal{C}_i$, $\mathcal{C}_i^{\text{imp}}$ to denote the corresponding matrices and subspaces associated to the ith mode.

4.1. Impulse controllability

As mentioned above, we will first investigate the concept of impulse controllability of a switched DAE, of which the definition is formalized as follows.

Definition 13. The switched DAE (1) with some fixed switching signal σ is called *impulse controllable on the interval* (t_0, t_f), if for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists a solution $(x, u) \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^{n+m}$ of (1) with $x(t_0^+) = x_0$ which is impulse free.

Remark 14. As an alternative for Definition 13, impulse controllability could also be defined in terms of arbitrary initial values $x_0 \in \mathbb{R}^n$. This would result in the immediate necessary condition that the first mode of a switched DAE needs to be impulse controllable. However, given a higher index DAE, Dirac impulses cannot be avoided for initial conditions in $(\mathcal{C}_0^{\text{imp}})^\perp$. Therefore it is reasonable to consider initial conditions in $\mathcal{C}_0^{\text{imp}} = \mathcal{V}_{(E_0,A_0,B_0)} + \ker E_0$. Considering the linearity of solutions and the fact that initial conditions in $\ker E_0$ result in trajectories that jump to zero in an impulse free manner, the initial conditions of interest are those contained in $\mathcal{V}_{(E_0,A_0,B_0)}$.

Remark 15. If the interval (t_0, t_f) does not contain a switch, then the corresponding switched DAE is *always* impulse controllable on that interval due the definition of the augmented consistency space in terms of smooth (in particular, impulse free) solutions. This seems counter intuitive, because the active mode on that interval is not necessarily impulse controllable; however, recall that impulse controllability for a single mode (see Definition 6) is formulated in terms of the ITP (7), which can be interpreted as a switched system with one switch at $t_1 = 0$. In fact, letting $t_0 =$

[1] A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is called a class $\mathcal{KL}$ function if (1) for each $t \geqslant 0$, $\beta(\cdot, t)$ is continuous, strictly increasing, with $\beta(0, t) = 0$; (2) for each $r \geqslant 0$, $\beta(r, \cdot)$ is decreasing and converging to zero as $t \to \infty$.

$-\varepsilon$, $t_f = \varepsilon$, $(E_0, A_0, B_0) = (I, 0, 0)$ and $(E_1, A_1, B_1) = (E, A, B)$, the DAE (8) is impulse controllable if, and only if, the corresponding ITP (reinterpreted as a switched DAE) is impulse controllable on $(-\varepsilon, \varepsilon)$.

A solution of a switched DAE can only be impulse free, if at each switching instance the solution evaluated at t_i^- is in the impulse controllable space $\mathcal{C}_i^{\text{imp}}$. Therefore we consider the largest set of points from which the impulse controllable space of the next mode can be reached impulse freely from the preceding mode. To that extent we define the following sequence of subspaces regarding the switched DAE (1) with switching signal (10):

$$\begin{aligned}\mathcal{K}_{\mathtt{n}}^b &= \mathcal{C}_{\mathtt{n}}^{\text{imp}},\\ \mathcal{K}_{i-1}^b &= \operatorname{im} \Pi_{i-1} \cap \left(e^{-A_{i-1}^{\text{diff}}(t_{i-1}-t_i)}\mathcal{K}_i^b + \mathcal{R}_{i-1}\right)\\ &\quad + \langle E_{i-1}^{\text{imp}} \mid B_{i-1}^{\text{imp}}\rangle + \ker E_{i-1},\\ &\quad i = \mathtt{n}, \mathtt{n}-1, \ldots, 1.\end{aligned} \tag{11}$$

Note that $\operatorname{im} \Pi_{i-1} = \mathcal{V}_{(E_i,A_i)}$ and that $\mathcal{C}_i^{\text{imp}} = \mathcal{V}_{(E_i,A_i)} + \mathcal{R}_i + \ker E_i$. Therefore we have that $\mathcal{K}_i^b \subseteq \mathcal{C}_i^{\text{imp}}$. Note furthermore, that the definition is *backwards* in time; the sequences start with the last mode $\mathtt{n}$ and end with the initial mode 0. With these sets, we can prove the following lemma.

Lemma 16. *Consider the (interval restricted) switched DAE $(E_\sigma \dot{x})_{[t_{i-1},t_i)} = (A_\sigma)_{[t_{i-1},t_i)} + (B_\sigma u)_{[t_{i-1},t_i)}$. Then $\mathcal{K}_{i-1}^b$ is the largest set of points at time t_{i-1}^- from which $\mathcal{K}_i^b$ can be reached (at t_i^-) in an impulse free way.*

The proof is similar to the proof of Lemma 19 in [8] and therefore omitted.

Corollary 17. *Consider the switched system (1) with switching signal (10). The system is impulse controllable if and only if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^b.$$

The proof is similar to the proof of Theorem 21 in [8] and therefore omitted.

Example 18. Consider the example given in the introduction on the interval $(0, t_f)$ with a switch at $t = t_1$. The matrices (E_0, A_0, B_0) correspond the system matrices given in (2) and (E_1, A_1, B_1) are the system matrices given in (3). The projectors Π_i and selectors $\Pi_i^{\text{diff}}, \Pi_i^{\text{imp}}$, $i \in \{0, 1\}$ can be calculated from the Wong sequences. Then it follows that

$$\mathcal{C}_1^{\text{imp}} = \operatorname{span}\left\{\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}\right\},\quad \Pi_0 = \operatorname{span}\left\{\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}\right\},$$

$$\mathcal{R}_0 = \operatorname{span}\left\{\begin{bmatrix}0\\-1\\R\\0\\0\end{bmatrix}, \begin{bmatrix}-L\\0\\0\\RC\\0\end{bmatrix}\right\} = \langle E_0^{\text{imp}} \mid B_0^{\text{imp}}\rangle$$

and thus from (11) it is calculated that $\mathcal{K}_0^b = \mathbb{R}^5$. Furthermore, it can be calculated that

$$\mathcal{V}_{[E_0,A_0,B_0]} = \operatorname{span}\left\{\begin{bmatrix}0\\-1\\R\\0\\0\end{bmatrix}, \begin{bmatrix}-L\\0\\0\\RC\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}\right\}. \tag{12}$$

and hence we can conclude that the system is impulse-controllable.

4.2. Impulse-free stabilizability

As shown in the introduction, a switched DAE which is impulse controllable and stabilizable is not necessarily impulse-free stabilizable. However, impulse-controllability is an obvious necessary condition for impulse-free stabilizability. In order to stabilize a state on a bounded interval in an impulse-free way, there needs to exist an impulse-free solution in the first place. To that extent, we will make the following standing assumptions throughout the rest of this section:

1. The switched DAE (1) is impulse-controllable.
2. The initial condition is consistent, i.e. $x(t_0^+) = x_0 \in \mathcal{V}_{E_0,A_0,B_0}$.

Under these assumptions, we will derive necessary and sufficient conditions for impulse-free stabilizability. The approach taken is as follows. First we consider the space of points that can be reached in an impulse free way from an initial value x_0. It will then be shown that this space is an affine subspace. We then consider an element of this affine subspace with minimal norm; if this norm is smaller than the norm of the corresponding initial value, we can conclude interval stabilizability.

Towards this goal, we consider the following sequence of (affine) subspaces (defined *forward* in time)

$$\begin{aligned}\mathcal{K}_0^f(x_0) &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \mathcal{R}_0,\\ \mathcal{K}_i^f(x_0) &= e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i(\mathcal{K}_{i-1}^f(x_0) \cap \mathcal{C}_i^{\text{imp}}) + \mathcal{R}_i,\ \ i > 0,\end{aligned} \tag{13}$$

For $x_0 = 0$ we drop the dependency on x_0, i.e.

$$\mathcal{K}_i^f := \mathcal{K}_i^f(0).$$

Remark 19. Note that the above $\mathcal{K}_i^f$ is different from $\mathcal{K}_i^f$ in [8], the latter is defined as the space of all points that can be reached in an impulse-free way, i.e., it is the union of $\mathcal{K}_i^f(x_0)$ over all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$.

The intuition behind the sequence is as follows: $\mathcal{K}_0^f(x_0)$ are all values for $x_u(t_1^-, x_0)$ which can be reached in an impulse free (in fact, smooth) way during the initial mode 0. Now, inductively, we calculate the set $\mathcal{K}_i^f(x_0)$ of points which can be reached just before the switching time t_{i+1} by first considering the points $\mathcal{K}_{i-1}^f(x_0)$ which can be reached in an impulse free way just before t_i, then pick those which can be continued in mode i impulse-freely by intersecting them with $\mathcal{C}_i^{\text{imp}}$, propagate this set forward according to the evolution operator and finally add the reachable space of mode i. This intuition is verified by the following lemma.

Lemma 20. *Consider the switched system (1) on some bounded interval (t_0, t_f) with the switching signal given by (10). Then for all $i = 0, 1, \ldots, \mathtt{n}$ and $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$*

$$\mathcal{K}_i^f(x_0) = \left\{\xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l}\exists \text{ an impulse-free solution } (x,u)\\ \text{of}(1) \text{ on } (t_0, t_{i+1}) \text{ s.t.}\\ x(t_0^+) = x_0 \wedge x(t_{i+1}^-) = \xi\end{array}\right\}.$$

Proof. First we will show that $x_u(t_i^-, x_0)$ is contained in $\mathcal{K}_i^f(x_0)$ if (x, u) is an impulse free solution on (t_0, t_f). To that extent, consider an impulse-free solutions (x, u) of (1) on (t_0, t_1), which by definition satisfies the solution formula (9), i.e.,

$$x_u(t_1^-, x_0) = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \eta_0,$$

for some $\eta_0 \in \mathcal{R}_0$ and $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. This shows that $x_u(t_1^-, x_0) \in \mathcal{K}_0^f(x_0)$. We proceed inductively by assuming that the statement holds for $i > 0$ and prove the statement for $i + 1$.

Let (x, u) be an impulse-free solution on (t_0, t_{i+1}). Then we have that $x_u(t_{i+1}^-, x_0)$ is of the form

$$x_u(t_{i+1}^-, x_0) = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \xi_{i-1} + \eta_i,$$

for some $\eta_i \in \mathcal{R}_i$ and $\xi_{i-1} \in \mathcal{C}_i^{\text{imp}}$. Furthermore, since (x, u) is impulse-free on (t_0, t_{i+1}), it follows that ξ_i can be reached impulse-freely from x_0 and hence $\xi_{i-1} \in \mathcal{K}^f_{i-1}(x_0)$. This proves that $x_u(t^-_{i+1}, x_0) \in \mathcal{K}^f_i(x_0)$.

In the following we will prove that for all elements of $\mathcal{K}^f_i(x_0)$ there exists an impulse-free solution (x, u) with initial condition $x_u(t^+_0, x_0) = x_0$. We will again prove this inductively. Therefore, consider $\xi_0 \in \mathcal{K}^f_0(x_0)$. Then for some $\eta_0 \in \mathcal{R}_0$ we have

$$\xi_0 = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \eta_0.$$

Since $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{C}^{\text{imp}}$, we have that there exists a $\tilde{u}$ such that $x_{\tilde{u}}(t, x_0)$ is impulse-free on $[t_0, t_1)$. Then it follows from the solution formula (9) that

$$x_{\tilde{u}}(t^-_1, x_0) = e^{A^{\text{diff}}t_1}\Pi x_0 + \tilde{\eta}_0,$$

for some $\tilde{\eta}_0 \in \mathcal{R}_0$. Since $\eta_0 \in \mathcal{R}_0$, there exists a smooth input $\hat{u}$ such that $x_{\hat{u}}(t^-_1, 0) = \eta_0 - \tilde{\eta}_0$ and $x_{\hat{u}}(t, 0)$ is impulse-free on $[t_0, t_1)$.

If we define $u = \hat{u}+\tilde{u}$ it then follows from linearity of solutions that $x_u(t^-_1, x_0) = \xi_0$ and is impulse-free on (t_0, t_1). Assuming that the statement holds for $i > 0$ we continue by proving the statement for $i + 1$.

Let $\xi_i \in \mathcal{K}^f_{i+1}(x_0)$, then we have for some $\xi_{i-1} \in \mathcal{K}^f_i(x_0) \cap \mathcal{C}^{\text{imp}}_{i-1}$ that

$$\xi_i = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \xi_{i-1} + \eta_i.$$

It follows from the induction assumption that there exists an impulse-free solution (x, u) on (t_0, t_i) with $x_u(t^-_i, x_0) = \xi_{i-1}$, because $\xi_{i-1} \in \mathcal{K}^f_i(x_0)$. Furthermore, $\xi_{i-1} \in \mathcal{C}^{\text{imp}}_{i-1}$ and $\eta_i \in \mathcal{R}_i$ implies that the impulse-free input u can be altered on the interval $[t_i, t_i + 1)$ such that $x_u(t^-_{i+1}, x_0) = \xi_i$ and $x_u(\cdot, x_0)$ is impulse-free. ∎

Remark 21. The assumption that $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ is of crucial importance for Lemma 16. If the zeroth mode is not impulse controllable and we would choose $x_0 \in (\mathcal{V}_{(E_0,A_0,B_0)} + \ker E_0)^\perp$ the occurrence of a Dirac impulse would be inevitable. This means that $\mathcal{K}^f_0(x_0)$ should be empty. However, the algorithm (13) would state that $\mathcal{K}^f_0(x_0)$ is nonempty, which is not true.

Remark 22. If the system is not impulse controllable, then there exists x_0 for which $\mathcal{K}^f_i(x_0) = \varnothing$ as follows from the definition. This also follows from the subspace algorithm because $\mathcal{K}^f_{i-1}(x_0) \cap \mathcal{C}^{\text{imp}}_i$ would be empty for some mode i and the sum of an empty set and a subspace is empty.

Lemma 20 gives rise to another characterization of impulse controllability, which follows as a corollary.

Corollary 23. *Consider the switched system (1) on some interval (t_0, t_f) with the switching signal given by (10) and the sequence of affine subspaces $\mathcal{K}^f_i(x_0)$ given by (13). Then (1) is impulse controllable on (t_0, t_f) if and only if*

$$\forall x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}: \quad \mathcal{K}^f_{\mathrm{n}}(x_0) \neq \varnothing.$$

Proof. If the system is impulse controllable, then for every initial condition x_0 there exists an impulse free solution (x, u) on (t_0, t_f). Therefore $x(t^-_f) \in \mathcal{K}^f_{\mathrm{n}+1}(x_0)$ (recall the convention that $t_{\mathrm{n}+1} := t_f$) and hence $\mathcal{K}^f_{\mathrm{n}+1}(x_0) \neq \varnothing$. Conversely, if $\mathcal{K}_{\mathrm{n}}(x_0) \neq \varnothing$, then let $\xi \in \mathcal{K}^f_{\mathrm{n}+1}(x_0)$. By definition there exists an impulse free solution (x, u) on (t_0, t_f) with $x(t^-_0) = x_0$ and $x(t^-_f) = \xi$. This holds for every $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ and hence (1) is impulse controllable. ∎

Note that in contrast to Corollary 17 the computations in Corollary 23 run forward in time. Hence this result is useful in the case that not all modes are determined yet and the next mode is to be chosen. If Corollary 17 would be used, all computations would need to be redone, whereas with a forward computation only parts need to be redone.

In the following we will show that $\mathcal{K}^f_i(x_0)$ is an affine shift of $\mathcal{K}^f_f$ and hence $\mathcal{K}^f_i(x_0)$ is an affine subspace. In proving this statement, we will use some general results which can be found in the Appendix.

Lemma 24. *Consider the switched system (1) with switching signal (10) and assume it is impulse-controllable. The impulse-free-reachable space from x_0 at t_i is an affine shift from the impulse-free reachable space, i.e., there exists a matrix M_i, such that*

$$\mathcal{K}^f_i(x_0) = M_i x_0 + \mathcal{K}^f_i. \tag{14}$$

Proof. First we simplify the notation introducing the following short hand notation $Y_i := e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i$. Then we prove the statement inductively. The statement holds trivially for $n = 0$, for $\mathcal{K}^f_0 = Y_0 x_0 + \mathcal{R}_0$ and hence we assume that the statement holds for n. Since we assumed that the system is impulse controllable, we have that $\mathcal{K}^f_i(x_0) \cap \mathcal{C}^{\text{imp}}_{i+1} \neq \varnothing$ for all x_0. Then for $n+1$ we obtain that

$$\begin{aligned}
\mathcal{K}^f_{i+1}(x_0) &= Y_{i+1}(\mathcal{K}^f_i(x_0) \cap \mathcal{C}^{\text{imp}}_{i+1}) + \mathcal{R}_{i+1} \\
&\overset{*}{=} Y_{i+1}((M_i x_0 + \mathcal{K}^f_i) \cap \mathcal{C}^{\text{imp}}_{i+1}) + \mathcal{R}_{i+1}, \\
&\overset{**}{=} Y_{i+1}(N_i M_i x_0 + (\mathcal{K}^f_i \cap \mathcal{C}^{\text{imp}}_{i+1})) + \mathcal{R}_{i+1}, \\
&= Y_{i+1}N_i M_i x_0 + \mathcal{K}^f_{i+1},
\end{aligned}$$

for some matrix N_i, $i \in \{0, 1, \dots, \mathrm{n}\}$, where $(*)$ follows from the induction step and $(**)$ follows from Proposition 42 in the Appendix. Defining $M_{i+1} = Y_{i+1}N_i M_i$ yields the result. ∎

Note that the matrix M_i in (14) exists only in case that the system is impulse-controllable, otherwise M_i would also need to map to the empty set. In the case M_i does exist, this matrix can be chosen independently of x_0. It is however not necessarily unique, because M_{i+1} is dependent on N_i obtained from Proposition 42 in a nonunique way. It follows from Lemma 43 from the Appendix that N_i can be any matrix for which

$$\begin{aligned}
&1.\ \ \operatorname{im}(N_i - I)M_i \subseteq \mathcal{R}_i, \\
&2.\ \ \operatorname{im} N_i M_i \subseteq \mathcal{C}^{\text{imp}}_{i+1}.
\end{aligned} \tag{15}$$

Thus, from the proof of Lemma 24 together with Lemma 43 from the Appendix the following constructive result can be obtained.

Corollary 25. *Consider the switched system (1) with switching signal (10) and assume it is impulse-controllable. Let $M_0 = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0$. Then for any choice of N_i satisfying (15), a matrix M_{i+1} satisfying (14) can be calculated sequentially as follows:*

$$M_{i+1} = e^{A_{i+1}^{\text{diff}}(t_{i+2}-t_{i+1})\Pi_{i+1}} N_i M_i.$$

Remark 26. In order to compute an N_i that satisfies (15) we can invoke Lemma 44 from Appendix. This means that given projectors onto $\mathcal{R}_i$ and $\mathcal{C}^{\text{imp}}_{i+1}$, an N_i that satisfies the conditions (15) can be constructed by solving

$$(I - \Pi_{\mathcal{R}_i})\Pi_{\mathcal{C}^{\text{imp}}_{i+1}} Q_i M_i = (I - \Pi_{\mathcal{R}_i})M_i \tag{16}$$

for Q_i and defining $N_i := \Pi_{\mathcal{C}^{\text{imp}}_{i+1}} Q_i$. Since the existence of a solution of (16) is guaranteed by the assumption of impulse-controllability, such a matrix equation can be solved using a linear programming solver.

Since $\mathcal{K}_i^f(x_0)$ contains all the states that can be reached from x_0 in an impulse free way, it follows that the norm of the state with minimal norm is given by the distance $\mathrm{dist}(\mathcal{K}_i^f(x_0), 0)$. The computation of this distance is straightforward, because $\mathcal{K}_i^f(x_0)$ is an affine subspace. It follows from elementary linear algebra that the distance between an affine subspace and the origin, is equal to the norm of any element projected to the orthogonal complement of the vector space associated to the affine subspace. In the case of $\mathcal{K}_i^f(x_0)$ we would need to project onto $(\mathcal{K}_i^f)^\perp$ with a projector $\Pi_{(\mathcal{K}_i^f)^\perp}$. An important property of these projectors is that their restriction to the corresponding augmented consistency space is well defined.

Lemma 27. *Consider the DAE (1) with switching signal (10). For any $i \in \{0, 1, \dots, \mathtt{n}\}$ let $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$, then*

$$\Pi_{(\mathcal{K}_i^f)^\perp}\xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Proof. From $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$ and $\Pi_{(\mathcal{K}_i^f)^\perp} + (I - \Pi_{(\mathcal{K}_i^f)^\perp}) = I$, it follows that

$$\Pi_{(\mathcal{K}_i^f)^\perp}\xi + (I - \Pi_{(\mathcal{K}_i^f)^\perp})\xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Since $\mathrm{im}(I - \Pi_{(\mathcal{K}_i^f)^\perp}) = \mathcal{K}_i^f$ and $\mathcal{K}_i^f \subseteq \mathcal{V}_{(E_i,A_i,B_i)}$ we obtain

$$\Pi_{(\mathcal{K}_i^f)^\perp}\xi \in \mathcal{V}_{(E_i,A_i,B_i)} - (I - \Pi_{(\mathcal{K}_i^f)^\perp})\xi \subseteq \mathcal{V}_{(E_i,A_i,B_i)}.$$

as was to be shown. ■

Consequently, the following result follows.

Lemma 28. *Consider the DAE (1) with switching signal (10) and assume it is impulse-controllable. For any M_i satisfying (14) we have that*

$$\min_{x \in \mathcal{K}_i^f(x_0)} |x| = |\Pi_{\left(\mathcal{K}_i^f\right)^\perp} M_i x_0|$$

It follows that we can consider $\Pi_{\left(\mathcal{K}_i^f\right)^\perp} M_i$ as a linear map from the initial condition x_0 to the state with minimal norm in $\mathcal{K}_i^f(x_0)$. This allows us to formulate the following characterization of impulse-free stabilizability, which is independent of the initial condition x_0 and independent of any coordinate system.

Theorem 29. *Consider the switched DAE (1) with switching signal (10) and assume it is impulse controllable. Then the system is impulse-free interval-stabilizable on (t_0, t_f) if and only if*

$$\|\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}\|_2 = \sup_{x \neq 0} \frac{|\Pi_{\left(\mathcal{K}_\mathtt{n}^f\right)^\perp} M_\mathtt{n} x|_2}{|x|_2} < 1$$

Proof. It follows from Lemma 28 that $\Pi_{\left(\mathcal{K}_\mathtt{n}^f\right)^\perp} M_\mathtt{n}$ is the linear operator that maps x_0 to the element in $\mathcal{K}_\mathtt{n}^f(x_0)$ with minimal norm. Therefore we see that if $\|\Pi_{\left(\mathcal{K}_i^f\right)^\perp} M_i\|_2 < 1$ that for all x_0 there exists an input u such that

$$|x_u(t_f, x_0)| = |\Pi_{\left(\mathcal{K}_i^f\right)^\perp} M_i x_0| < |x_0|.$$

From this we can conclude that there exists a class $\mathcal{KL}$ function $\beta(|x_0|, t_f - t_0)$ such that the system is impulse-free interval stabilizable in the sense of Definition 8.

Conversely, if the system is impulse-free interval stabilizable, then there exists a trajectory for each initial condition $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ such that $|x_u(t_f^-, x_0)| \leqslant \beta_i(|x_0|, t_f - t_0) < |x_0|$. This means that for the operator $\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}$ that maps $|x_0|$ to the element with minimal norm that can be reached in an impulse-free way it must hold that

$$\|\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}\|_2 = \sup_{x \neq 0} \frac{|\Pi_{\left(\mathcal{K}_\mathtt{n}^f\right)^\perp} M_\mathtt{n} x|_2}{|x|_2} < 1,$$

which proves the result. ■

For many applications it is not sufficient to reduce the norm of the state, but it is necessary to control the state to zero without any Dirac impulses occurring. If a state can be steered to zero in an impulse free way, we call this state *impulse-free null-controllable*. A formal definition of this concept is as follows.

Definition 30. Consider the system (1) with switching signal (10). An initial condition x_0 is called *impulse-free null-controllable* if there exists an input u such that $x_u(t_f^-, x_0) = 0$ and the trajectory is impulse-free. We call the system impulse-free null-controllable if every $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ is impulse-free null-controllable.

Using the method from the previous section, the following characterization can readily be stated.

Theorem 31. *Consider the system (1) with switching signal (10). An initial value x_0 is impulse-free null-controllable, if and only if for some $i \geqslant 0$*

$$\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f.$$

Proof. If an initial condition is impulse-free null-controllable, there exists an input u such that $x_u(t_f^-, x_0) = 0$ and the trajectory is impulse free. This means that $0 \in \mathcal{K}_{\mathtt{n}+1}^f(x_0)$. As a consequence

$$0 \subseteq M_{\mathtt{n}+1}x_0 + \mathcal{K}_{\mathtt{n}+1}^f,$$

from which it follows that $M_{\mathtt{n}+1}x_0 \in \mathcal{K}_{\mathtt{n}+1}^f$ and therefore $\mathcal{K}_{\mathtt{n}+1}^f(x_0) \subseteq \mathcal{K}_{\mathtt{n}+1}^f$.

Conversely if for some $i = k \geqslant 0$ $\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f$, it follows that $M_i x_0 \in \mathcal{K}_i^f$. As a consequence $0 \in M_i x_0 + \mathcal{K}_i^f = \mathcal{K}_i^f(x_0)$. It follows from the sequence (13) if $\mathcal{K}_i^f \subseteq \mathcal{K}_i^f$ for $i = k \geqslant 0$ that it holds for all $i \geqslant k$. ■

As a direct consequence we can state the following result.

Corollary 32. *Consider the switched system (1) with switching signal (10) and assume it is impulse controllable. Then the system is impulse-free null-controllable on (t_0, t_f) if, and only if, for some $i \in \{0, 1, \dots, \mathtt{n}\}$*

$$\Pi_{(\mathcal{K}_i^f)^\perp} M_i = 0.$$

Proof. If the system is impulse-null controllable, we have that $\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f$ for all x_0. Then it follows that

$$M_i x_0 + \mathcal{K}_i^f \subseteq \mathcal{K}_i^f,$$

for all x_0 and hence $\mathrm{im}\, M_i \subseteq \mathcal{K}_i^f$. The result then follows.

Conversely, if $\Pi_{(\mathcal{K}_i^f)^\perp} M_i = 0$, then $\Pi_{(\mathcal{K}_i^f)^\perp}\mathcal{K}_i^f(x_0) = 0$ for all x_0, which implies that $\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f$ for all x_0. ■

$\mathcal{K}_i^f$ and M_i can both be computed sequentially forward in time. This means that it might not be necessary to have knowledge of all the modes of the switched system. According to Corollary 32 we can conclude impulse-free null-controllability already if the conditions are satisfied for some $i \in \mathbb{N}$.

4.3. Impulsive stabilizability and impulse-controllability

In the case that Dirac impulses are allowed in the trajectory similar results as in the above can be formulated. The crucial condition for impulse-free trajectories is that the state is in the impulse controllable space of the next mode at each switching instance. If this condition is dropped, a similar lemma as Lemma 20 can be formulated after considering the following sequence of sets

$$\begin{aligned} \tilde{\mathcal{K}}_0^f(x_0) &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \mathcal{R}_0, \\ \tilde{\mathcal{K}}_i^f(x_0) &= e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \tilde{\mathcal{K}}_{i-1}^f(x_0) + \mathcal{R}_i, \ \ i > 0, \end{aligned} \tag{17}$$

For $x_0 = 0$ we drop the dependency on x_0, i.e.

$$\tilde{\mathcal{K}}_i^f := \tilde{\mathcal{K}}_i^f(0).$$

Lemma 33. *Consider the switched system (1) on some bounded interval (t_0, t_f) with the switching signal given by (10). Then for all $i = 0, 1, \ldots, \mathrm{n}$*

$$\tilde{\mathcal{K}}_i^f(x_0) = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ a solution } (x,u) \\ \text{of (1) on } (t_0, t_{i+1}) \text{ s.t.} \\ x(t_0^+) = x_0 \wedge x(t_{i+1}^-) = \xi \end{array} \right\}.$$

Proof. The proof is along similar lines as the proof of Lemma 20 when C_i^{imp} is replaced by $\mathbb{R}^n$ for all $i \in \{1, 2, \ldots, \mathrm{n}\}$. ■

It follows directly that $\tilde{\mathcal{K}}_i^f(x_0)$ is an affine shift from $\tilde{\mathcal{K}}_i^f$, whether the system is impulse controllable or not. This is formalized in the next lemma.

Lemma 34. *Consider the switched system (1) with switching signal (10). Then $\tilde{\mathcal{K}}_i^f(x_0)$ is an affine shift of $\tilde{\mathcal{K}}_i^f$,* i.e. *for all i there exists a matrix $\tilde{M}_i$ such that*

$$\tilde{\mathcal{K}}_i^f(x_0) = \tilde{M}_i x_0 + \tilde{\mathcal{K}}_i^f. \tag{18}$$

Proof. Denote $Y_i = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i$ for shorthand notation. Then for $i = 0$ we have $\tilde{M}_0 = Y_0$ satisfies (18). Hence assume the statement holds for i. Then if we define $\tilde{M}_{i+1} = Y_i \tilde{M}_i$ for $i+1$ we have that

$$\begin{aligned} \tilde{\mathcal{K}}_{i+1}^f(x_0) &= Y_{i+1}\tilde{\mathcal{K}}_i^f(x_0) + \mathcal{R}_i, \\ &= Y_i(\tilde{M}_i x_0 + \tilde{\mathcal{K}}_i^f) + \mathcal{R}_i, \\ &= Y_i \tilde{M}_i x_0 + \tilde{\mathcal{K}}_{i+1}^f \\ &= \tilde{M}_{i+1} x_0 + \tilde{\mathcal{K}}_i^f \end{aligned}$$

which proves the statement. ■

Lemma 35. *Consider the DAE (1) with switching signal (10). For any $\tilde{M}_i$ satisfying (18) we have that*

$$\min_{x \in \tilde{\mathcal{K}}_i^f(x_0)} |x| = |\Pi_{\left(\tilde{\mathcal{K}}_i^f\right)^\perp} \tilde{M}_i x_0|$$

Theorem 36. *Consider the switched DAE (1) with switching signal (10). Then the system is stabilizable if and only if for any $\tilde{M}_{\mathrm{n}}$ satisfying (18)*

$$\|\Pi_{(\tilde{\mathcal{K}}_{\mathrm{n}}^f)^\perp} \tilde{M}_{\mathrm{n}}\|_2 = \sup_{x \neq 0} \frac{|\Pi_{\left(\tilde{\mathcal{K}}_{\mathrm{n}}^f\right)^\perp} \tilde{M}_{\mathrm{n}} x|_2}{|x|_2} < 1$$

Proof. The proof follows the proof of Theorem 29 analogously. ■

As was already shown in the introduction, not every stabilizable system that is also impulse-controllable, is automatically impulse-free stabilizable. This can be explained by viewing $\mathcal{K}_i^f(x_0)$ and $\tilde{\mathcal{K}}_i^f(x_0)$ as affine subspaces. Note that since every state that can be reached impulse-free from x_0 is by definition also an element of $\tilde{\mathcal{K}}_i^f(x_0)$. This leads to the following result.

Lemma 37. *Consider the switched system (1) with switching signal (10) and assume the system is impulse-controllable. Then*

$$\mathcal{K}_i^f(x_0) \subseteq \tilde{\mathcal{K}}_i^f(x_0).$$

Proof. This follows immediately from Lemmas 20 and 33. ■

As a consequence, we can state the following corollary.

Corollary 38. *Consider the system (1) with switching signal (10) and assume it is impulse-controllable. Then for any M_i satisfying (14) we have*

$$\tilde{\mathcal{K}}_i^f(x_0) = M_i x_0 + \tilde{\mathcal{K}}_i^f,$$

i.e. M_i satisfies (18).

Proof. For any two $x, y \in \tilde{\mathcal{K}}_i^f(x_0)$ we have that $x - y \in \tilde{\mathcal{K}}_i^f$. This means that $x = y + \tilde{\eta}$ for some $\tilde{\eta} \in \tilde{\mathcal{K}}_i^f$. By Lemma 37 we have that $y = M_i x_0 + \eta \in \mathcal{K}_i^f(x_0) \subseteq \tilde{\mathcal{K}}_i^f(x_0)$. This means that for any $x \in \tilde{\mathcal{K}}_i^f(x_0)$ we obtain that $x = M_i x_0 + \eta + \tilde{\eta} \subseteq M_i x_0 + \tilde{\mathcal{K}}_i^f(x_0)$. This proves that $\mathcal{K}_i^f(x_0) \subseteq M_i x_0 + \tilde{\mathcal{K}}_i^f$.

Consider $\alpha = M_i x_0 + \tilde{\eta}$ for some $\tilde{\eta} \in \tilde{\mathcal{K}}_i^f$. Then since $\mathcal{K}_i^f \subseteq \tilde{\mathcal{K}}_i^f$ there exist an $\bar{\eta} \in \tilde{\mathcal{K}}_i^f$ and an $\eta \in \mathcal{K}_i^f$ such that $\tilde{\eta} = \bar{\eta} + \eta$. Hence we obtain that $\alpha = M_i x_0 + \bar{\eta} + \eta = \beta + \eta$ for some $\beta \in \mathcal{K}_i^f(x_0) \subseteq \tilde{\mathcal{K}}_i^f(x_0)$. But this means that for some $\tilde{M}_i$ satisfying (18) and $\hat{\eta} \in \tilde{\mathcal{K}}_i^f$ that $\alpha = \tilde{M}_i x_0 + \hat{\eta} + \eta$. Because $\hat{\eta} + \eta \in \tilde{\mathcal{K}}_i^f$ we have that $\alpha \in \tilde{\mathcal{K}}_i^f(x_0)$. Since α was chosen arbitrary, it follows that $M_i x_0 + \tilde{\mathcal{K}}_i^f \subseteq \tilde{\mathcal{K}}_i^f(x_0)$. ■

Given that a system is impulse-controllable and stabilizable, we have that there exists an M_i satisfying (14) and we know that $\|\Pi_{(\tilde{\mathcal{K}}_{\mathrm{n}}^f)^\perp} M_{\mathrm{n}}\|_2 < 1$. However, the system is impulse-free stabilizable if and only if $\|\Pi_{(\mathcal{K}_{\mathrm{n}}^f)^\perp} M_{\mathrm{n}}\|_2 < 1$. This is however not implied by the statement that $\|\Pi_{(\tilde{\mathcal{K}}_{\mathrm{n}}^f)^\perp} M_{\mathrm{n}}\|_2 < 1$. Indeed, since $\mathcal{K}_i^f \subseteq \tilde{\mathcal{K}}_i^f$ we have that $\operatorname{im} \Pi_{(\tilde{\mathcal{K}}_{\mathrm{n}}^f)^\perp} \subseteq \operatorname{im} \Pi_{(\mathcal{K}_{\mathrm{n}}^f)^\perp}$, which means that it could happen that there exists an initial condition $x_0 \neq 0$ for which

$$\frac{|\Pi_{(\mathcal{K}_{\mathrm{n}}^f)^\perp} M_{\mathrm{n}} x_0|}{|x_0|} \geqslant 1, \quad \text{and} \quad \frac{|\Pi_{(\tilde{\mathcal{K}}_{\mathrm{n}}^f)^\perp} M_{\mathrm{n}} x_0|}{|x_0|} < 1.$$

Example 39. Again consider the example given in the introduction on the interval $(0, t_f)$ with a switch at $t = t_1$. The matrices (E_0, A_0, B_0) correspond the system matrices given in (2) and (E_1, A_1, B_1) are the system matrices given in (3). Then it follows from the algorithm (17) that the reachable space of the switched system $\tilde{\mathcal{K}}_1^f$, a suitable matrix $\tilde{M}_1$ and $\Pi_{(\tilde{\mathcal{K}}_1^f)^\perp}$ are given respectively by

$$\tilde{\mathcal{K}}_1^f = \operatorname{span}\left\{\begin{bmatrix} 0\\0\\1\\0\\0 \end{bmatrix}\right\}$$

$$\tilde{M}_1 = \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1 \end{bmatrix}, \quad \Pi_{\left(\tilde{\mathcal{K}}_1^f\right)^\perp} = \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}.$$

From which it follows that $\|\Pi_{(\tilde{\mathcal{K}}_1^f)^\perp} \tilde{M}_1\| = \frac{1}{\sqrt{2}} < 1$ and hence the system is stabilizable. However, the impulse-free reachable space

$\mathcal{K}_1^f$ can be calculated from (13) and is given by

$$\mathcal{K}_1^f = 0,\ \Pi_{(\tilde{\mathcal{K}}_1^f)^\perp} = I,\ M_1 = \begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{bmatrix} = \Pi_{(\mathcal{K}_1^f)^\perp} M_1$$

From which it follows that $\|\Pi_{(\tilde{\mathcal{K}}_1^f)^\perp}\tilde{M}_1\| = \frac{\sqrt{2}}{\sqrt{2}} = 1$ and hence the system is not impulse-free stabilizable.

Remark 40. In the case V_0 becomes a control input after the switch the system would be null-controllable, but not impulse-free null-controllable. Furthermore, since the state of the initial condition can be reduced via an impulse-free trajectory, the system would also become impulse-free (interval) stabilizable. However, since there is no way of discharging the capacitor, it follows that there exists no input such that $\lim_{t\to\infty} x(t) = 0$.

Remark 41. All the results on stabilizability in this paper can be applied to switched ordinary differential equations (ODEs) without difficulty. In the case of a switched ODE we have $E_i = I$, $\Pi_i = I$, $B_i^{\text{diff}} = B_i$ and $A_i^{\text{diff}} = A_i$. Note that all solutions are trivially impulse-free, hence, impulse-free stabilizability is equivalent to stabilizability.

5. Conclusion

In this paper stabilization of switched differential algebraic equations was considered, where Dirac impulses in both the input and state-trajectory were to be avoided. Necessary and sufficient conditions for the existence of impulse-free solutions were given, followed by characterizations of (impulse-free) interval stabilizability. The results rely on the fact that the points that can be reached from an initial condition form an affine subspace. It followed that the system is (impulse-free) interval stabilizable if and only if the operator that maps the initial condition to the element of minimal norm (that can be reached in an impulse-free manner) has a norm strictly smaller than one.

A natural future direction of research would be the investigation of controllers achieving interval stabilizability for switched systems. The theory established in this paper could be used as starting point in the search (for feedback) controllers. Furthermore, a natural extension would be to consider stabilizability properties of switched systems with unknown switching signals.

CRediT authorship contribution statement

Paul Wijnbergen: Formal analysis, Investigation, Methodology, Visualization, Writing - original draft, Writing - review & editing. **Stephan Trenn:** Conceptualization, Methodology, Supervision, Writing - original draft, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Here we recap some general results on (affine) subspaces that result from linear algebra.

Proposition 42. *Let $\mathcal{V}$ and $\mathcal{S}$ be subspaces of $\mathbb{R}^n$ and let $M \in \mathbb{R}^{n\times n}$ be of rank $r \leqslant n$. If $(Mx_0 + \mathcal{S}) \cap \mathcal{V} \neq \varnothing$ for all $x_0 \in \mathbb{R}^n$, then there exists a matrix $N \in \mathbb{R}^{n\times n}$ such that for all x_0*

$$(Mx_0 + \mathcal{S}) \cap \mathcal{V} = NMx_0 + \mathcal{S} \cap \mathcal{V}. \tag{19}$$

Proof. Let $m_1, m_2, \ldots, m_p$ be a basis for the image of M. Then the statement is proven if we can prove that

$$(m_i + \mathcal{S}) \cap \mathcal{V} = Nm_i + \mathcal{S} \cap \mathcal{V}, \qquad \forall i \in \{1, 2, \ldots, p\}$$

Since we have that $(m_i + \mathcal{S}) \cap \mathcal{V} \neq \varnothing$ we have that for all i we have that there exists an $\eta_i \in \mathcal{S}$ such that $m_i + \eta_i \in \mathcal{V}$. Let $\hat{N}$ be a linear map such that

$$\hat{N}m_i = \eta_i.$$

Then if we define $N = I + \hat{N}$ we have that

$$\begin{aligned} Nm_i &= m_i + \hat{N}m_i,\\ &= m_i + \eta_i,\\ &\in \mathcal{V} \cap (m_i + \mathcal{S}) \end{aligned}$$

Since subspaces are closed under addition, it follows that for all $\bar{\eta} \in \mathcal{S} \cap \mathcal{V} \subseteq \mathcal{V}$ we have that

$$Nm_i + \bar{\eta} = m_i + \eta_i + \bar{\eta} \in \mathcal{V}.$$

and

$$m_i + \eta_i + \bar{\eta} = m_i + \hat{\eta} \in m_i + \mathcal{S},$$

for some $\eta_i + \bar{\eta} = \hat{\eta} \in \mathcal{S}$, which proves that there exists an N such that $Nm_i + \mathcal{S} \cap \mathcal{V} \subseteq (m_i + \mathcal{S}) \cap \mathcal{V}$.

Conversely, we have for $\xi \in (m_i + \mathcal{S}) \cap \mathcal{V}$ and for some $\beta \in \mathcal{S}$ that $\xi = m_i + \beta \in \mathcal{V}$. Let $\beta = \hat{N}m_i + \gamma$, for some $\gamma \in \mathcal{S}$. Then we obtain

$$\begin{aligned} m_i + \beta &= m_i + \hat{N}m_i + \gamma,\\ &= Nm_i + \gamma,\\ &= \xi \in (m_i + \mathcal{S}) \cap \mathcal{V}. \end{aligned}$$

It remains to prove that $\gamma \in \mathcal{S} \cap \mathcal{V}$. Since $Nm_i \in (m_i + \mathcal{S}) \cap \mathcal{V} \subseteq \mathcal{V}$ by definition, we have that $\xi - Nm_i = \gamma \in \mathcal{V}$. Furthermore, by definition, we had $\gamma \in \mathcal{S}$ and hence $\gamma \in \mathcal{S} \cap \mathcal{V}$. Hence we have proven that $(m_i + \mathcal{S}) \cap \mathcal{V} \subseteq Nm_i + \mathcal{S} \cap \mathcal{V}$. With the inclusion in both direction proven, the equality follows. ∎

It follows from Proposition 42 that if the intersection $(Mx_0 + \mathcal{S}) \cap \mathcal{V} \neq \varnothing$ for all x_0, that this matrix N is not unique. In fact, this observation results in the next lemma.

Lemma 43. *With the same notation as in Proposition 42 we have that $N \in \mathbb{R}^{n\times n}$ satisfies (19) if and only if*

1. *$\operatorname{im}(N - I)M \subseteq \mathcal{S}$,*
2. *$\operatorname{im} NM \subseteq \mathcal{V}$,*

Proof. Assume that N satisfies $\operatorname{im}(N-I) \subseteq \mathcal{S}$ and $\operatorname{im} NY \subseteq \mathcal{V}$. This means that $\operatorname{im}(N-I)Y \subseteq \mathcal{S}$. Hence $NMx_0 \in \mathcal{S} + Mx_0$. Furthermore, by assumption we had that $NMx_0 \in \operatorname{im} N \subseteq \mathcal{V}$ and hence $NMx_0 \in (Mx_0 + \mathcal{S}) \cap \mathcal{V}$. Hence it follows that $NMx_0 + \mathcal{S} \cap \mathcal{V} \subseteq (Mx_0 + \mathcal{S}) \cap \mathcal{V}$.

On the otherhand, let $\xi \in (Mx_0 + \mathcal{S}) \cap \mathcal{V}$. Then $\xi = Mx_0 + \eta$ for some $\eta \in \mathcal{S}$ and $\xi \in \mathcal{V}$. Since $NMx_0 \in \mathcal{V}$ we have that $NMx_0 - \xi \in \mathcal{V}$. From which it follows that $(N - I)Mx_0 \in \mathcal{V}$ and also $(N - I)Mx_0 \in \mathcal{S}$. Thus we have that $NMx_0 - \xi \in \mathcal{S} \cap \mathcal{V}$. From this it follows that $\xi \in NMx_0 + \mathcal{S} \cap \mathcal{V}$ and thus it is proven that under the assumptions (14) holds.

Next assume that (14) holds. Then it follows that

$$\begin{aligned} NMx_0 &\in (Mx_0 + \mathcal{S}) \cap \mathcal{V} + \mathcal{S} \cap \mathcal{V}\\ &= (Mx_0 + \mathcal{S}) \cap \mathcal{V}. \end{aligned}$$

Since this holds for all x_0 it follows that $\operatorname{im} NM \subseteq \mathcal{V}$. Furthermore, it follows that $NMx_0 \in Yx_0 + \mathcal{S}$, from which it follows that $(N - I)Mx_0 \in \mathcal{S}$ for all x_0, and thus $\operatorname{im}(N - I)M \subseteq \mathcal{S}$. Which proves the result. ∎

Given the subspaces $\mathcal{V}$, $\mathcal{S}$ and the matrix M, a matrix N satisfying the conditions of Lemma 43 can constructively be computed.

Lemma 44. *Let $\Pi_{\mathcal{V}}$ and $\Pi_{\mathcal{S}}$ be projectors onto $\mathcal{V}$ and $\mathcal{S}$ respectively. For any Q that solves*

$$(I - \Pi_{\mathcal{S}})\Pi_{\mathcal{V}} Q M = (I - \Pi_{\mathcal{S}})M$$

the matrix $N = \Pi_{\mathcal{V}} Q$ solves (19).

Proof. Since $\operatorname{im} N \subseteq \operatorname{im} \Pi_{\mathcal{V}} = \mathcal{V}$ the condition $\operatorname{im} NM \subseteq \mathcal{V}$ is satisfied. Furthermore, we have that

$$\begin{aligned}
\operatorname{im}(N - I)M &= \operatorname{im}(\Pi_{\mathcal{V}} Q - I)M, \\
&= \operatorname{im}(\Pi_{\mathcal{S}} + (I - \Pi_{\mathcal{S}}))(\Pi_{\mathcal{V}} Q - I)M \\
&\subseteq \mathcal{S} + \operatorname{im}(I - \Pi_{\mathcal{S}})(\Pi_{\mathcal{V}} Q - I)M, \\
&= \mathcal{S} + \operatorname{im}\left((I - \Pi_{\mathcal{S}})M - (I - \Pi_{\mathcal{S}})M\right) = \mathcal{S}
\end{aligned}$$

Hence N satisfies the conditions of Lemma 43, which proves the result. ∎

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