Minimal Realization for Linear Switched Systems

Md Sumon Hossain and Stephan Trenn

***Abstract*— We discuss the problem of minimal realization for linear switched systems with a given switching signal and present some preliminary results for the single switch case. The key idea is to extend the reachable subspace of the second mode to include nonzero initial values (resulting from the first mode) and also extend the observable subspace of the first mode by taking information from the second mode into account. We provide some simple example to illustrate the approach.**

I. Introduction

Realization theory is one of the most important and central topics of system theory. In general, the realization problem deals with finding an equivalent internal description of a dynamical system from an external one. In general, the aim of realization theory is to understand the relationship between an observed behavior and dynamical systems producing this observed behavior. Realization theory provides a theoretical foundation for model reduction, system identification and filtering/observer design. Indeed, transforming a system to a minimal order by preserving its input-output behavior could be seen as the first step towards model reduction.

The realization theory of switched systems have been discussed in the couple of papers [1], [2], [3] and the references therein, [4], [5], [6]. The author in [3] combines the theory of rational formal power series with the classical automata theory to discuss the realization theory of hybrid systems. The cases of arbitrary and constrained switching are discussed where the switching signal are viewed as input to the switched system. Moreover, some work have been done on realization theory of switched systems as well as on observability and reachability, [7], [8], [9], [10], [11], [12].

In this paper, we are motivated to find the minimal realization of linear switched systems (LSS) with a given switching signal. Consider the LSS of the form:

$$\Sigma_\sigma : \begin{cases} \dot{x}_q(t) = A_{\sigma(t)}x_q(t) + B_{\sigma(t)}u(t),\ t \in (t_q, t_{q+1}) \\ x_q(t_q^+) = J_{\sigma(t_q^+)}x_{q-1}(t_q^-), \\ y(t) = C_{\sigma(t)}x_q(t), \quad t \in \mathbb{R}, \end{cases} \tag{1}$$

where $x_q : (t_q, t_{q+1}) \to \mathbb{R}^{n_q}$ is the q-th piece of the state, $\sigma : \mathbb{R} \to \mathscr{Q} = \{1, 2, \cdots, f\} \subset \mathbb{N}$ is a given piecewise constant function with finitely many switching times: $\{t_q \,|\, q \in \mathscr{Q}, t_1 < t_2 < \cdots < t_f\}$ in the bounded interval $[t_1, t_{f+1})$ of interest and let the initial condition be zero, i.e., $x(t_1^-) = 0$. To simplify notation, we assume that $\sigma(t) = q$ on (t_q, t_{q+1}). In some slight abuse of notation we will simply speak in the following of the solution $x(\cdot)$ instead of the different solution pieces $x_q(\cdot)$.

The authors are with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, Nijenborgh 9, 9747 AG, Groningen, The Netherlands.

Email: s.hossain@rug.nl s.trenn@rug.nl

For each $q \in \mathscr{Q}$, the system matrices A_q, B_q, C_q, J_q, are of appropriate size, describing the dynamics correspond to the linear system active in mode $q \in \mathscr{Q}$. Here, $u : \mathbb{R} \to \mathbb{R}^u$ is the input, x is the state trajectory, and y is the measured output. Here, the q-th mode is actived in the interval (t_q, t_{q+1}), for $q \in \{1, 2, \cdots, f\}$, so the duration of q-th mode is $\tau_q = t_{q+1} - t_q$. Furthermore, J_q is the jump map from one mode to another mode. Indeed, it can happen that the jump map is identity, i.e., $J_q = I$, $q \in \mathscr{Q}$, then the system (1) is known as a classical switched system, we call them switched ODE without jumps.

Remark 1: The jump map allow having different dimensions for the subsystems active in different modes.

Recently, in paper [13], we have presented a time-varying model reduction approach for linear switched system by considering it as a time-varying piecewise constant system. But, this was computationally infeasible approach for higher-order systems and we want to investigate whether efficient approximations methods can be derived. In this paper, we propose a minimal realization theory which is still piecewise constant and more efficient than the previous approach. Some preliminary results are derived for the single switch case. To the authors' knowledge, this is the first study of minimal realization for switched systems viewed as a time-varying system.

Notation 1: We use following notation for LSS (1):

$$\Sigma_\sigma = (\{(A_q, B_q, C_q, J_q) | q \in \mathscr{Q}\}, N, \mathscr{Q}),$$

where $N = (n_1, n_2, \cdots, n_f)$, n_q are the dimension of each mode. We denote the overall state-space dimension of the system by

$$\dim \Sigma_\sigma = \sum_{q \in \mathscr{Q}} n_q.$$

The paper is organized as follows. In Section II, the problem formulation and preliminaries are given with the idea of decomposition structure. Section III discusses the minimal realization of switched system for the single switch case. Finally, some numerical results are shown in Section IV.

II. Problem Setting and Preliminaries

In this section, we introduce some notions and properties related to minimality of linear switched system of the form (1).

We start with a small illustrative example.

Example 2: Consider a switched system with modes:

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^\top \right),$$

$$(A_2, B_2, C_2, J_2) = \left(\begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^\top, \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \right),$$

and the single switching signal:

$$\sigma(t) = \begin{cases} 1, \text{ on } [t_1, t_2), \\ 2, \text{ on } [t_2, t_f]. \end{cases} \tag{2}$$

It is clear that this switched system is reachable (i.e. $x(t_f^-)$ can be steared from zero to any value in $\mathbb{R}^3$ by the input) and observable (i.e. for a vanishing input any nonzero initial value $x(t_1)$ leads to a non-zero output for a vanishing input). Now if we remove the second state then it can be shown that it still preserve same input-output behavior (for initial value $x(t_1) = 0$). Therefore, we can conclude that for switched systems (in contrast to classical linear systems) reachability and observability are not anymore a sufficient condition for minimality.

This motivates us to study for suitable notion of minimal realization. Let Φ_σ be the set of all finite dimensional realization with same input-output relation of (1). Recall the following definitions [3].

Definition 3: A linear switched system $\widehat{\Sigma}_\sigma \in \Phi_\sigma$ with $\widehat{N} = (\widehat{n}_1, \widehat{n}_2, \cdots, \widehat{n}_f)$ is said to be a minimal realization of switched system Σ_σ if for any realization $\widetilde{\Sigma}_\sigma \in \Phi_\sigma$ with $\widetilde{N} = (\widetilde{n}_1, \widetilde{n}_2, \cdots, \widetilde{n}_f)$, it holds

$$\dim \widehat{\Sigma}_\sigma \leq \dim \widetilde{\Sigma}_\sigma.$$

In the following, we consider an arbitrary example to analyse more general case.

Example 4: Consider a switched system with subsystems:

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} -2 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 & -1 & 2 \\ -1 & 1 & -3 & -2 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ 0 \\ 0 \\ 7 \\ 6 \\ 0 \end{bmatrix}^\top \right)$$

$$(A_2, B_2, C_2, J_2) = \left(\begin{bmatrix} -1 & 0 & 0 & 0 & 2 & 0 \\ 1 & -2 & 0 & 0 & -1 & 0 \\ -1 & 0 & -3 & -1 & 0 & -2 \\ 2 & -1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 10 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}^\top, I \right)$$

The single switching signal $\sigma(t)$ is given by (2).

In first subsystem, we see that the first three states are reachable in which one of them are also observable. Again, in the second subsystem the first four states are reachable and two of them are also observable. So the open question is that how we can compute the minimal realization of the switched system.

As discussed above, we have seen that the second subsystem starts with nonzero initial value. So it makes sense that we have to preserve the energy that play role both for first subsystem as well as second subsystem in whole time axis. A question arises: Under which conditions, one can find minimal switched systems of a certain class generating the specified input-output behavior?

A. Two different notions of minimal realization

In the following, we present two different notions of a minimal realization of the system class (1).

Notion I: Under the assumption that all state-space dimensions are equal, find a common coordinate transformation matrix T to whole system class so that the original state variables $x(t)$ is represented by $\widetilde{x}(t)$ (in smaller size). In this case, the original system (1) can be represented by

$$\widetilde{\Sigma}_\sigma : \begin{cases} \dot{\widetilde{x}}(t) = \widetilde{A}_q \widetilde{x}(t) + \widetilde{B}_q u(t), \quad \widetilde{x}(t_1) = 0, \\ \widetilde{x}(t_q^+) = \widetilde{J}_q \widetilde{x}(t_q^-), \\ \widetilde{y}(t) = \widetilde{C}_q \widetilde{x}(t). \end{cases} \tag{3}$$

The system matrices: $\widetilde{A}_q$, $\widetilde{B}_q$, $\widetilde{C}_q$, $q \in \mathscr{Q}$ are obtained by partitioning as follows:

$$(T^{-1}A_qT, T^{-1}B_q, C_qT) \overset{T}{:=} \left(\begin{bmatrix} \widetilde{A}_q & * \\ * & * \end{bmatrix}, \begin{bmatrix} \widetilde{B}_q \\ * \end{bmatrix}, [\widetilde{C}_q \; *] \right), \tag{4}$$

where $*$ represents submatrices that are immaterial for further analysis and each subsystem have same dimension. Note that one obtains a switched ODE without jumps if the original switched systems did not exhibit jumps. Some results in this direction are reported in [3].

Notion II: Find a family of transformation matrices:

$$T_\sigma = \{T_q | q \in \mathscr{Q}, x_q = T_q \widetilde{x}_q\}.$$

The minimal realization of (1) can be represented by:

$$\widetilde{\Sigma}_\sigma : \begin{cases} \dot{\widetilde{x}}_q(t) = \widetilde{A}_q \widetilde{x}_q(t) + \widetilde{B}_q u(t), \text{ on } [t_q, t_{q+1}), \\ \widetilde{x}_q(t_q^+) = \widetilde{J}_q \widetilde{x}_{q-1}(t_q^-), \; \widetilde{x}_q(t_1) = 0, \\ \widetilde{y}(t) = \widetilde{C}_q \widetilde{x}_q(t). \end{cases} \tag{5}$$

Then the reduced representation has the same system class as the original system. The system matrices: $\widetilde{A}_q$, $\widetilde{B}_q$, $\widetilde{C}_q$, $q \in \mathscr{Q}$ are obtained by partitioning the system matrices as follows:

$$\left(T_q^{-1}A_qT_q, T_q^{-1}B_q, C_qT_q\right) \overset{T_q}{:=} \left(\begin{bmatrix} \widetilde{A}_q & * \\ * & * \end{bmatrix}, \begin{bmatrix} \widetilde{B}_q \\ * \end{bmatrix}, [\widetilde{C}_q \; *] \right). \tag{6}$$

Hence in this case, the subsystems might have different dimensions. To the best of the authors knowledge, there are no results yet published for this case and the remainder of the note will focus on this notion. In the following, at first we find the decomposition structure for non-switch case where the initial value is nonzero. And then, we implement the techniques for switched system.

B. Decomposition Structures: non zero initial value

Assume a realization that is not minimal, then we can always find a similarity transformation that rearranges the four possible grouping of modes: reachable and observable, reachable and unobservable, unreachable and observable, unreachable and unobservable.

Consider a linear system of the form:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{7}$$

which is neither reachable nor observable. We can find a transformation matrix

$$T = [T_{co} \; T_{c\bar{o}} \; T_{\bar{c}o} \; T_{\bar{c}\bar{o}}]$$

such that

$$(T^{-1}AT, T^{-1}B, CT) \overset{T}{=} \left(\begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}, [C_{co} \; 0 \; C_{\bar{c}o} \; 0] \right).$$

The columns of submatrices T_{co}, $T_{c\bar{o}}$, $T_{\bar{c}o}$ and $T_{\bar{c}\bar{o}}$ form basises for the subspace that are both reachable and observable, reachable and unobservable, unreachable and observable and unreachable and unobservable, respectively. If the system Σ is both reachable and observable, the matrix T_{co} has full dimension and all other sub-matrices disappear.

The result first established by Kalman [14], known as *Kalman decomposition* (KD) and is used for a direct test of minimality. From the decomposition, the subsystem (A_{co}, B_{co}, C_{co}) is both reachable and observable and hence minimal.

Lemma 5: A state space realization with zero initial value is minimal iff it is reachable and observable.

Unfortunately, the results of minimality is no longer true for nonzero initial values, i.e., if $x(0) = x_0 \neq 0$. The reason for the properties is that KD takes into account the structure of the matrices A, B and C but not that of the initial value. If a system in nonzero initial value is not minimal, then it is possible to find its minimal representation.

We can solve the underlying problem by extending the idea of KD. Assume that the nonzero initial value x_0 is contained in a subspace $\mathscr{X}_0 \subseteq \mathbb{R}^n$ which is spanned by the columns of $X_0 \in \mathbb{R}^{n,n_0}$ with $x_0 \in \operatorname{im} X_0$. The simple idea is to compute A-invariant extended reachable subspace by including the non zero initial value. In system (7), we can compute extended system in which $B \in \mathbb{R}^{n,m}$ is replaced by $[B\, X_0] \in \mathbb{R}^{n,m+n_0}$. Then the extended system can be written as

$$\Sigma_e : \begin{cases} \dot{x}_e(t) = Ax_e(t) + \begin{bmatrix} B & X_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix}, \; x_e(0) = 0, \\ y_e(t) = Cx_e(t). \end{cases} \tag{8}$$

Apply KD for both reachable and observable subspace $\widehat{T}_e$ in extended system (8).

Then the system (8) has following form

$$\widehat{\Sigma}_e : \begin{cases} \dot{\widehat{x}}_e(t) = \widehat{A}_e \widehat{x}_e(t) + \begin{bmatrix} \widehat{B}_e & \widehat{X}_0 \end{bmatrix} \begin{bmatrix} u(t) \\ u_0(t) \end{bmatrix}, \; \widehat{x}_e(0) = 0, \\ \widehat{y}_e(t) = \widehat{C}_e \widehat{x}_e(t), \end{cases} \tag{9}$$

where the subsystem $(\widehat{A}_e, \begin{bmatrix} \widehat{B}_e & \widehat{X}_0 \end{bmatrix}, \widehat{C}_e)$ is both reachable and observable.

Thus the system (9) can be represented by

$$\widehat{\Sigma} : \begin{cases} \dot{\widehat{x}}(t) = \widehat{A}_e \widehat{x}(t) + \widehat{B}_e u(t), \; \widehat{x}(0) = \widehat{x}_0 \in \operatorname{im} \widehat{X}_0, \\ \widehat{y}(t) = \widehat{C}_e \widehat{x}(t), \end{cases} \tag{10}$$

where $\widehat{x}_0$ restricts the lower dimensional subspace $\widehat{X}_0$.

To find an equivalence system, we need to relate systems (7) and (8) and the reduced version (9) and (10). We have that only in the limit when u_0 is a Dirac delta, the system (8) can reproduce same input-output behavior as the original system (7). Similarly, the reduced system (10) reproduces same input-output behavior as (7).

In the following, we introduce two types of equivalence relation.

Definition 6 (ε-equivalence): Assume the system $\Sigma : (A, B, C)$ with $x(0) = x_0 \in \operatorname{im} X_0 = \mathscr{X}_0$. Then the extended system $\Sigma_e : (A, [B\, X_0], C)$ is said to be ε-equivalent if for all $x_0 \in \operatorname{im} X_0$, there exists a family of inputs u_0^ε parameterized by $\varepsilon > 0$ such that for all input $u(\cdot)$, satisfies

$$y_e^\varepsilon(\cdot, \begin{bmatrix} u \\ u_0^\varepsilon \end{bmatrix}, 0) \underset{\varepsilon \to 0}{\longrightarrow} y(\cdot, u, x_0).$$

We denote the equivalence relation by

$$\Sigma_e \equiv_{i,o}^{\mathscr{X}_0} \Sigma.$$

It is not difficult to see that systems (7) and (8) are ε-equivalent corresponding to the Dirac delta input u_0^ε.

Again, we introduce another equivalence relation.

Definition 7 (input-output equivalence): Consider the system $\Sigma : (A, B, C)$ with the initial value $x_0 \in \mathscr{X}_0$. Then a system $\widehat{\Sigma} : (\widehat{A}, \widehat{B}, \widehat{C})$ is said to be input-output equivalent if for all $x_0 \in \mathscr{X}_0$, there exists $\widehat{x}_0 \in \widehat{\mathscr{X}_0}$ such that for all input $u(\cdot)$, satisfies

$$\widehat{y}(\cdot, u, x_0) = y(\cdot, u, \widehat{x}_0).$$

This equivalence relation can be denoted by

$$\Sigma \equiv_{i,o} \widehat{\Sigma}.$$

We see that the systems (7) and (10), and (8) and (9) are input-output equivalent.

For state space dimension, we conclude that

$$\dim \widehat{\Sigma} = \dim \widehat{\Sigma}_e \leq \dim \Sigma_e = \dim \Sigma.$$

Similarly, we can find following relation for output spaces,

$$\dim\{y_e(t_f, \begin{bmatrix} 0 \\ u_0^\varepsilon(\cdot) \end{bmatrix}, 0)\} = \dim\{y(\cdot, 0, x_0) | x_0 \in \operatorname{im} X_0\},$$

where $u_0^\varepsilon(\cdot)$ restricts the reachable subspace of system Σ_e.

Combining the connections between the equations (7), (8), (9) and (10), we have that

$$\widehat{\Sigma} \equiv_{i,o} \Sigma \equiv_{i,o}^{\mathscr{X}_0} \Sigma_e \equiv_{i,o} \widehat{\Sigma}_e \equiv_{i,o}^{\widehat{\mathscr{X}_0}} \widehat{\Sigma}.$$

In the following, some preliminary results are presented without details.

Lemma 8: Assume the system Σ with $x(0) = x_0 \in \operatorname{im} X_0$. Then any system $\widehat{\Sigma}$ constructed by (10), is input-output equivalent to Σ.

Lemma 9: Consider a system $\widetilde{\Sigma}_e$ constructed by (9) of a system Σ_e. Then the following implication holds:

$$\widetilde{\Sigma}_e \equiv_{i,o}^{\widehat{\mathscr{X}_0}} \widetilde{\Sigma} \equiv_{i,o} \Sigma \equiv_{i,o}^{\mathscr{X}_0} \Sigma_e \implies \widetilde{\Sigma}_e \equiv_{i,o} \Sigma_e.$$

Corollary 10: Assume that $\widehat{\Sigma} \equiv_{i,o} \Sigma$. Then there is a system $\widetilde{\Sigma}$ in which $\widetilde{\Sigma} \equiv_{i,o} \Sigma$ such that $\widetilde{\Sigma} \equiv_{i,o} \widehat{\Sigma}$.

Lemma 11: Consider a system Σ with $x(0) = x_0 \in \operatorname{im} X_0$, given by (7). Then any system $\widehat{\Sigma}_e$ in which $\widehat{\Sigma}_e \equiv_{i,o} \Sigma_e \equiv_{i,o}^{\mathscr{X}_0} \Sigma$, satisfies

$$\dim \widehat{\Sigma}_e \geq \dim\{y(\cdot, 0, x_0) | x_0 \in \operatorname{im} X_0\}.$$

Lemma 12: Assume a system $\widehat{\Sigma}$ with $\widehat{x}(0) = \widehat{x}_0 \in \operatorname{im} \widehat{X}_0$, given by (10) in which

$$\widehat{\Sigma} \equiv_{i,o} \Sigma.$$

Then any system $\widetilde{\Sigma}$ in which $\widetilde{\Sigma} \equiv_{i,o} \Sigma$, satisfies

$$\dim \widetilde{\Sigma} \geq \dim \widehat{\Sigma}.$$

Proof: We have a system $\widehat{\Sigma}$ with $\widehat{x}(0) = \widehat{x}_0 \in \operatorname{im} \widehat{X}_0$ in which $\widehat{\Sigma} \equiv_{i,o} \Sigma$.
Again, assume a system $\widetilde{\Sigma}$ such that

$$\widetilde{\Sigma} \equiv_{i,o} \Sigma.$$

Now consider

$$\dim \widetilde{\Sigma} < \dim \widehat{\Sigma} = \dim \widehat{\Sigma}_e.$$

We can construct an extended system $\widetilde{\Sigma}_e$ for $\widetilde{\Sigma}$ such that

$$\dim \widetilde{\Sigma}_e = \dim \widetilde{\Sigma}.$$

Then from lemma 9, we find Σ_e such that $\widetilde{\Sigma}_e \equiv_{i,o} \Sigma_e$, and then

$$\widetilde{y}_e(\cdot, \left[\begin{smallmatrix} u \\ u_0^{\varepsilon} \end{smallmatrix}\right], 0) = y_e(\cdot, \left[\begin{smallmatrix} u \\ u_0^{\varepsilon} \end{smallmatrix}\right], 0),\ \forall u, u_0^{\varepsilon}.$$

This implies that

$$\dim \widetilde{\Sigma}_e = \dim \Sigma_e.$$

Hence we have that

$$\dim \widetilde{\Sigma} = \dim \Sigma_e > \dim \widehat{\Sigma},$$

which makes a contradiction.
Therefore,

$$\dim \widetilde{\Sigma} \not< \dim \widehat{\Sigma}.$$

■

Summarizing above results, we can conclude a theorem.

Theorem 13: Consider a system Σ given by (7). Then a system $\widehat{\Sigma}$ given by (10) is a minimal realization of Σ if, and only if its extended system $\widehat{\Sigma}_e$ given by (9) is reachable and observable.

Proof: The proof is a simple consequence of above results. ■

We will now apply this results to linear switched system for single switch case. It is clear that the second subsystem might be started with nonzero initial value, coming from the reachable subspace of the first subsystem.

III. Minimal realization of switched system

In this section, we find the minimal realization of switched systems for single switch case.
Consider a single switched system in two subsystems:

$$\begin{aligned} &(A_1, B_1, C_1), \quad \text{on } t \in [t_1, t_2), \\ &(A_2, B_2, C_2, J_2), \quad \text{on } t \in [t_2, t_f]. \end{aligned} \tag{11}$$

Assume that the first subsystem starts with zero initial value and second subsystem starts with nonzero initial value.
First subsystem: Assume the first subsystem:

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t), \quad x_1(t_1) = 0, \\ y(t) &= C_1 x_1(t), \end{aligned} \tag{12}$$

where $t \in [t_1, t_2)$.
We find the projection matrix $\mathbb{T}_1 \in \mathbb{R}^{n \times r_1}$ with $x_1(t) = \mathbb{T}_1 z_1(t)$. Then the minimal realization is given by

$$\begin{aligned} \dot{z}_1(t) &= \widehat{A}_1 z_1(t) + \widehat{B}_1 u(t), \quad z_1(t_1) = 0, \\ y(t) &= \widehat{C}_1 z_1(t), \end{aligned} \tag{13}$$

where $\widehat{A}_1 = \mathbb{T}_1{}^{\dagger} A_1 \mathbb{T}_1$, $\widehat{B}_1 = \mathbb{T}_1{}^{\dagger} B_1$, $\widehat{C}_1 = C_1 \mathbb{T}_1$, $\mathbb{T}_1^{\dagger} \mathbb{T}_1 = I$.
It is true that the second subsystem activates at t_2 with final value; $x_1(t_2^-)$.
Second subsystem: Consider the second subsystem:

$$\begin{aligned} \dot{x}_2(t) &= A_2 x_2(t) + B_2 u(t), \\ y(t) &= C_2 x_2(t), \end{aligned} \tag{14}$$

where $t \in [t_2, t_f]$ and $x_2(t_2) = J_2 x_1(t_2^-)$.
We find the projection matrix $\mathbb{T}_2 \in \mathbb{R}^{n \times r_2}$ with $x_2(t) = \mathbb{T}_2 z_2(t)$. Then the minimal realization is given by

$$\begin{aligned} \dot{z}_2(t) &= \widehat{A}_2 z_2(t) + \widehat{B}_2 u(t), \\ y(t) &= \widehat{C}_2 z_2(t). \end{aligned} \tag{15}$$

where $\widehat{A}_2 = \mathbb{T}_2^{\dagger} A_2 \mathbb{T}_2$, $\widehat{B}_2 = \mathbb{T}_2^{\dagger} B_2$, $\widehat{C}_2 = C_2 \mathbb{T}_2$, $\mathbb{T}_2^{\dagger} \mathbb{T}_2 = I$.
In particular,

$$x_1(t_2^-) = \mathbb{T}_1 z_1(t_2^-), \quad x_2(t_2^+) = \mathbb{T}_2 z_2(t_2^+).$$

Then we have

$$z_2(t_2^+) = \mathbb{T}_2^{\dagger} x_2(t_2^+) = \mathbb{T}_2^{\dagger} J_2 x_1(t_2^-) = \mathbb{T}_2^{\dagger} J_2 \mathbb{T}_1 z_1(t_2^-).$$

We have seen that both the transformation matrices depend on each other, and they are unique and well defined in fixed switching signal. Next, we will show that the system matrices of each minimal subsystem are nothing but the submatrices of their transformed system matrices.
In the following, we present an algorithm to find the transformation matrices $\mathbb{T}_1$ and $\mathbb{T}_2$.

A. Algorithmic computation

The proposed method has two phases: firstly, extend reachable subspace of second subsystem along with its non zero initial value; secondly, extend observable subspace of first subsystem by including the states which are important in second subsystem. This will ensure the reachability and observability for the minimal realization. Overall, the algorithm is summarized as follows.

1. Compute the reachable subspace $\mathscr{R}_1 = \operatorname{im} R_1$ of the first subsystem (A_1, B_1, C_1) and use this to extend the reachable subspace of second subsystem: $(A_2, B_{2,e}, C_2)$ where the input matrix is given by

$$B_{2,e} := \operatorname{im}[B_2, J_2 R_1].$$

2. Apply KD to $(A_2, B_{2,e}, C_2)$ and compute the transformation matrix: $T_2 = [\mathbb{T}_2, \mathbb{T}_{2,rest}]$, where $\mathbb{T}_2$ is for the reachable and observable part. Then the partitioned system matrices are given by

$$(T_2^{-1} A_2 T_2, T_2^{-1} B_2, C_2 T_2) \overset{T_2}{:=} \left(\begin{bmatrix} \widehat{A}_2 & * \\ * & * \end{bmatrix}, \begin{bmatrix} \widehat{B}_2 \\ * \end{bmatrix}, \begin{bmatrix} \widehat{C}_2 & * \end{bmatrix} \right) \tag{16}$$

3. Compute the intersection:

$$\mathscr{R}_1 \cap \operatorname{im} \mathbb{T}_2 =: \operatorname{im} C_{11}^{\top}$$

and construct extended first subsystem $(A_1, B_1, C_{1,e})$ where the output matrix is given by

$$C_{1,e} := \operatorname{im} \begin{bmatrix} C_1 \\ C_{11} \end{bmatrix}.$$

4. Apply KD to $(A_1,B_1,C_{1,e})$ and compute the transformation matrix: $T_1=[\mathbb{T}_1,\mathbb{T}_{1,rest}]$ where $\mathbb{T}_1$ is for the reachable and observable part. Then the partitioned of system matrices are given by:

$$(T_1^{-1}A_1T_1,T_1^{-1}B_1,C_1T_1)\overset{T_1}{:=}\left(\begin{bmatrix}\widehat{A}_1 & * \\ * & *\end{bmatrix},\begin{bmatrix}\widehat{B}_1 \\ *\end{bmatrix},\begin{bmatrix}\widehat{C}_1 & *\end{bmatrix}\right) \tag{17}$$

and the jump map is given by

$$T_2^{-1}J_2T_1:=\begin{bmatrix}\widehat{J}_2 & *\end{bmatrix}.$$

In the following, we derive some results.

Lemma 14: Assume the switched system Σ_σ with modes (A_1,B_1,C_1) and (A_2,B_2,C_2,J_2), and the reduced switched system $\widehat{\Sigma}_\sigma$ with modes $(\widehat{A}_1,\widehat{B}_1,\widehat{C}_1)$ and $(\widehat{A}_2,\widehat{B}_2,\widehat{C}_2,\widehat{J}_2)$ with projection matrices $\mathbb{T}_1$ and $\mathbb{T}_2$, following the procedure in subsection III-A. Then Σ_σ and $\widehat{\Sigma}_\sigma$ have same input-output behavior.

Proof: The output equation of the switched system Σ_σ is given by

$$y(t)=\begin{cases}\int_{t_1}^{t}C_1e^{A_1(t-\tau)}B_1u(\tau)d\tau,\ t\in[t_1,t_2),\\ C_2e^{A_2(t-t_2)}J_2x(t_2^-)+\int_{t_2}^{t}C_2e^{A_2(t-\tau)}B_2u(\tau)d\tau\end{cases} \tag{18}$$

Assume T_1 and T_2 are the transformation matrices of KD for extended first and second subsystems, respectively, discussed in subsection III-A. Then the output equation (18) can be written as

$$y(t)=\begin{cases}\int_{t_1}^{t}\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}C_1\\ C_{11}\end{bmatrix}T_1T_1^{-1}e^{A_1(t-\tau)}T_1T_1^{-1}B_1u(\tau)d\tau,\\ C_2T_2T_2^{-1}e^{A_2(t-t_2)}T_2T_2^{-1}J_2R_1\widetilde{x}(t_2^-)\\ \quad+\int_{t_2}^{t}C_2T_2T_2^{-1}e^{A_2(t-\tau)}T_2T_2^{-1}B_2u(\tau)d\tau\end{cases}$$

where $\mathscr{R}_1=imR_1$ is the reachable subspace of first subsystem and $\widetilde{x}(t)$ is the solution of first subsystem for the transformation matrix T_1.

Now we have that

$$\begin{aligned}\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}C_1\\ C_{11}\end{bmatrix}T_1&=\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}C_1^{co} & 0 & C_1^{\bar{c}o} & 0\\ C_{11}^{co} & 0 & C_{11}^{\bar{c}o} & 0\end{bmatrix}\\ &=\begin{bmatrix}C_1^{co} & 0 & C_1^{\bar{c}o} & 0\end{bmatrix},\end{aligned}$$

$$T_1^{-1}B_1=\begin{bmatrix}B_1^{co}\\ B_1^{c\bar{o}}\\ 0\\ 0\end{bmatrix},\ T_2^{-1}\begin{bmatrix}B_2 & J_2R_1\end{bmatrix}=\begin{bmatrix}B_2^{co} & B_{2,e}^{co}\\ B_2^{c\bar{o}} & B_{2,e}^{c\bar{o}}\\ 0 & 0\\ 0 & 0\end{bmatrix},$$

$$C_2T_2=\begin{bmatrix}C_2^{co} & 0 & C_2^{\bar{c}o} & 0\end{bmatrix},\quad T_1^{-1}e^{A_1}T_1=e^{T_1^{-1}A_1T_1},$$

$$\begin{aligned}&R_1\widetilde{x}(t_2^-)=x(t_2^-)=\mathbb{T}_1\widehat{x}(t_2^-),\\ &T_2^{-1}J_2R_1\widetilde{x}(t_2^-)=T_2^{-1}J_2\mathbb{T}_1\widehat{x}(t_2^-)=\widehat{J}_2\widehat{x}(t_2^-).\end{aligned}$$

Finally, we have that

$$\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}C_1\\ C_{11}\end{bmatrix}T_1T_1^{-1}e^{A_1(t-\tau)}T_1T_1^{-1}B_1=C_1^{co}e^{A_1^{co}(t-\tau)}B_1^{co},$$

$$\begin{aligned}C_2T_2T_2^{-1}e^{A_2\tau}T_2T_2^{-1}J_2R_1\widetilde{x}(t_2^-)&=C_2^{co}e^{A_2^{co}\tau}J_2^{co}\widehat{x}(t_2^-),\\ C_2T_2T_2^{-1}e^{A_2(t-\tau)}T_2T_2^{-1}B_2&=C_2^{co}e^{A_2^{co}(t-\tau)}B_2^{co}.\end{aligned}$$

Then the output equation (18) can be written as

$$y(t)=\begin{cases}\int_{t_1}^{t}C_1^{co}e^{A_1^{co}(t-\tau)}B_1^{co}u(\tau)d\tau,\ t\in[t_1,t_2)\\ C_2^{co}e^{A_2^{co}(t-t_2)}J_2^{co}\bar{x}(t_2^-)+\int_{t_2}^{t}C_2^{co}e^{A_2^{co}(t-\tau)}B_2^{co}u(\tau)d\tau\end{cases} \tag{19}$$

Again, assume $\mathbb{T}_1(\subseteq T_2)$ and $\mathbb{T}_2(\subseteq T_2)$ are the reachable and observable subspaces of extended first and second subsystems, respectively. Then the transformed system matrices are given by

$$\begin{aligned}(\mathbb{T}_1^{\dagger}A_1\mathbb{T}_1,\mathbb{T}_1^{\dagger}B_1,C_1\mathbb{T}_1)&:=(\widehat{A}_1,\widehat{B}_1,\widehat{C}_1),\\ &:=(A_1^{co},B_1^{co},C_1^{co}),\\ (\mathbb{T}_2^{\dagger}A_2\mathbb{T}_2,\mathbb{T}_2^{\dagger}B_2,C_2\mathbb{T}_2,\mathbb{T}_2^{\dagger}J_2\mathbb{T}_1)&:=(\widehat{A}_2,\widehat{B}_2,\widehat{C}_2,\widehat{J}_2),\\ &:=(A_2^{co},B_2^{co},C_2^{co},J_2^{co}).\end{aligned}$$

Let $\widehat{y}(t)$ is the output of the reduced system $\widehat{\Sigma}_\sigma$. Then using the above results and the equation (19), we can conclude that

$$y(t)=\widehat{y}(t),\ \forall t.$$

This complete the proof. ∎

Now, we present the main theorem for the minimal realization of switched system in single switching signal.

Theorem 15: Assume the switched system Σ_σ with the single switching signal with subsystems (A_1,B_1,C_1) and (A_2,B_2,C_2,J_2). Then the pair of transformation matrices $(\mathbb{T}_1\subseteq T_2,\mathbb{T}_2\subseteq T_2)$, described in subsection III-A gives a realization $\widehat{\Sigma}_\sigma$, and $\widehat{\Sigma}_\sigma$ is a minimal realization of Σ_σ.

Proof: Lemma 14 shows that the constructed switched system is input-output equivalent to the original switched system.

Furthermore, Theorem 13 shows that the state-dimension of the second mode is minimal for the possible non-zero initial values which are possible to reach via the first mode.

Finally, its not difficult to see that reduced first mode is minimal under all systems which are input-output equivalent systems to original first mode under the constraint that the observable initial states from the second mode are not removed. ∎

IV. NUMERICAL RESULTS

In this section, we apply the proposed method to find the minimal realization of switched system. We consider some examples to illustrate the proposed method.

Example 16: Recall the switched system in example 2 with the switching signal $\sigma(t)$ given in (2).

We apply our proposed method. The computed KD transformation matrices $(\mathbb{T}_1\subseteq)T_1$ and $(\mathbb{T}_2\subseteq)T_2$ in the partitions (17) and (16) are given by:

$$T_1=T_2=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix},\quad \mathbb{T}_1=\mathbb{T}_2=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}.$$

The subsystems for minimal switched system are given by

$$
\begin{aligned}
(A_1,B_1,C_1) &:= \left(\mathbb{T}_1^{\dagger}A_1\mathbb{T}_1, \mathbb{T}_1^{\dagger}B_1, C_1\mathbb{T}_1\right)\\
&:= \left(\begin{bmatrix}-0.1 & 0\\ 0 & -0.3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1 & 0\end{bmatrix}\right),
\end{aligned}
$$

$$
\begin{aligned}
(A_2,B_2,C_2,J_2) &:= \left(\mathbb{T}_2^{\dagger}A_2\mathbb{T}_2, \mathbb{T}_2^{\dagger}B_2, C_2\mathbb{T}_2, \mathbb{T}_2^{\dagger}J_2\mathbb{T}_1\right)\\
&:= \left(\begin{bmatrix}-0.2 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\right).
\end{aligned}
$$

It is easy to show that the minimal system gives exactly same input-output behavior.

Again, recall the original switched system in example 2 with the following switching signal:

$$
\sigma_2(t) = \begin{cases} 2, \text{ on } [t_1,t_2), \\ 1, \text{ on } [t_2,t_3]. \end{cases} \tag{20}
$$

The computed minimal subsystems are given by

$$
\begin{aligned}
(A_2,B_2,C_2) &\overset{\mathbb{T}_1}{:=} (-0.1,1,0), \quad J_2 := \begin{bmatrix}0\\1\end{bmatrix},\\
(A_1,B_1,C_1) &\overset{\mathbb{T}_2}{:=} \left(\begin{bmatrix}-0.1 & 0\\ 0 & -0.2\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}^{\top}\right).
\end{aligned}
$$

where the transformation matrices are

$$
T_1 = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \; T_2 = I, \mathbb{T}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \; \mathbb{T}_2 = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}.
$$

It can be shown that computed minimal realization in the switching cases also give exactly same input-output behavior to their original switched system.

Now we consider another example for more detail.

Example 17: Recall the switched system in example 4. We see that in this example, the subspaces for KD are aligned to the coordinate axis so it is easy to find the transformation matrices $\mathbb{T}_1$ and $\mathbb{T}_2$ for the minimal system by applying our algorithm.

For numerical purposes, we change the system by randomly generated following invertible matrix:

$$
M = \begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & -1 & 0\\ -1 & -1 & -1 & -1 & 0 & 1\\ -1 & 1 & 0 & 0 & 0 & 1\\ 1 & 1 & 0 & -1 & -1 & 1\\ -1 & 1 & -1 & -1 & 1 & 0\end{bmatrix},
$$

Take similarity transformation by M on the original switched system, given by

$$
\begin{aligned}
(A_1,B_1,C_1) &\overset{M}{\cong} (M^{-1}A_1M, M^{-1}B_1, C_1M),\\
(A_2,B_2,C_2,J_2) &\overset{M}{\cong} (M^{-1}A_2M, M^{-1}B_2, C_2M, I).
\end{aligned}
$$

We apply the proposed method and find the transformation matrices $\widetilde{\mathbb{T}_1}$ and $\widetilde{\mathbb{T}_2}$ for the minimal subsystems of the transformed switched system, given as follows:

$$
(A_1,B_1,C_1) \overset{M\widetilde{\mathbb{T}_1}}{:=} \left(\begin{bmatrix}-4 & -12 & 10\\ 1 & 10 & -11\\ 2 & 12 & -12\end{bmatrix} \begin{bmatrix}34\\ -31\\ -94/3\end{bmatrix}, \begin{bmatrix}60\\0\\60\end{bmatrix}^{\top}\right),
$$

$$
\begin{aligned}
(A_2,B_2,C_2) &\overset{M\widetilde{\mathbb{T}_1}}{:=} \left(\begin{bmatrix}-5/4 & 3/2\\ 1/8 & -7/4\end{bmatrix} \begin{bmatrix}-2\\ -1/2\end{bmatrix}, \begin{bmatrix}-25\\ 14\end{bmatrix}^{\top}\right),\\
J_2 &:= \begin{bmatrix}-5/8 & 1 & -3/2\\ -19/16 & -1/2 & -3/4\end{bmatrix}.
\end{aligned}
$$

Then we have the relations: $\mathbb{T}_1 := M\widetilde{\mathbb{T}_1}$ and $\mathbb{T}_2 := M\widetilde{\mathbb{T}_2}$. The computed minimal realization gives exactly same input-output behavior as the original switched system.

Remark 18: Clearly, in this example if we alter the switching signal, then we can get minimal switched system which is different dimension than earlier.

V. Conclusion

In this paper, firstly we have presented a notion of minimal realization of a system by considering an arbitrary non zero initial value. We have shown that the non zero initial value can consider as an auxiliary input which gives an extended reachable subspace. Then the original and minimal systems have same input-output behavior. The same techniques have been used to find the minimal realization of switched system. We have shown that each subsystem has their own minimal version, hence the overall switched system is minimal. However, this is our on going research, our long term goal is to generalize the results for many switching signal.

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