# Observability of Switched Linear Singular System in Discrete Time: Single Switch Case

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***Abstract*— In this paper, we investigate the observability of switched linear singular systems in discrete time. As a preliminary study, we restrict the systems with a single switch switching signal, i.e. the system switches from one mode to another mode at a certain switching time. We provide two necessary and sufficient conditions for observability characterization. The first condition is applied for arbitrary switching time and the second one is for switching times that are far enough from the initial time and the final time of observation. These two conditions explicitly contain the switching time variable that indicates that generally, the observability is dependent on the switching time. However, under some sufficient conditions we provide, the observability will not depend on the switching time anymore. Furthermore, for two-dimensional systems, it is fully independent of the switching time. In addition, from the example we discussed, an observable switched system can be built from two unobservable modes and different mode sequences may produce different observability property i.e. swapping the mode sequence may destroy the observability.**

## I. INTRODUCTION

We consider in this paper discrete time switched linear singular systems (switched LSS) of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1a}$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) \tag{1b}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$, is the input vector, $y(k) \in \mathbb{R}^p, p \in \mathbb{N}$, is the output vector, $\sigma : \mathbb{N} \to \{1, 2, ..., \mathrm{p}\}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant $k$, matrices $E_i, A_i, B_i, C_i, D_i$ are real constant with appropriate dimension with $E_i$ is not assumed to be non-singular. In case of $E_i$ is non-singular for all modes, (1) belongs to the class of (non-singular) switched linear systems, where the study about the solution, controllability, and observability is well established [1], [2], [3]. The presence of singular matrices $E_i$ occur in some dynamical processes that are subject to algebraic constraints, see e.g. [4].

In continuous time domain, switched LSSs have been studied extensively. The solvability issue was well established in [5], [6] while the stability issue was fully determined in [5], [7], [8], [9]. Furthermore, the controllability and observability theorems were satisfactorily formulated in [10], [11], [12], and some strategies were developed to control this system has appeared in [13], [14], [15]. Moreover, some stability studies for models of physical systems have been conducted, for example for power systems [16].

In discrete time, some studies about switched LSSs are available in the literature. Most of them study the stability property as can be found in [17], [18], [19], [20], [21], [22], [23], [24]. Control methods were proposed in terms of iterative learning for trajectory tracking purposes [25], and filtering and state feedback control [26]. The latest study for this system appeared in [27] where a one-step-map was formulated that can be used to find the solution. The derived one-step-map matrix maps the state at any time instant $k$ to the state at $k+1$ so that this singular system could be represented as a non-singular system.

For observability study, we expect significant differences for the discrete time case compared to continuous time case. In continuous case, the observability property for the single switch case is independent to the switching time (see e.g. [11]) whereas in discrete time case, the observability property is dependent to the switching time. To illustrate this, we consider the following example for non-singular linear switching system.

*Example 1:* Consider non-singular switched linear system with two modes

$$x(k+1) = A_{\sigma(k)}x(k), \tag{2a}$$

$$y(k) = C_{\sigma(k)}x(k) \tag{2b}$$

with $\sigma : \mathbb{N} \to \{1, 2\}$, $A_1 = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right], A_2 = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$, $C_1 = \left[\begin{smallmatrix}1&1\end{smallmatrix}\right]$, $C_2 = \left[\begin{smallmatrix}1&0\end{smallmatrix}\right]$, $x(k) \in \mathbb{R}^2$ is the state, and $y(k) \in \mathbb{R}$ is the output at time $k \in \mathbb{N}$. Both individual modes are observable because the observability matrices

$$\begin{bmatrix} C_1 \\ C_1A_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_2 \\ C_2A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

have full rank and the state can be recovered in each mode by observing the output for two time steps (i.e. on the time interval $[0, K]$ with $K = 1$). Consider now the switched system (2) on the same time interval $[0, K]$ where we switch at $k_s = 1$ from mode 1 to mode 2. Then the output satisfies

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C_1x(0) \\ C_2A_1x(0) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2A_1 \end{bmatrix} x(0)$$

and since $\left[\begin{smallmatrix} C_1 \\ C_2A_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$ does have a non-trivial kernel, we *cannot* determine the initial state from the output on the interval $[0, K]$, i.e. the switched system is *not* observable!

On the other hand, this switched system is observable if the system starts from mode 2 and switches to mode 1 at time

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instant $k_s = 1$. This means that the observability property is not symmetric with respect to the switching signals. ⋄

In this paper, we consider the observability characterization for single switch case of switched linear singular systems in discrete time. It is assumed that the mode switching is triggered only by time. Other discussions in the literature may include mode switching which is triggered by the state and/or the output. The discussion is structured as follows. We recall the existence and uniqueness of the solution first in Preliminaries section and write the observability definition in Section III. We present the main results in Section IV containing the observability theorem and some examples.

## II. Preliminaries

We are revisiting the solution and observability characterization of switched linear (non-singular) systems and the solution of switched linear singular systems in this section as the foundation to characterize the observability notion discussed in this paper.

### A. Linear Singular System

Consider the homogeneous Linear Singular System (LSS) in the form of

$$Ex(k+1) = Ax(k), \tag{3a}$$
$$y(k) = Cx(k) \tag{3b}$$

for $k \in \mathbb{N}$ and where $E, A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{p\times n}$ are known, $E$ is singular with $\operatorname{rank} E = r < n$, $x : \mathbb{N} \to \mathbb{R}^n$ is the state, and $y : \mathbb{N} \to \mathbb{R}^p$ is the output. We omit the input because observability of linear systems does not depend on the input.

If (3a) is regular, i.e. $\det(sE - A)$ is not identically zero, then there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{4}$$

where $N \in \mathbb{R}^{n_N\times n_N}$ is nilpotent, for some $J \in \mathbb{R}^{r\times r}$. Following [28], we call (4) a quasi Weirstrass form (QWF) of the matrix pair $(E, A)$. If the nilpotency index of $N$ is 1 (i.e. $N = 0$), then system (3a) is called index-1 system.

*Lemma 2.1 ([27]):* Consider the matrix pair $(E, A)$ with $E, A \in \mathbb{R}^{n\times n}$ and let $\mathcal{S} := A^{-1}(\operatorname{im} E) = \{\ \xi \in \mathbb{R}^n \mid A\xi \in \operatorname{im} E\ \}$. Then $(E, A)$ is regular and index-1 if, and only if,

$$\mathcal{S} \cap \ker E = \{0\}.$$

Furthermore, choose full rank matrices $V$ and $W$ such that $\operatorname{im} V = \mathcal{S}$ and $\operatorname{im} W = \ker E$, then if $(E, A)$ is regular and index-1, then $T = [V, W]$ and $S = [EV, AW]^{-1}$ transform $(E, A)$ into QWF (4).

*Corollary 2.2 ([27]):* Consider (3a), let $\mathcal{S} := A^{-1}(\operatorname{im} E)$ and assume $(E, A)$ is regular and index-1 with QWF (4). Then $x$ is a solution of (3a) if, and only if, $x(0) \in \mathcal{S}$ and

$$x(k + 1) = \Phi_{(E,A)}x(k),$$

where

$$\Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Note that $\Phi_{(E,A)}$ is independent from the specific choice of $T$ and is called one-step-map for system (3a). Furthermore, if (3a) is actually non-singular (i.e. $E = I$), then $\Phi_{(E,A)} = A$.

In view of the forthcoming extension to switched system, we want to highlight that the interpretation of $\Phi_{(E,A)}$ as a one-step-map for the LSS (3a) is only valid if (3a) holds for at least two time steps, see [27, Rem. 2.6].

*Definition 2.3:* LSS (3) is observable on the interval $[0, K]$, $K \in \mathbb{N}$, if its state $x$ is uniquely determined on $[0, K]$ by its output $y$ on $[0, K]$.

By using linearity, the observability definition of LSS (3) is equivalent to zero distinguishability; i.e. it is observable if, and only if, the following implication holds on $[0, K]$:

$$y \equiv 0 \quad \Longrightarrow \quad x \equiv 0.$$

Furthermore, in view of Corollary 2.2, if (3) is index-1, then $x \equiv 0$ on $[0, K]$ if, and only if, $x(0) = 0$, in particular, the observability condition can be reduced to

$$y(k) = 0\ \forall k \in [0, K] \quad \Longrightarrow \quad x(0) = 0. \tag{5}$$

From Corollary 2.2 we can also derive that

$$y(k) = C\Phi^k_{(E,A)}x(0) \quad k \in [0, K]$$

as well as $x(0) \in \mathcal{S}$. Hence for index-1 LSS we immediately have the following observability characterization:

*Lemma 2.4:* Index-1 LSS (3) is observable on $[0, K]$ if, and only if,

$$\mathcal{S} \cap \mathcal{O}^K = \{0\}, \tag{6}$$

where $\mathcal{O}^K := \ker[C^\top, (C\Phi_{(E,A)})^\top, \dots, (C\Phi^K_{(E,A)})^\top]^\top$.

Note that due to Cayley-Hamilton, $\mathcal{O}^K = \mathcal{O}^{n-1}$ if $K \geq n - 1$, but $\mathcal{O}^K \supsetneq \mathcal{O}^{n-1}$ is possible; in particular, the unobservable space (i.e. the subspace of all initial values $x(0)$ which produce a zero output) depends on the length $K$ of the considered interval when $K$ is small compared to the system dimension $n$. This is a major difference to the continuous time case, where the unobservable space (given by $\mathcal{O}^{n-1}$) is independent from the length of the observation interval. This makes the observability analysis for switched systems more challenging in the discrete time case compared to the continuous time case.

### B. Switched Linear Singular Systems

Consider a homogeneous switched LSS i.e. system (1) without input of the form

$$E_{\sigma(k)}x(k + 1) = A_{\sigma(k)}x(k), \quad k \in \mathbb{N}, \tag{7a}$$
$$y(k) = C_{\sigma(k)}x(k) \tag{7b}$$

where $\sigma : \mathbb{N} \to \{1, 2, ..., \mathtt{p}\}$ is the switching signal, determining which mode $\sigma(k)$ is active at time $k \in \mathbb{N}$. Each mode is given by the matrix triple $(E_p, A_p, C_p) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times m}$ for $p \in \{1, 2, \dots, \mathtt{p}\}$. Again, we omit the input because we assume that the switching signal is fixed, therefore (1) is linear and observability will not depend on $B_{\sigma(k)}$ and $D_{\sigma(k)}$.

To ensure existence and uniqueness of solutions of (7) for general switching signals it is in general not enough to assume that each matrix pair $(E_p, A_p)$ is regular (in contrast to the continuous time case), even assuming that each matrix pair is index-1 is not sufficient, see the example in the introduction of [27]. In order to have a well-posed switched LSS it is actually necessary to assume that the family of matrix pairs $(E_p, A_p)$ is jointly index-1 in the following sense:

*Definition 2.5 ([27]):* A family of matrix pairs $\{(E_1, A_1), \dots, (E_{\mathtt{p}}, A_{\mathtt{p}})\}$ or the corresponding system (7) is called (jointly) index-1 if, and only if,

$$\mathcal{S}_i \cap \ker E_j = \{0\}, \quad \forall i, j \in \{1, 2, \dots, \mathtt{p}\},$$

where $\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$.

As shown in [27], a consequence of the (jointly) index-1 assumption for (7) is that $\operatorname{rank} E_i = \text{constant} =: r$ and that $\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n$ for all $i, j \in \{1, 2, \dots, \mathtt{p}\}$. Furthermore, for every index-1 switched LSS, the individual modes are also index-1 (just choose $i = j$ in the definition and apply Lemma 2.1) hence we find nonsingular matrices $S_i$ and $T_i$ that satisfy

$$(S_i E_i T_i, S_i A_i T_i) = \left( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix} \right), \tag{8}$$

for some $J_i \in \mathbb{R}^{r \times r}$.

Moreover, it was shown in [27] that $x$ is a solution of the index-1 system (7) if, and only if, $x(0) \in \mathcal{S}_{\sigma(0)}$ and

$$x(k+1) = \Phi_{\sigma(k+1), \sigma(k)} x(k), \forall k \in \mathbb{N}$$

where $\Phi_{i,j}$ is the *one-step map* from mode $j$ to mode $i$ given by

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \Phi_j \tag{9}$$

where $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is a unique projector onto $\mathcal{S}_i$ along $\ker E_j$ and

$$\Phi_j := \Phi_{(E_j, A_j)} = T_j \begin{bmatrix} J_j & 0 \\ 0 & 0 \end{bmatrix} T_j^{-1}. \tag{10}$$

*Remark 2.6:* Under the assumption that the switching signal is fixed and known (as is the case in our observability study), it may not be necessary to assume that (7) is jointly index-1 to have existence and uniqueness of solutions, because not all mode combination $(i, j)$ will occur. This extension is a topic of future research.

## III. Single switch results

In the following we will restrict our attention to the single switch case, i.e. we consider (7) with

$$\sigma(k) = \begin{cases} 1, & 0 \le k < k_s, \\ 2, & k_s \le k \le K, \end{cases} \tag{11}$$

see also Figure 1.

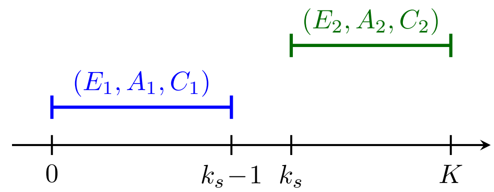


Fig. 1. Single switch linear singular system (7)

The observability definition from the nonswitched case carries over without change. Furthermore, under the index-1 assumption for (7) existence and uniqueness of solutions for all initial values $x(0) \in \mathcal{S}_1$ can also be characterized by the implication (5).

We can now formulate our main result about characterizing observability of a switched LSS with a single switch.

### A. Arbitrary switching time and observation time

*Theorem 3.1:* Consider the switched LSS (7) with the single switch switching signal (11) and assume it is index-1 with corresponding one-step maps $\Phi_1, \Phi_2$ given by (5) and $\Phi_{2,1}$ as in (9). Then (7) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_1 \cap \mathcal{O}_1^{k_s - 1} \cap \left[ \Phi_{2,1} \Phi_1^{k_s - 1} \right]^{-1} \left( \mathcal{O}_2^{K - k_s} \right) = \{0\}, \tag{12}$$

where, for $i = 1, 2$ and $k \in \mathbb{N}$,

$$\mathcal{O}_i^k := \ker[C_i^\top, (C_i \Phi_i)^\top, \dots, (C_i \Phi_i^k)^\top]^\top.$$

*Proof:* The solution of the given system can be written in matrix form as

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k_s - 1) \\ y(k_s) \\ y(k_s + 1) \\ \vdots \\ y(K) \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 \\ C_1 \Phi_1 \\ \vdots \\ C_1 \Phi_1^{k_s - 1} \\ C_2 \Phi_{2,1} \Phi_1^{k_s - 1} \\ C_2 \Phi_2 \Phi_{2,1} \Phi_1^{k_s - 1} \\ \vdots \\ C_2 \Phi_2^{K - k_s} \Phi_{2,1} \Phi_1^{k_s - 1} \end{bmatrix}}_{O^{k_s, K}} x_0 = O^{k_s, K} x_0.$$

Then we have

$$\ker O^{k_s, K} = \mathcal{O}^{k_s, K} = \mathcal{O}_1^{k_s - 1} \cap \left[ \Phi_{2,1} \Phi_1^{k_s - 1} \right]^{-1} \left( \mathcal{O}_2^{K - k_s} \right),$$

where we use the fact, that $\ker(O\Phi) = \Phi^{-1}(\ker O)$ for any matrices $O$ and $\Phi$ of appropriate size (note that here $\Phi^{-1}$ denotes the preimage and not the inverse matrix). ($\Rightarrow$) Assume $0 \neq x_0 \in \mathcal{S}_1 \cap \mathcal{O}^{k_s, K}$. Then there exists a unique, non-trivial solution $x$ of (6) with $x(0) = x_0$. Since $x(0) \in \mathcal{O}^{k_s, K}$ then $y(k) = 0$, $0 \le k \le K$. This means that there exists a non-trivial solution of $x$ with zero output. Hence, (3) is not observable.
($\Leftarrow$) Consider a solution of (7) then $x(0) \in \mathcal{S}_1$. Furthermore, if $y(k) = 0$ for all $k \in [0, K]$, then $x(0) \in \mathcal{O}^{k_s, K}$. Hence $x(0) \in \mathcal{S}_1 \cap \mathcal{O}^{k_s, K} = \{0\}$, which shows the desired implication (5). ■

It is interesting to compare the observability condition in continuous time and discrete time. While the observability condition in continuous time does not depend on the switching time (see [11, Theorem 9]), the obervability condition in discrete time in general depends on the switching time as stated in (12). This means that changing the switching time might change the observability property.

### B. Large enough observation time

The dependence on the switching time $k_s$ in the observability characterization (12) can be reduced by exploiting the Cayley-Hamilton-Theorem as follows:

*Corollary 3.2:* Consider the index-1 switched LSS (7) with the switching signal (11) and assume $n \leq k_s \leq K - n$. Then (7) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_1 \cap \mathcal{O}_1 \cap \left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}(\mathcal{O}_2) = \{0\} \tag{13}$$

where for $i = 1, 2$

$$\mathcal{O}_i := \ker[C_i^\top, (C_i\Phi_i)^\top, \ldots, (C_i\Phi_i^{n-1})^\top]^\top.$$

Furthermore, if $\Phi_1$ is idempotent, then (7) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_1 \cap \mathcal{O}_1 \cap \Phi_{2,1}^{-1}(\mathcal{O}_2) = \{0\}. \tag{14}$$

So in general even if each mode is active long enough such that no additional information about the state from the output can be deduced, the observability property of the switched system still depends on the switching time unless very strict assumptions are made on the first mode.

Note that the observability characterization (14) is almost identical to the one obtained in continuous time (under an impulse-free, i.e. index-1, assumption), however, the assumption that $\Phi_1$ is idempotent (i.e. $\Phi_1$ is a projector) is extremely restrictive (and implies that the first mode has constant state trajectories).

The following lemma will be used to provide some conditions in which the observability does not depend on the switching time.

*Lemma 3.3:* Let $M, \Phi \in \mathbb{R}^{n\times n}$.

(i) If $\ker M \cap \operatorname{im}\Phi = \{0\}$ then there is a $k$, $0 \leq k \leq n$, such that

$$\ker(M\Phi^k) = \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

(ii) If, for some $k$, $\ker(M\Phi^k) \supseteq \ker(M\Phi^{k+1})$ then

$$\ker(M\Phi^k) \supseteq \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

(iii) If, for some $k$, $\ker(M\Phi^k) \subseteq \ker(M\Phi^{k+1})$ then

$$\ker(M\Phi^k) \subseteq \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

(iv) If, for some $k$, $\ker(M\Phi^k) = \ker(M\Phi^{k+1})$ then

$$\ker(M\Phi^k) = \ker(M\Phi^{k+j}) \quad \forall j \in \mathbb{N}.$$

*Proof:*

(i) Since there exists $k$, $0 \leq k \leq n$, such that $\ker\Phi^{k+1} = \ker\Phi^k$, we conclude

$$\begin{aligned} x \in \ker(M\Phi^{k+1}) &\Leftrightarrow M\Phi^{k+1}x = 0 \\ &\overset{*}{\Leftrightarrow} \Phi^{k+1}x = 0 \\ &\Leftrightarrow x \in \ker\Phi^{k+1} = \ker\Phi^k \\ &\Leftrightarrow \Phi^k x = 0 \\ &\overset{*}{\Leftrightarrow} M\Phi^k x = 0 \\ &\Leftrightarrow x \in \ker(M\Phi^k), \end{aligned}$$

where the equivalences marked with $*$ are a consequence from the assumption $\ker M \cap \operatorname{im}\Phi = \{0\}$.

(ii) Under the given assumption we have

$$\begin{aligned} &x \in \ker(M\Phi^{k+2}) \\ \Leftrightarrow\ &\Phi x \in \ker(M\Phi^{k+1}) \subseteq \ker(M\Phi^k) \\ \Rightarrow\ &M\Phi^k\Phi x = 0 \Leftrightarrow x \in \ker(M\Phi^{k+1}). \end{aligned}$$

(iii) Under the given assumption we have

$$\begin{aligned} &x \in \ker(M\Phi^{k+1}) \\ \Leftrightarrow\ &\Phi x \in \ker(M\Phi^k) \subseteq \ker(M\Phi^{k+1}) \\ \Rightarrow\ &M\Phi^{k+1}\Phi x = 0 \Leftrightarrow x \in \ker(M\Phi^{k+2}). \end{aligned}$$

(iv) This is an immediate consequence from (ii) and (iii). ∎

From Lemma 3.3 part (i) and (iv), we can derive the following corollary explaining the independence of observability on the switching time under some certain conditions.

*Corollary 3.4:*

(i) Assume that $\ker(O_2\Pi_{\mathcal{S}_2}^{\ker E_1}) \cap \operatorname{im}\Phi_1 = \{0\}$. Then the observability of switched LSS (7) does not depend on $k_s$ for any $k_s > n - 1$. Moreover, the third subspace in (13) can be replaced by the simpler condition $\left[\Pi_{\mathcal{S}_2}^{\ker E_1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2)$.

(ii) If, for some $k \in \mathbb{N}$,

$$\mathcal{O}_1^{k-1} = \mathcal{O}_1^k \text{ and} \tag{15}$$

$$\left[\Phi_{2,1}\Phi_1^{k-1}\right]^{-1}(\mathcal{O}_2) = \left[\Phi_{2,1}\Phi_1^k\right]^{-1}(\mathcal{O}_2) \tag{16}$$

then the observability of switched LSS (7) does not depend on the switching time for any $k_s \geq k$. Moreover, if for some $k > n-1$, (16) holds then the observability of switched LSS (7) does not depend on the switching time for any $k_s \geq n - 1$.

From part (ii) and (iii) in Lemma 3.3, we can derive some situations where the observability property would be independent of the switching time as explained in the following remark.

*Remark 3.5:* If, for some $k_s > n - 1$, the third subspace in (13) satisfies

$$\left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}(\mathcal{O}_2) \supseteq \left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2)$$

then

$$\left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2) \supseteq \left[\Phi_{2,1}\Phi_1^{k_s+1}\right]^{-1}(\mathcal{O}_2) \supseteq \cdots.$$

In this situation the third subspace in (13) will not grow for bigger $k_s$ resulting that if the switched system is observable with the given switching time $k_s$ then it will always be observable for any bigger switching time. Similarly, if for some $k_s > n-1$

$$\left[\Phi_{2,1}\Phi_1^{k_s-1}\right]^{-1}(\mathcal{O}_2) \subseteq \left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2)$$

then

$$\left[\Phi_{2,1}\Phi_1^{k_s}\right]^{-1}(\mathcal{O}_2) \subseteq \left[\Phi_{2,1}\Phi_1^{k_s+1}\right]^{-1}(\mathcal{O}_2) \subseteq \cdots .$$

In this situation the third subspace in (13) will not shrink for bigger $k_s$ resulting that if the switched system is not-observable for the given switching time $k_s$ then the switched system will always be not-observable for any bigger switching time.

So far, we have obtained some sufficient conditions as in 3.4 to verify the independence of the observability on the switching time. When restricting to a two-dimensional state space ($n = 2$), we are actually able to prove that observability cannot depend on the switching time as long as both modes are active for at least two time steps.

*Theorem 3.6:* Consider the switched LSS (7) of index with state space dimension $n = 2$. Assume that both modes are active for at least two time steps, i.e. $2 \leq k_s \leq K-2$. Then the observability of the switched system does not depend on $k_s$. In fact, the observability condition (13) can be reduced to

$$\mathcal{S}_1 \cap \mathcal{O}_1 \cap [\Phi_{2,1}\Phi_1]^{-1}(\mathcal{O}_2) = \{0\} \tag{17}$$

where $\mathcal{O}_i := \ker[C_i^\top, (C_i\Phi_i)^\top]^\top$ for $i = 1, 2$.

*Proof:* The observability condition is

$$\mathcal{S}_1 \cap \ker\begin{bmatrix} C_1 \\ C_1\Phi_1\end{bmatrix} \cap \ker\begin{bmatrix} C_2\Pi_{2,1}\Phi_1^{k_s} \\ C_2\Phi_2\Pi_{2,1}\Phi_1^{k_s}\end{bmatrix} = \{0\} \tag{18}$$

With some invertible matrix $P$ we have

$$\ker\begin{bmatrix} C_1 \\ C_1\Phi_1 \\ C_2\Pi_{2,1}\Phi_1^{k_s} \\ C_2\Phi_2\Pi_{2,1}\Phi_1^{k_s}\end{bmatrix} := \mathcal{O}^{k_s} \text{depends on } k_s$$

$$\Leftrightarrow \ker\begin{bmatrix} C_1P \\ C_1\Phi_1P \\ C_2\Pi_{2,1}\Phi_1^{k_s}P \\ C_2\Phi_2\Pi_{2,1}\Phi_1^{k_s}P\end{bmatrix} = \ker\begin{bmatrix}\widetilde{C}_1 \\ \widetilde{C}_1\widetilde{\Phi}_1 \\ \widetilde{M}\widetilde{\Phi}_1^{k_s}\end{bmatrix} \text{ depends on } k_s$$

where $\widetilde{C}_1 := C_1P$, and $\widetilde{\Phi}_1 := P^{-1}\Phi_1P$ is in Jordan canonical form, and $\widetilde{M} := \begin{bmatrix} C_2\Pi_{2,1}P \\ C_2\Phi_2\Pi_{2,1}P\end{bmatrix}$. Assume w.l.o.g. $\widetilde{C}_1 = [\gamma_1, \gamma_2]$. Consider $\lambda, \lambda_1, \lambda_2$ and $\alpha \pm \beta i$ denote all possible eigenvalues of $\widetilde{\Phi}_1$. If $\widetilde{\Phi}_1$ is arbitrary then $\widetilde{\Phi}_1$ will be either $\widetilde{\Phi}_1 = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$, $\widetilde{\Phi}_1 = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}$, or $\widetilde{\Phi}_1 = \begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}$.

*Case 1.* $\widetilde{\Phi}_1 = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$. Assume first $C_1 \neq 0$.

$$\begin{pmatrix} v_1 \\ v_2\end{pmatrix} \in \ker\begin{bmatrix}\widetilde{C}_1 \\ \widetilde{C}_1\widetilde{\Phi}_1\end{bmatrix} = \ker\begin{bmatrix}\gamma_1 & \gamma_2 \\ \gamma_1\lambda_1 & \gamma_2\lambda_2\end{bmatrix}$$

leads to $\gamma_2(\lambda_2 - \lambda_1)v_2 = 0$ resulting three possible situations as follow:

(i) $\gamma_2 = 0, \lambda_1 \neq \lambda_2$. Since $C_1 \neq 0$ by assumption, $\gamma_1 \neq 0$ and hence $v_1 = 0$ and $v_2 \neq 0$ and consequently

$$\ker\begin{bmatrix}\widetilde{C}_1 \\ \widetilde{C}_1\widetilde{\Phi}_1\end{bmatrix} = \operatorname{span}\begin{pmatrix}0 \\ 1\end{pmatrix}.$$

Different observability for different $k_s$ could be achieved if $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \operatorname{span}\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)$ for some $k_s$ and $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} \neq \operatorname{span}\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)$ for some other $k_s$. Let $\widetilde{M} = \left[\begin{smallmatrix}m_{11} & m_{12} \\ m_{21} & m_{22}\end{smallmatrix}\right]$ then

$$\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \ker\begin{bmatrix} m_{11}\lambda_1^{k_s} & m_{12}\lambda_2^{k_s} \\ m_{21}\lambda_1^{k_s} & m_{22}\lambda_2^{k_s}\end{bmatrix}.$$

If $\lambda_1 = 0$ then $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \operatorname{span}\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)$ cannot be true for any $k_s > 0$, on the other hand, if $\lambda_1 \neq 0$ then

$$\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \ker\begin{bmatrix} m_{11} & (\frac{\lambda_2}{\lambda_1})^{k_s}m_{12} \\ m_{21} & (\frac{\lambda_2}{\lambda_1})^{k_s}m_{22}\end{bmatrix} = \operatorname{span}\begin{pmatrix}0 \\ 1\end{pmatrix}$$

implies either $\lambda_2 = 0$ or $m_{12} = m_{22} = 0$. In both case, $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s}$ doesn't not depend on $k_s$.

(ii) $\lambda_1 = \lambda_2 = \lambda \neq 0$ that gives us

$$\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \ker\begin{bmatrix} m_{11}\lambda^{k_s} & m_{12}\lambda^{k_s} \\ m_{21}\lambda^{k_s} & m_{22}\lambda^{k_s}\end{bmatrix} = \ker\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}$$

which clearly does not depend on $k_s$.

(iii) $v_2 = 0$, $\lambda_1 \neq \lambda_2, \gamma_2 \neq 0$.
As a consequence, $v_1 \neq 0$ hence $\gamma_1 = 0$. In this case we have

$$\ker\begin{bmatrix}\widetilde{C}_1 \\ \widetilde{C}_1\widetilde{\Phi}_1\end{bmatrix} = \operatorname{span}\begin{pmatrix}1 \\ 0\end{pmatrix}.$$

Different observability for different $k_s$ could be achieved if $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \operatorname{span}\left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right)$ for some $k_s$ and $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} \neq \operatorname{span}\left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right)$ for some other $k_s$. Analogue arguments as in (i) shows that this is not possible.

If $C_1 = 0$ then the second subspace in (18) will be $\mathbb{R}^2$ which means that we need to proof that the intersection of the first and the third subspace in (18) does not depend on $k_s$. Only if $\dim\mathcal{S}_1 = 1$ this is possible. Since $\operatorname{im}\Phi_1 \subseteq \mathcal{S}_1$ it follows that $\Phi_1$ has at least one eigenvalue 0. In particular, by applying an appropriate coordinate transformation we obtain $\widetilde{\Phi}_1 = \left[\begin{smallmatrix}\lambda & 0 \\ 0 & 0\end{smallmatrix}\right]$ and $\widetilde{\mathcal{S}}_1 = \operatorname{im}\left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right)$. With the same arguments as above it can be shown that it is not possible that $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} = \operatorname{span}\left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right)$ for some $k_s$ and $\ker\widetilde{M}\widetilde{\Phi}_1^{k_s} \neq \operatorname{span}\left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right)$ for some other $k_s$.

*Case 2.* $\widetilde{\Phi}_1 = \left[\begin{smallmatrix}\lambda & 1 \\ 0 & \lambda\end{smallmatrix}\right]$.
Similar as in Case 1 we first consider $C_1 \neq 0$ and observe that

$$\begin{pmatrix} v_1 \\ v_2\end{pmatrix} \in \ker\begin{bmatrix}\widetilde{C}_1 \\ \widetilde{C}_1\widetilde{\Phi}_1\end{bmatrix} = \ker\begin{bmatrix}\gamma_1 & \gamma_2 \\ \gamma_1\lambda & \gamma_1 + \gamma_2\lambda\end{bmatrix}$$

implies $\gamma_1 v_2 = 0$ and we distinguish the following two cases

(i) $v_2 = 0$.
Then $v_1 \neq 0$ and hence $\gamma_1 = 0$. In this case we have

$$\ker \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_1 \widetilde{\Phi}_1 \end{bmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Different observability for different $k_s$ could be achieved if $\ker \widetilde{M} \widetilde{\Phi}_1^{k_s} = \operatorname{span} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ for some $k_s$ and $\ker \widetilde{M} \widetilde{\Phi}_1^{k_s} \neq \operatorname{span} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ for some other $k_s$. If $\lambda = 0$ then $\Phi^{k_s} = 0$ for all $k_s > 1$, hence we can assume that $\widetilde{\Phi}_1$ is invertible and we have

$$\begin{aligned} \ker \widetilde{M} \widetilde{\Phi}_1^{k_s} = \operatorname{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\Leftrightarrow \ker \widetilde{M} = \operatorname{span} \widetilde{\Phi}_1^{k_s} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \ker \widetilde{M} = \operatorname{span} \begin{pmatrix} \lambda^{k_s} \\ 0 \end{pmatrix} \end{aligned}$$

which is independent of $k_s$. Thus it's not possible to have different observability for different $k_s$.

(ii) $\gamma_1 = 0$.
Then $\gamma_2 \neq 0$ and hence $v_2 = 0$ and this case was already treated in (i).
If $C_1 = 0$ then similar as above we can conclude that then $\Phi_1$ must have an eigenvalue zero, which means that $\Phi_1$ is nilpotent in the current case and hence the observability condition cannot depend on $k_s$.

*Case 3.* $\widetilde{\Phi}_1 = \left[\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right]$.

First note that $\beta \neq 0$ otherwise we would be in Case 1. Hence $\widetilde{\Phi}_1$ is invertible and therefore $\mathcal{S}_1 = \mathbb{R}^2$. Then the intersection (18) can only depend on $k_s$ if $\widetilde{C}_1 = [\gamma_1, \gamma_2] \neq 0$. Observe now that

$$\begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_1 \widetilde{\Phi}_1 \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 \\ \alpha\gamma_1 - \beta\gamma_2 & \gamma_1\beta + \gamma_2\alpha \end{bmatrix}$$

and $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in \ker \left[\begin{smallmatrix} \widetilde{C}_1 \\ \widetilde{C}_1 \widetilde{\Phi}_1 \end{smallmatrix}\right]$ leads to $\beta(\gamma_1 v_2 - \gamma_2 v_1) = 0$. Since $\beta \neq 0$ we have therefore $\gamma_1 v_2 = \gamma_2 v_1$. Furthermore, we have that

$$\ker \begin{bmatrix} \widetilde{C}_1 \\ \widetilde{C}_1 \widetilde{\Phi}_1 \end{bmatrix} \subseteq \operatorname{span} \begin{pmatrix} \gamma_2 \\ -\gamma_1 \end{pmatrix},$$

so we can assume that $v_1 = \gamma_2$ and $v_2 = -\gamma_1$. This leads to the contradiction $\gamma_1^2 + \gamma_2^2 = 0$. ∎

At the moment it is not clear to the authors whether Theorem 3.6 remains valid for higher dimensional systems.

## IV. Illustrative Example

We have seen already in the introduction that even if all modes of the switched systems are observable on a given interval $[0, K]$, the switched system considered on the same interval is not necessarily observable.

On the other hand it is also possible that the switched system is observable although the individual modes are not observable, this is illustrated in the following example.

*Example 2:* Consider SiSwLSS (7) with

$$E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix},$$

with $\mathcal{S}_1 = \operatorname{span}\left\{ \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \\ -1 \end{smallmatrix}\right) \right\}, \mathcal{S}_2 = \operatorname{span}\left\{ \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right) \right\}, \ker E_1 = \ker E_2 = \operatorname{span}\left\{ \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 1 \end{smallmatrix}\right) \right\}$, which is jointly index-1 since $\mathcal{S}_i \cap \ker E_j = \{0\}$, $i, j = 1, 2$, with corresponding one-step-map matrices

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \Phi_{1,2} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \Phi_{2,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Each individual system i.e. mode-1 and mode-2 are both not-observable on $[0, K]$ $\forall K \geq n$ since $\mathcal{S}_1 \cap \mathcal{O}_1 = \operatorname{span} \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \\ -1 \end{smallmatrix}\right) \neq \{0\}$, and $\mathcal{S}_2 \cap \mathcal{O}_2 = \operatorname{span} \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \\ -1 \end{smallmatrix}\right) \neq \{0\}$.

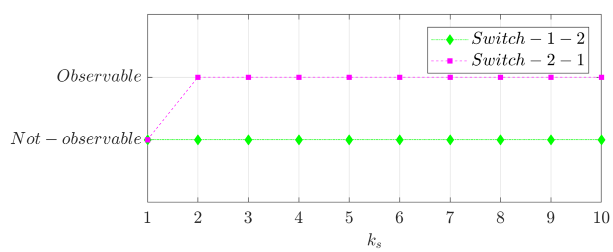


Fig. 2. Switching time vs observability Example 2

We observe here the observability on $[0, K], K \geq k_s + n - 1$ for $1 \leq k_s \leq 10$ (see Fig. 2). Mode sequence 1-2 i.e. the system starts from mode-1 and switches to mode-2 always produced a not-observable switched system since the observability condition (12) for $k_s < 4$ or (13) for $k_s \geq 4$ holds. This is not surprising because each individual system is not-observable which is obvious to produce a not-observable switched system. Furthermore, by checking the condition in part (i) of Corollary 3.4 we have that

$$\ker(O_2 \Pi_{\mathcal{S}_2}^{\ker E_1}) \cap \operatorname{im} \Phi_1 = \{0\}$$

which means that we can make sure that mode sequence 1-2 is not observable for any $k_s$ i.e. it doesn't depend on the switching time.

In the other hand, mode sequence 2-1 i.e. the system starts from mode-2 and switches to mode-1 produced a not-observable switched system with switching time $k_s = 1$ since for $k_s = 1$

$$\mathcal{S}_2 \cap \mathcal{O}_2^{k_s-1} \cap \left[\Phi_{1,2}\Phi_2^{k_s-1}\right]^{-1}(\mathcal{O}_1) = \operatorname{span}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \neq \{0\}.$$

For switching time $k_s = 2, 3, ..., 10$ the switched system is observable. This shows that even though each mode is not-observable, the switched system could be observable which means that switching the system from one unobservable mode to another unobservable mode can build an observable switched system. Fig. 2 also shows that the observability condition is not symmetric meaning that swapping the mode sequence may change the observability property. For mode sequence 2-1, condition in part (i) of Corollary 3.4 is not satisfied but condition in part (ii) is satisfied i.e. we have that

$$\left[\Phi_{1,2}\Phi_2^{9}\right]^{-1}(\mathcal{O}_1) = \left[\Phi_{1,2}\Phi_2^{10}\right]^{-1}(\mathcal{O}_1).$$

This implies that we also can make sure that the mode sequence 2-1 is always observable for any $k_s > 10$ i.e. it doesn't depend on the switching time anymore.

## V. Summary

We have presented two necessary and sufficient conditions for observability characterization of switched linear singular systems in discrete time with single switch switching signal. The first characterization corresponds to arbitrary switching time whereas the second characterization corresponds to large enough switching time (mode 1 is active at least $n-1$ time steps. The switching time variable appears explicitely in the characterization which means that in general the observability property seems to dependent on the switching time. However, we have provided some sufficient conditions when the observability is not depend on the switching time anymore. Furthermore, for two-dimensional systems, we have proved that it does not depend on the switching time. Finally, we illustrate via examples, that 1) even if each individual subsystem is not-observable, the switching can make the system observable, and 2) swapping the mode sequence can change the observability.

The results of this paper still leave many open problems that will be studied in our future works. First, the dependence of observability on the switching time for higher dimensional systems. Second, the characterization for multiple switches case. Furthermore, forward observability and observer design are also interesting to study.

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