A smooth model for periodically switched descriptor systems ⋆

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Abstract

Switched descriptor systems characterized by a repetitive finite sequence of modes can exhibit state discontinuities at the switching time instants and the discontinuities' amplitudes depend on the consistency projectors of the modes. A switched ordinary differential equations model whose continuous state evolution approximates the state of the original system is proposed. Sufficient conditions based on linear matrix inequalities on the modes projectors ensure that the approximation error is of linear order of the switching period. The theoretical findings are applied to a switched capacitor circuit and numerical results illustrate the practical usefulness of the proposed model.

Key words: Descriptor systems, differential algebraic equations, non-smooth and discontinuous problems, linear/nonlinear models, switched systems, power electronics.

1 Introduction

Switched descriptor systems represent the dynamic behavior of several physical apparatus, e.g. mechanical systems [12] and electronic circuits [10]. The dynamics of switched descriptor systems is determined by the switching among different modes, where each mode is characterized by a set of linear differential equations and algebraic constraints. A mathematical representation of this class of systems can be obtained in terms of switched linear differential algebraic equations (DAEs).

Several modeling and control aspects related to the analysis of switched linear descriptor systems have been considered in the literature, e.g. averaging [8], observer design [16], stability [12,14]. In particular, the use of switched ordinary differential equations (ODEs) with a continuous state evolution that approximates the dynamic behavior of the switched DAE has been shown to be useful for the analysis of switched descriptor systems. One of these situations is the simulation of descriptor systems with singularities, e.g. inconsistent initial conditions [13], where numerical issues could be amplified by the presence of switching modes. In this case approximated switched ODEs models could help to obtain numerical results by using standard software suited for systems with a continuous state evolution. The approximation of a switched descriptor system with a switched ODE has also been used for stability analysis [5] and observer design [11].

The results presented in the literature regarding switched ODEs which approximate switched DAEs consider a constant switching period and do not provide explicit expressions for the approximation errors [5,7]. On the other hand, in many practical applications, such as for power converters, the switching among the different modes are not repetitive in the sense that different duty cycles and different switching periods are required for the system operations [6,9].

In this paper, by extending the model proposed in [5], we propose a switched ODE model which approximates the state evolution of switched linear descriptor systems, also in the presence of state discontinuities at switching time instants. The dynamic matrix of each mode of the proposed switched system depends on the switching period. In spite of [5], we provide a design rule for the formalization of the dynamic matrices strictly related to the switching period. It is proved that the error between

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the solution of the proposed model and that of the original switched DAE is of the same order as the switching period.

The note is organized in several section. In Sections 2 and 3 some mathematical preliminary properties are recalled and the class of switched descriptor systems of interest is presented, respectively. Section 4 presents the new switched ODE model. The main result of the paper is shown in Section 5: a suitable set of linear matrix inequalities depending on the system projectors provide a sufficient condition to the approximation result. In Section 6 a numerical verification of the theoretical results obtained by considering a practical switched capacitor circuit is proposed. In Section 7 the results are summarized. All proofs of the lemmas and theorem proposed in the paper are collected in the Appendix.

2 Preliminaries

In this section the notation adopted and some preliminary definitions and properties are briefly recalled.

The following notation is adopted throughout the paper: $\mathbb{N}$ is the set of positive integers; $\mathbb{R}_+$ is the set of positive real numbers; the product of any $\mathtt{q}$ matrices $G_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots, \mathtt{q}$, is defined as (note the order)

$$\prod_{i=1}^{\mathtt{q}} G_i = G_{\mathtt{q}} G_{\mathtt{q}-1} \cdot \dots \cdot G_2 G_1;$$

$\lfloor x \rfloor$ is the largest integer less than or equal to $x \in \mathbb{R}$, the symbol $\|\cdot\|$ indicates the Euclidean norm.

We are interested in switched descriptor systems in which the sequence of $\mathtt{q}$ modes is repeated with some period $p \in \mathbb{R}_+$. The proposed approximating smooth model will be shown to have a state error which goes to zero when p tends to zero. The arguments used in the proofs make extensive use of the following property.

Definition 1 ($\mathrm{O}(p^m)$) *For any finite integer $m \in \mathbb{N}$, a matrix function $G : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is said to be an $\mathrm{O}(p^m)$ function as $p \to 0$ ($G(p) = \mathrm{O}(p^m)$ for short), if there exist positive constants α and $\bar{p}$ such that*

$$\|G(p)\| \le \alpha p^m, \quad \forall p \in (0, \bar{p}).$$

Any linear combination of functions which are $\mathrm{O}(p^m)$ is an $\mathrm{O}(p^m)$ function itself.

Let us consider a time interval $(0, T)$ with some finite time instant $T \in \mathbb{R}_+$. The number of intervals of length p from 0 to T, say $\ell : \mathbb{R}_+ \to \mathbb{N}$, can be written as

$$\ell(p) = \left\lfloor \frac{T}{p} \right\rfloor. \tag{1}$$

Clearly, when p goes to zero $\ell(p)$ goes to infinity. In particular, it is

$$\frac{1}{\ell(p)} = \mathrm{O}(p).$$

Indeed $\frac{1}{\ell(p)} \le \frac{1}{\frac{T}{p}-1} = \frac{p}{T-p} \le \alpha p$ with $\alpha \ge \frac{1}{T-\bar{p}}$.

We can now state some properties of matrix functions which satisfy Definition 1. In particular, for any finite integer $m \in \mathbb{N}$, the following implications hold:

$$G(p) = \mathrm{O}(p^m) \implies G(p)\ell(p) = \mathrm{O}(p^{m-1}) \tag{2a}$$
$$G(p) = \mathrm{O}(p) \implies G(p)^{\ell(p)} = \mathrm{O}(p^m) \tag{2b}$$
$$G(p) = \mathrm{O}(p^2) \implies (G(p)\ell(p))^{\ell(p)} = \mathrm{O}(p^m) \tag{2c}$$
$$G(p)^{\ell(p)} = \mathrm{O}(1) \implies G(p) = \mathrm{O}(1). \tag{2d}$$

The proofs of the implications above are reported in Sec. A.1. Note that the opposite of (2d) do not hold, in general.

3 Switched descriptor system

The switched descriptor system of interest can be represented as an homogeneous switched DAE with $\mathtt{q}$ modes, i.e.

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x \tag{3}$$

where $\sigma : \mathbb{R}_+ \to \Sigma$ is a piecewise constant right-continuous function, that selects at each time instant the index of the active mode from the finite index set $\Sigma := \{1, 2, \dots, \mathtt{q}\}$. We assume that σ is periodic with switching period $p > 0$, i.e.,

$$\sigma(t) = \begin{cases} 1, & t \in [t_k, s_{k,2}), \\ 2, & t \in [s_{k,2}, s_{k,3}), \\ \vdots & \\ \mathtt{q}, & t \in [s_{k,\mathtt{q}}, t_{k+1}) \end{cases} \tag{4}$$

with $k \in \mathbb{N}$, $i \in \Sigma$, the time instants t_k being the multiple of the period p, the switching time instants $s_{k,i}$ being the time instant when the i-th mode starts within the k-th period. In particular, we assume $s_{k,1} = t_k$ for all $k \in N$. Then we have

$$t_k := kp, \qquad s_{k,i} := t_k + \sum_{j=1}^{i-1} d_{j,k}p, \tag{5}$$

where $d_{i,k} \in D$, $D = (0, 1)$, is the duty cycle of the i-th mode for the k-th period; in particular, $\sum_{i=1}^{\mathtt{q}} d_{i,k} = 1$, see Fig 1.

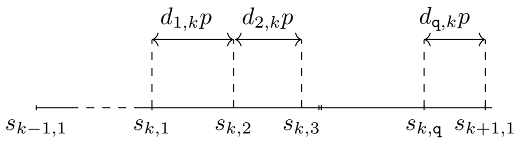

Fig. 1. Illustration of the switching times notation.

For any regular matrix pair (E_i, A_i) there exist transformation matrices S_i and T_i which put (E_i, A_i) into the quasi Weierstrass form

$$(S_iE_iT_i, S_iA_iT_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix}\right), \tag{6}$$

with $T_i = [V_i,\ W_i]$, $\quad S_i = [E_iV_i,\ A_iW_i]^{-1}$ where N_i is a nilpotent matrix, I is the identity matrix, J_i, V_i and W_i are matrices of appropriate size. For any regular matrix pair (E_i, A_i) it is possible to define the consistence projector Π_i and the flow matrix F_i as follow:

$$\Pi_i = T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1} \tag{7a}$$

$$F_i = T_i \begin{bmatrix} J_i & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1} \tag{7b}$$

The consistency projector is an idempotent matrix, i.e., $\Pi_i^2 = \Pi_i$. It is easy to verify that $F_i\Pi_i = F_i = \Pi_iF_i$.

In [8, Theorem 12] it is shown that $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the impulse free part of any (distributional) solution of (3) if and only if it is a solution of the following *switched ODE model* with jumps

$$\dot{x}(t) = F_ix(t), \quad t \in (s_{k,i}, s_{k,i+1}) \tag{8a}$$

$$x(s_{k,i}^+) = \Pi_i x(s_{k,i}^-) \tag{8b}$$

with $x(0^-) = x_0$, for $k \in \mathbb{N}$, $i \in \Sigma$, where the matrices F_i are given by (7b) and considering $s_{k,q+1} := t_{k+1} = s_{k+1,1}$.

4 Proposed switched ODE model

The proposed *frequency-dependent switched model* is obtained by smoothly approximate each jump.

$$\dot{x}_s(t) = F_i^{\varepsilon_p} x_s(t), \quad t \in [s_{k,i}, s_{k,i+1}) \tag{9}$$

with $x_s(0) = x_0$, for $k \in \mathbb{N}$, $i \in \Sigma$ and

$$F_i^{\varepsilon_p} = F_i + \Phi_i(p) \tag{10}$$

where the matrices F_i are defined in (7b) and

$$\Phi_i(p) = T_i \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{\varepsilon_p} I \end{bmatrix} T_i^{-1} \tag{11}$$

with

$$\varepsilon_p = -\frac{\Delta_p p}{\log p^2} \tag{12}$$

where without loss of generality we assumed $p < 1$, and

$$\Delta_p \leq \Delta p, \quad 0 < \Delta \ll \min\{d_{i,k}\}_{i \in \Sigma, k \in \mathbb{N}} \tag{13}$$

for all $p < \bar{p}$. In other words it must be $\Delta_p = O(p)$. Since $p < 1$, the logarithm will be negative, so ε_p is actually positive. The structure of Φ descend by the model in [5], but here we improve the model by introducing the dependence on the switching period p.

In order to motivate the definition of the proposed model let us consider (11). With simple algebraic manipulations one can write

$$\begin{aligned} \frac{\Delta_p p}{\log p^2} \Phi_i(p) &= T_i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T_i^{-1} \\ &= T_i \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} T_i^{-1} - T_i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_i^{-1} \\ &= I - \Pi_i. \end{aligned} \tag{14}$$

The choice (11) with (12) allows an approximation with order $\mathrm{O}(p^2)$ of the state jump. In particular, by using (14) one can write:

$$\begin{aligned} e^{\Phi_i(p)\Delta_p p} &= \sum_{n=0}^{\infty} \frac{(I - \Pi_i)^n \log^n p^2}{n!} \\ &= I + \sum_{n=1}^{\infty} \frac{(I - \Pi_i)^n \log^n p^2}{n!} \\ &= I + (I - \Pi_i) \sum_{n=1}^{\infty} \frac{\log^n p^2}{n!} \\ &= I + (I - \Pi_i) \sum_{n=0}^{\infty} \frac{\log^n p^2}{n!} - (I - \Pi_i) \\ &= (I - \Pi_i) e^{\log p^2} + \Pi_i = \Pi_i + (I - \Pi_i) p^2. \end{aligned} \tag{15}$$

By considering the state at the end of the $(i-1)$-th mode, say x_{i-1}^-, from (15) it follows that

$$\begin{aligned} x_s(s_{k,i} + \Delta_p p) &= e^{(\Phi_i(p)+F_i)\Delta_p p} x_{i-1}^- \\ &\overset{a}{=} e^{\Phi_i(p)\Delta_p p} e^{F_i \Delta_p p} x_{i-1}^- \\ &= (\Pi_i + (I - \Pi_i)p^2)(I + F_i \Delta_p p + O(p^3)) x_{i-1}^- \\ &= \Pi_i x_{i-1}^- + O(p^2) \end{aligned} \tag{16}$$

where in $\overset{a}{=}$ has been used the commutativity property between F_i and $\Phi_i(p)$ which follows from

$$F_i(I - \Pi_i) = (I - \Pi_i)F_i = F_i - F_i\Pi_i = F_i - F_i = 0.$$

The expression (16) means that the state of the smooth system after a time interval $\Delta_p p$ of the i-th mode approximates the state value after the jump with $O(p^2)$, provided that $x_s(s_{k,i}) = x_{i-1}^-$. This last equality is not verified, in general, when considering the sequence of switching periods. In next section it will be proved that the approximation result between x_s and x holds without such strict assumption too.

To do this, it is useful to provide some preliminary expressions for the exponential of the systems flow matrices which are proved through the following lemma.

Lemma 2 *Given a set of matrices defined as in* (7) *and* (10)–(13)*, the following relations hold*

$$e^{F_i d_{i,k} p} = I + F_i d_{i,k} p + O(p^2) \tag{17a}$$

$$e^{F_i^{\varepsilon p} d_{i,k} p} = \Pi_i + F_i d_{i,k} p + O(p^2) \tag{17b}$$

$$\prod_{i=1}^{\mathtt{q}} e^{F_i^{\varepsilon p} d_{i,k} p} = \prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p} \Pi_i + O(p^2) \tag{17c}$$

with $i \in \Sigma$*, for any* $k \in \{1, \dots, \ell(p)\}$ *and for all* p.

PROOF. See Sec. A.2.

5 Main result

The switched ODE system with $\mathtt{q}$ modes described by (9) can be considered as an $\mathrm{O}(p)$ approximation of the switched DAE (3). In this section we provide sufficient conditions such that $x(t) - x_s(t) = \mathrm{O}(p)$ holds uniformly for any $t \in [0, T] \setminus \{(s_{k,i}, s_{k,i} + \Delta_p p)\}_{k\in\mathbb{N}, i\in\Sigma}$. Note that in principle it is not possible to approximate a discontinuous function (solution of switched descriptor system) with a continuous function (solution of switched ODE) uniformly for all $t \in [0, T]$ unless the jump magnitude converges to zero, which we do not assume here. However, with the proposed approximation method we are able to show uniform convergence of order p outside a set (a union of small intervalls following the switched) whose measure is also of order p.

For $k \in \mathbb{N}$ let us introduce the vector signal $\delta_k = [d_{1,k}, \dots, d_{\mathtt{q},k}]^\top \in D^{\mathtt{q}}$, where $D = (0, 1)$, which indicates the duty cycles of all modes over time and let us consider the Lipschitz continuous matrix function $M : D^{\mathtt{q}} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ defined by the following expression

$$M(\delta_k, p) = \prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p} \Pi_i. \tag{18}$$

The proof of the approximation result exploits some preliminary conditions on products of $\ell(p)$ terms in the form (18), so as shown by the following lemma.

Lemma 3 *Consider a finite* $T \in \mathbb{R}$*,* $\ell(p)$ *as in* (1)*, a discrete time signal* $(\delta_k)_{k\in\mathbb{N}}$ *with values in* $D^{\mathtt{q}} = (0, 1)^{\mathtt{q}}$*, and generic Lipschitz continuous matrix functions* $M : D^{\mathtt{q}} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ *and* $G : D^{\mathtt{q}} \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$*. Assume that there exists a* $\gamma_1 \geq 0$ *such that*

$$\|M(\delta_k, p)\| \leq 1 + \gamma_1 p \tag{19a}$$

$$G(\delta_k, p) = \mathrm{O}(p^2), \tag{19b}$$

for all $k \in \{1, \dots, \ell(p)\}$*. Then*

$$\prod_{k=1}^{\ell(p)} M(\delta_k, p) = \mathrm{O}(1) \tag{20a}$$

$$\prod_{k=1}^{\ell(p)} (M(\delta_k, p) + G(\delta_k, p)) = \prod_{k=1}^{\ell(p)} M(\delta_k, p) + \mathrm{O}(p). \tag{20b}$$

PROOF. See Sec. A.3.

It is interesting to compare the results in Lemma 3 with those in Lemma 2 in [2]. Therein, by considering constant duty cycles, i.e. $\delta_k = \delta$ for all $k \in \mathbb{N}$, it is shown that if $M(p)^{\ell(p)} = \mathrm{O}(1)$ and $G(p) = \mathrm{O}(p)$ it is $(M(p) + G(p))^{\ell(p)} = \mathrm{O}(1)$. The assumption (19a) in the particular case that δ_k is constant, is more restrictive than $M(p)^{\ell(p)} = \mathrm{O}(1)$, so as it can be deduced from (2d). On the other hand, from Lemma 3 it follows that if (19) hold one can write the more explicit expression $(M(p) + G(p))^{\ell(p)} = M(p)^{\ell(p)} + \mathrm{O}(p)$.

The condition (19a) cannot be easily checked apriori from the structure of the model (9). The following lemma provides more operative conditions based on linear matrix inequalities which must be satisfied by the system projectors.

Lemma 4 *Consider a set of matrices defined as in* (7) *and* (10)–(13). *Assume that there exists a symmetric matrix P such that the following set of linear matrix inequalities*

$$P \succ 0 \tag{21a}$$
$$\Pi_i^\top P \Pi_i - P \preceq 0 \tag{21b}$$

with $i = 1, \ldots, \mathtt{q}$, has a solution, then there exists a $\gamma_1 \geq 0$ such that the following condition holds

$$\left|\!\left|\!\left| \prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p} \Pi_i \right|\!\right|\!\right| \leq 1 + \gamma_1 p \tag{22}$$

for any $k \in \{1, \ldots, \ell(p)\}$ and for all p with $|||\cdot|||$ being the norm induced by the matrix P.

PROOF. See Sec. A.4.

We are now ready to prove our main result.

Theorem 5 *Consider the switched DAE system* (3) *and the smooth model* (9) *with the same initial conditions $x(0^-) = x_s(0^-) = x_0$. Assume that there exists a symmetric matrix P such that the set of LMIs* (21) *is satisfied. Then*

$$x(t) - x_s(t) = O(p) \tag{23}$$

holds for any $t \in [0, T] \setminus \{(s_{k,i}, s_{k,i} + \Delta_p p)\}_{k \in \mathbb{N}, i \in \Sigma}$.

PROOF. See Sec. A.5

Clearly conditions (21) can be relaxed by exploiting the fact that the sequence of the projectors is fixed and by finding a symmetric matrix P which satisfies (21a) and $\Pi_\cap^\top P \Pi_\cap - P \preceq 0$, where $\Pi_\cap = \prod_{i=1}^{\mathtt{q}} \Pi_i$. Note that if $\operatorname{im} \Pi_\cap \subseteq \operatorname{im} \Pi_i$ and $\ker \Pi_\cap \supseteq \ker \Pi_i$ then $\Pi_\cap$ is a projector himself, see [8].

6 Example

In this section we verify the approximation (23) in Theorem 5 by using numerical results obtained by considering a practical switched capacitor electrical circuit.

Let us consider the typical elementary cell of a ladder step-up switched capacitor shown in Fig. 2. The circuit consists of two capacitors and four switches that are controlled in a complementary way. Then the modes of the system are two, corresponding to the pair $\{\mathcal{S}_1, \mathcal{S}_2\}$ turned on together with the pair $\{\mathcal{S}_3, \mathcal{S}_4\}$ turned off and viceversa.

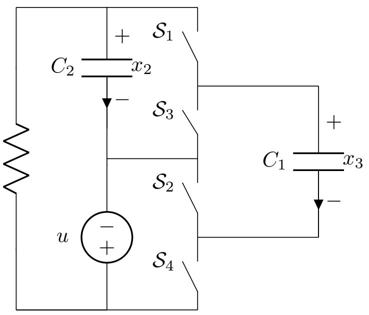

Fig. 2. Elementary cell of a ladder step-up switched capacitor converter.

By considering as input a constant voltage source u the circuit can be modeled by adding a dummy state variable, say x_1, together with x_2 and x_3 being the state variables corresponding to the voltages on the capacitors C_1 and C_2, respectively. Then the matrices pairs of the two modes are:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_2 R & C_1 R \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_2 R & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

The consistence projectors are

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 0 & C_2\rho & C_1\rho \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\Pi_2\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 1 & 0 & 0 \end{bmatrix} \qquad \Pi_1\Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ C_1\rho & C_2\rho & 0 \\ C_1\rho & C_2\rho & 0 \end{bmatrix}$$

where $\rho = \frac{1}{C_1 + C_2}$. It can be easily verified that the linear matrix inequalities (21) are satisfied, even though the projectors have euclidean norms larger than 1.

The matrices F_i and Φ_i, $i = 1, 2$, are:

$$F_1 = \frac{\rho^2}{R} \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{\rho} & -C_2 & -C_1 \\ -\frac{1}{\rho} & -C_2 & -C_1 \end{bmatrix} \quad \Phi_1 = \frac{\rho \log p^2}{p\Delta_p} \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & -C_1 \\ 0 & -C_2 & C_2 \end{bmatrix}$$
$$F_2 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{C_2 R} & -\frac{1}{C_2 R} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Phi_2 = \frac{\log p^2}{p\Delta_p} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The simulation has been carried out by selecting the following parameters: $C_1 = C_2 = 120\,\mu\text{F}$, $R = 10\,\text{k}\Omega$ and $\Delta_p = 0.9p$. In Fig. 3 is shown the behavior of the state variables for different switching periods, i.e. $p = 0.1\,\text{s}$ and $p = 0.07\,\text{s}$ respectively, over six periods. Note that the duty cycles are different for each period. The state variable x_2 presents jumps when the system switches from mode 1 to mode 2 and viceversa. The state evolution of the error related to the second state variable is shown in Fig. 4. At the switching time instants state jumps occur and the error becomes quite small after few switching periods. The root mean square of the state error between the solutions of the model (9) and that of the switched

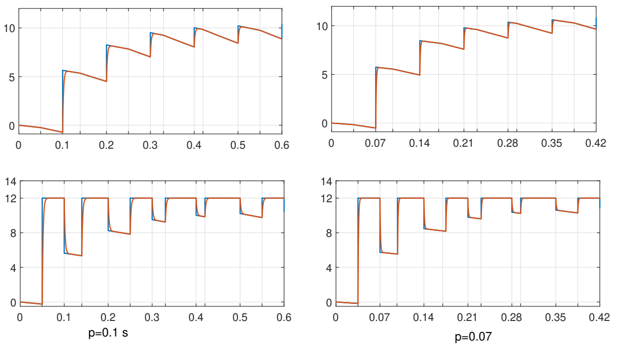

Fig. 3. Time evolution of the state variables (first top, second bottom) of the switched capacitor circuit with $p = 0.1$ s and $p = 0.07$ s: switched DAE system (blue lines) and proposed model (red lines).

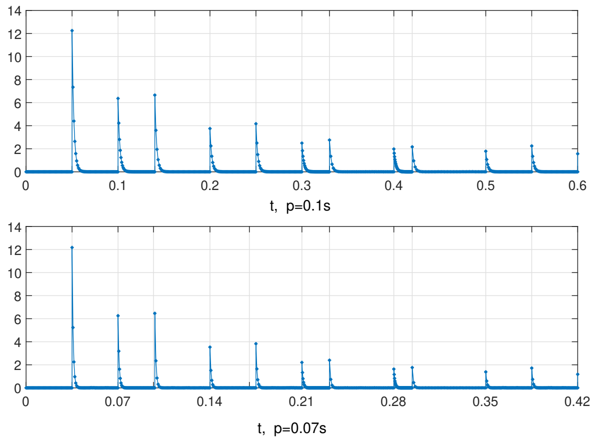

Fig. 4. Error time evolution over six periods for the state variable x_3 of the switched capacitor circuit for $p = 0.1$ s (top) and $p = 0.07$ s (bottom).

descriptor system (3) solution are 0.1178 and 0.0752 for $p = 0.1$ s and $p = 0.07$ s, respectively. By implementing the same scenarios with the model proposed in [15] with $\varepsilon = 0.004$, the resulting root mean square of the state errors are larger, i.e. 0.1983 and 0.2324, respectively.

7 Conclusion

Many practical switched systems are characterized by a repetitive sequence of a finite number of modes and can be represented as descriptor systems. For switched descriptor systems which present jumps in the state at the switching time instants it is of practical interest to find possible smooth models which approximate the behavior of the discontinuous system. In this paper a switched ODE model whose state continuous solution approximates the evolution of the switched descriptor system solution has been proposed. Linear matrix inequalities depending on the system projectors provide sufficient conditions for proving that the approximation error between the two models is of order of the switching period. The practical operating conditions of not constant duty cycles and varying switching periods have been considered. Numerical results obtained on switched descriptor models of switched capacitor circuits have validated the theoretical results.

A Appendix

A.1 Proofs of the properties (2)

The implication (2a) follows from

$$\|G(p)\ell(p)\| \leq \alpha p^m \left\lfloor \frac{T}{p} \right\rfloor \leq \alpha p \frac{T}{p} = \alpha T p^{m-1}.$$

The implication (2b) follows from

$$\begin{aligned} \|G(p)^{\ell(p)}\| &\leq \|G(p)\|^{\ell(p)} \leq (\alpha p)^{\ell(p)} \\ &= \alpha^m p^m (\alpha p)^{\ell(p)-m} \leq \alpha^m p^m \end{aligned}$$

and one can choose $\hat{p} \leq \bar{p}$ such that $\alpha\hat{p} \leq 1$, for some $\hat{p} \leq \bar{p}$.

The implication (2c) follows from

$$\begin{aligned} \|(G(p)\ell(p))^{\ell(p)}\| &\leq \|G(p)\ell(p)\|^{\ell(p)} \leq (\alpha p T)^{\ell(p)} \\ &= \alpha^m T^m p^m (\alpha p T)^{\ell(p)-m} \leq \alpha^m T^m p^m \end{aligned}$$

and one can choose $\hat{p} \leq \bar{p}$ such that $\alpha\hat{p}T \leq 1$, for some $\hat{p} \leq \bar{p}$.

The implication (2d) is a direct consequence of Definition 1.

A.2 Proof of Lemma 2

PROOF. The condition (17a) is straightforward by using a Taylor expansion of the exponential matrix.

The condition (17b) is obtained as follows

$$\begin{aligned} e^{F_i^{\varepsilon p} d_{i,k} p} &= e^{F_i d_{i,k} p + \Phi_i(p) d_{i,k} p} \\ &\overset{a}{=} e^{F_i d_{i,k} p} e^{\Phi_i(p) d_{i,k} p} \\ &= (I + F_i d_{i,k} p + O(p^2)) e^{\Phi_i(p) d_{i,k} p} \\ &\overset{b}{=} (I + F_i d_{i,k} p + O(p^2))(\Pi_i + O(p^2)) \\ &= \Pi_i + F_i d_{i,k} p + O(p^2) \end{aligned}$$

where in $\stackrel{a}{=}$ has been used the commutativity property between F_i and $\Phi_i(p)$ and in $\stackrel{b}{=}$ has been used the following result:

$$\begin{aligned}
e^{\Phi_i(p)d_{i,k}p} &= \sum_{n=0}^{\infty} \frac{(I-\Pi_i)^n \log^n p^2}{n!} \frac{d_{i,k}^n}{\Delta_p^n} \\
&= I + \sum_{n=1}^{\infty} \frac{(I-\Pi_i)^n \log^n p^2}{n!} \frac{d_{i,k}^n}{\Delta_p^n} \\
&= I + (I-\Pi_i) \sum_{n=1}^{\infty} \frac{\log^n p^2}{n!} \frac{d_{i,k}^n}{\Delta_p^n} \\
&= I + (I-\Pi_i) \sum_{n=0}^{\infty} \left(\frac{d_{i,k} \log p^2}{\Delta_p} \right)^n \frac{1}{n!} - (I-\Pi_i) \\
&= (I-\Pi_i) e^{\frac{d_{i,k}}{\Delta_p} \log p^2} + \Pi_i \\
&= \Pi_i + (I-\Pi_i) p^{\frac{2d_{i,k}}{\Delta_p}} = \Pi_i + O(p^2).
\end{aligned}$$

The condition (22) is obtained by applying (17a) and (17b). Indeed by using (17b) it follows that the left hand side of (22) can be written as

$$\begin{aligned}
\prod_{i=1}^{\mathsf{q}} e^{F_i^{\varepsilon p} d_{i,k} p} &= \prod_{i=1}^{\mathsf{q}} (\Pi_i + F_i d_{i,k} p + O(p^2)) \\
&= \prod_{i=1}^{\mathsf{q}} (\Pi_i + F_i d_{i,k} p) + O(p^2).
\end{aligned}$$

By using (17a) it follows that the first term in the right hand side of (22) can be written as

$$\begin{aligned}
\prod_{i=1}^{\mathsf{q}} e^{F_i d_{i,k} p} \Pi_i &= \prod_{i=1}^{\mathsf{q}} (I + F_i d_{i,k} p + O(p^2)) \Pi_i \\
&= \prod_{i=1}^{\mathsf{q}} (\Pi_i + F_i d_{i,k} p) + O(p^2)
\end{aligned}$$

where we exploited the property $F_i \Pi_i = F_i$ which holds for all $i \in \Sigma$. By combining the last two expressions the expression (22) directly follows.

A.3 Proof of Lemma 3

PROOF. For the sake of notation let us indicate $M(\delta_k, p)$ with $M_{k,p}$ and $G(\delta_k, p)$ with $G_{k,p}$, respectively.

Since the constant γ_1 is the same for all matrices $M_{k,p}$ one has $\left\| \prod_{k=1}^{\ell(p)} M_{k,p} \right\| \leq \prod_{k=1}^{\ell(p)} \|M_{k,p}\| \leq (1+\gamma_1 p)^{T/p} \leq e^{\gamma_1 T}$. Then condition (20a) directly follows.

In order to obtain (20b) one can write

$$\prod_{k=1}^{\ell(p)} (M_{k,p} + G_{k,p}) = \prod_{k=1}^{\ell(p)} M_{k,p} + \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} H_{i,k,p} \quad (24)$$

with

$$N(\ell(p), k) = \frac{\ell(p)!}{k!(\ell(p)-k)!} = \frac{\prod_{i=0}^{k-1} (\ell(p) - i)}{k!}$$

and $H_{i,k,p}$ suitable linear combinations of matrices where each $H_{i,k,p}$ contains a product with k matrices $G_{j,p}$ with $j \in \mathbb{N}$. Therefore one can write

$$\begin{aligned}
\left\| \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} H_{i,k,p} \right\| &\leq \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} \|H_{i,k,p}\| \\
&\leq \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} (1+\gamma_1 p)^{\ell(p)-k} (\alpha_{i,k} p^2)^k \\
&\leq \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} (1+\gamma_1 p)^{\ell(p)} (\alpha_{i,k} p^2)^k \\
&\leq \sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} e^{\gamma_1 T} (\alpha_{i,k} p^2)^k \\
&\leq e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} N(\ell(p), k) (\bar{\alpha}_k p^2)^k \\
&= e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{\prod_{i=0}^{k-1} (\ell(p) - i)}{k!} (\bar{\alpha}_k p^2)^k \\
&\leq e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{\ell(p)^k}{k!} (\bar{\alpha}_k p^2)^k \\
&= e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{(\bar{\alpha}_k \ell(p) p^2)^k}{k!} \leq e^{\gamma_1 T} \sum_{k=1}^{\ell(p)} \frac{(\bar{\alpha} p T)^k}{k!} \\
&\leq e^{\gamma_1 T} p \sum_{k=1}^{\ell(p)} \frac{(\bar{\alpha} T)^k}{k!} \leq e^{\gamma_1 T} p \sum_{k=1}^{\infty} \frac{(\bar{\alpha} T)^k}{k!} \\
&= e^{\gamma_1 T} \left(e^{\bar{\alpha} T} - 1 \right) p \qquad (25)
\end{aligned}$$

where for all k it is $\bar{\alpha}_k \geq \max\{\alpha_{i,k}\}_{i=1}^{N(\ell(p),k)}$, $\bar{\alpha} \geq \max\{\bar{\alpha}_k\}_{k=1}^{\ell(p)}$ and $p \geq p^k$ because without loss of generality one can assume $p \leq 1$.

From (25) it follows that

$$\sum_{k=1}^{\ell(p)} \sum_{i=1}^{N(\ell(p),k)} H_{i,k,p} = \mathrm{O}(p)$$

and then by using (24) the condition (20b) directly follows.

A.4 Proof of Lemma 4

PROOF. Let us consider the following

$$\prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p}\Pi_i = \prod_{i=1}^{\mathtt{q}}(I + F_i d_{i,k} p + \mathrm{O}(p^2))\Pi_i = \prod_{i=1}^{\mathtt{q}} \Pi_i + \mathrm{O}(p) \quad (26)$$

Let us consider the difference equation $\xi_{k+1} = F_k\xi_k$ with $\xi_k \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $F_k \in \mathcal{F}$ and $\mathcal{F} = \{\Pi_1, \dots, \Pi_{\mathtt{q}}\}$. By using the piecewise quadratic stability based on Lyapunov theory [3] it follows that the existence of a matrix P which solves the LMIs (21) imply that the system is absolutely stable for any sequence of matrices in $\mathcal{F}$, see Sec. 5 in [1]. Then, by using Theorem 3 in [4] it follows that

$$\left|\left|\left|\Pi_i\right|\right|\right| \leq 1 \quad (27)$$

for all $i = 1, \dots, \mathtt{q}$ with $|||\cdot|||$ being the norm induced by the matrix P. Therefore, by applying such norm to (26) and by using (27) one to write

$$\left|\left|\left|\prod_{i=1}^{\mathtt{q}} e^{F_i d_{i,k} p}\Pi_i\right|\right|\right| \leq \left|\left|\left|\prod_{i=1}^{\mathtt{q}} \Pi_i\right|\right|\right| + \gamma_1 p \leq \prod_{i=1}^{\mathtt{q}} \left|\left|\left|\Pi_i\right|\right|\right| + \gamma_1 p \leq 1 + \gamma_1 p \quad (28)$$

which completes the proof.

A.5 Proof of Theorem 5

PROOF. We first show that (23) holds at any switching time instant by considering for x the value before the possible jump, i.e.

$$x(s_{k,i}^-) - x_s(s_{k,i}) = O(p) \quad (29)$$

for any $k \in \{1, \dots, \ell(p)\}$ and $i \in \Sigma$.

By using Lemma 4 and Lemma 3 with $M(d_k, p) = \prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,k} p}\Pi_j$ it follows

$$\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,k} p}\Pi_j = \mathrm{O}(1) \quad (30a)$$

$$\prod_{j=1}^{\mathtt{q}} \left(e^{F_j d_{j,k} p}\Pi_j + O(p^2)\right) = \prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,k} p}\Pi_j + \mathrm{O}(p) \quad (30b)$$

for any $k \in \mathbb{N}_0$. Let us compute the solution of the smooth system:

$$\begin{aligned} x_s(s_{k,i}) &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j^{\varepsilon p} d_{j,m} p}\right)\prod_{j=1}^{i} e^{F_j^{\varepsilon p} d_{j,k} p} x_0 \\ &\stackrel{a}{=} \left[\prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j + \mathrm{O}(p^2)\right)\right. \\ &\quad \left.\cdot \prod_{j=1}^{i}(\Pi_j + F_j d_{j,k} p + O(p^2))\right] x_0 \\ &\stackrel{b}{=} \left[\prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\right. \\ &\quad \left.\cdot \prod_{j=1}^{i}(\Pi_j + F_j d_{j,k} p + \mathrm{O}(p^2)) + \mathrm{O}(p)\right] x_0 \\ &\stackrel{c}{=} \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i}\Pi_j x_0 + \mathrm{O}(p) \end{aligned} \quad (31)$$

where in $\stackrel{(a)}{=}$ has been used (22) and (17b) in Lemma 2, in $\stackrel{(b)}{=}$ has been used (30b), in $\stackrel{(c)}{=}$ has been used (30a).

The solution of the switched DAE can be written as

$$\begin{aligned} x(s_{k,i}^-) &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i} e^{F_j d_{j,m} p}\Pi_j x_0 \\ &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i}(\Pi_j + O(p))x_0 \\ &= \prod_{m=1}^{k-1}\left(\prod_{j=1}^{\mathtt{q}} e^{F_j d_{j,m} p}\Pi_j\right)\prod_{j=1}^{i}\Pi_j x_0 + O(p). \end{aligned} \quad (32)$$

By subtracting (31) to (32) one obtains (29).

Now, it can be proven that $x(t) - x_s(t) = O(p)$ holds for time instants different from switching time instants except for the time intervals $\tau \in (s_{k,i}, s_{k,i} + \Delta_p p)$ for any $k \in \mathbb{N}$ and $i \in \Sigma$. Let us assume that $\tau \in [s_{k,i} + \Delta_p p, s_{k,i+1})$. Then the solution of the switched DAE system is given by

$$x(\tau) = e^{F_i(\tau - s_{k,i})} \Pi_i x(s_{k,i}^-).$$

In the same time interval, the solution of the smooth system is given by

$$x_s(\tau) = e^{F_i^{\varepsilon p}(\tau - \Delta_p p - s_{k,i})} x_s(s_{k,i}^-)$$

Then one can write

$$\begin{aligned}
&x(\tau) - x_s(\tau) \\
&= e^{F_i(\tau - s_{k,i})} \Pi_i x(s_{k,1}^-) - e^{F_i^{\varepsilon p}(\tau - \Delta_p p - s_{k,i})} x_s(s_{k,i}^-) \\
&= (\Pi_i + O(\tau - s_{k,i})) x(s_{k,1}^-) \\
&- (\Pi_i + O((\tau - \Delta_p p - s_{k,i})) x_s(s_{k,i}^-) \\
&= \Pi_i (x(s_{k,i}^-) - x_s(s_{k,i}^-)) + O(p) = O(p)
\end{aligned} \tag{33}$$

for any $\tau \in [s_{k,i} + \Delta_p p, s_{k,i+1})$, $k \in \mathbb{N}$, $i \in \Sigma$, where in the last manipulation we used (29). By combining (29) and (33) the proof is complete.

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