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Normal forms and internal regularization of nonlinear differential-algebraic control systems †

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Summary

In this paper, we propose two normal forms for nonlinear differential-algebraic control systems DACSs under external feedback equivalence, using a notion called maximal controlled invariant submanifold. The two normal forms simplify the system structures and facilitate understanding the various roles of variables for nonlinear DACSs. Moreover, we study when a given nonlinear DACS is internally regularizable, i.e., when there exists a state feedback transforming the DACS into a differential-algebraic equation DAE with internal regularity, the later notion is closely related to the existence and uniqueness of solutions of DAEs. We also revise a commonly used method in DAE solution theory, called the geometric reduction method. We apply this method to DACSs and formulate it as an algorithm, which is used to construct maximal controlled invariant submanifolds and to find internal regularization feedbacks. Two examples of mechanical systems are used to illustrate the proposed normal forms and to show how to internally regularize DACSs.

KEYWORDS:
differential-algebraic equations; nonlinear control systems; normal forms; external feedback equivalence; internal regularization; mechanical systems

1 | INTRODUCTION

Consider a nonlinear differential-algebraic control system DACS of the form

$$\Xi^u : E(x)\dot{x} = F(x) + G(x)u, \tag{1}$$

where $x \in X$ is the generalized state, with X an n-dimensional differentiable manifold (or an open subset of $\mathbb{R}^n$), where $u \in \mathbb{R}^m$ is the control vector, and where $E : TX \to \mathbb{R}^l$, $F : X \to \mathbb{R}^l$ and $G : X \to \mathbb{R}^{l\times m}$ are smooth maps and the word "smooth" will always mean $\mathcal{C}^\infty$-smooth throughout the paper. For each $x \in X$, we have $E(x) : T_xX \to \mathbb{R}^l$, which is a linear map $\dot{x} \mapsto E(x)\dot{x}$. In particular, if X is an open subset of $\mathbb{R}^n$, then for each $x \in X$, we have $E(x) : \mathbb{R}^n \to \mathbb{R}^l$, i.e., $E(x) \in \mathbb{R}^{l\times n}$. A DACS of the form (1) will be denoted by $\Xi^u_{l,n,m} = (E, F, G)$ or, simply, Ξ^u. A particular case of (1) is a linear DACS of the form

$$\Delta^u : E\dot{x} = Hx + Lu, \tag{2}$$

where $E \in \mathbb{R}^{l\times n}$, $H \in \mathbb{R}^{l\times n}$, $L \in \mathbb{R}^{l\times m}$, denoted by $\Delta^u_{l,n,m} = (E, H, L)$. From a practical point of view, numerous physical systems can be modeled via DACSs of the form (1), e.g., constrained mechanics [1, 2], electrical circuits [3, 4], and chemical processes [5].

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For a DACS of the form (1), the map E is not necessarily square (i.e., in general, $l \neq n$) nor invertible (even if $l = n$) and, as a consequence, some variables of the generalized state x play different roles for the system. More specifically, the non-invertibility of E may imply the existence of algebraic constraints and some variables of x (even some u-variables) are constrained by those algebraic constraints. On the other hand, because of the non-squareness of E, some other variables of x may enter into system statically (since there are no differential equations implying their evolutions) and we may call them algebraic variables. Note that although the algebraic variables of x may perform "like" inputs of the system, throughout we will distinguish them from the original control inputs u. The variables u are predefined control inputs, such as external forces, which have "accesses" to acting on the system. However, the algebraic variables of x are predefined states which can not be changed actively and arbitrarily. Such algebraic variables may come from unknown constraint forces or some redundancies of mathematical modeling.

A typical example to illustrate that the control variables u and the algebraic variables of x are different is the following DACS which may represent the dynamics of a mechanical system under both nonholonomic and holonomic constraints (see e.g. [1] for the definitions of nonholonomic and holonomic constraints).

$$M(p)\ddot{p} + V(\dot{p}, p) = \tau + H^T(p)\lambda_n + N^T(p)\lambda_h \tag{3a}$$

$$H(p)\dot{p} = 0 \tag{3b}$$

$$C(p) = 0, \tag{3c}$$

where p is the vector of position variables, $M(p)$ is a matrix-valued function which is associated with masses (or inertia) and $V(\dot{p}, p)$ is a vector function which characterizes the Coriolis, the centrifugal and the gravity forces, and τ is a vector of external torques, $C(p)$ is a vector of scalar functions $c_i(p)$, $i = 1, \dots, k$ and $N(p) = \frac{\partial C(p)}{\partial p}$, and $H(p)$ is matrix-valued functions of appropriate size. Clearly, equation (3b) defines nonholonomic constraints, which depend on both velocities and positions, equation (3c) defines holonomic constraints, which depend on positions only. The variables λ_n and λ_h are the Lagrange multipliers with respect to the nonholonomic and holonomic constraints, respectively. We can regard system (3) as a DACS of the form (1), with the generalized state $x = (p, \dot{p}, \lambda_n, \lambda_h)$ and the control input $u = \tau$. Observe that the variables λ_h and λ_n are algebraic variables since there are no equations for $\dot{\lambda}_n$ and $\dot{\lambda}_h$ but they are not control inputs contrary to the external force τ. The later can be realized by some actuators (e.g., electric and hydraulic motors) while λ_h and λ_n are variables related to unknown constrained forces.

One purpose of this paper is to find normal forms under the external feedback equivalence (see Definition 3.1 below). We will construct our normal form using a notion called maximal controlled invariant submanifold, which is, roughly speaking, the locus where the solutions of the DACSs exist and is defined by the constraints which the system should respect (for the precise definition, see Definition 2.2 below). For linear DACSs of the form (2), a canonical form, which consists of six independent subsystems, was proposed in [6]. We can easily conclude the roles of the variables (e.g., which variables are algebraic or constrained) from the canonical structure of each subsystem. For nonlinear DACSs, although it is hard to find a fully decoupled normal form, we intend to simplify the system structures utmost such that the above mentioned various roles of variables can be explicitly and easily seen from our proposed form. The authors of [7] offered a nonlinear generalization of the Kronecker canonical form using an inversion algorithm for differential-algebraic equations DAEs of the general form $F(\dot{x}, x, t) = 0$. A zero dynamics form for DACSs with outputs was proposed in [8] using the notion of maximal output zeroing submanifold introduced in [9]. Note that our system Ξ^u is different in two ways from the DACSs studied in [8] and [9]. First, in [9],[8], the distribution $\ker E(x)$ is assumed to be involutive while we consider any $E(x)$. Second, systems in [9],[8] are equipped with outputs. Calculating the zero dynamics of a DACS Ξ^u with zero output $y = h(x) = 0$ can be seen as studying an extended DACS Ξ^u_{ext} : $E(x)\dot{x} = F(x) + G(x)u$, $0 = h(x)$, because the maximal output zeroing submanifold of Ξ^u (with the output $y = h(x)$) coincides with the maximal controlled invariant submanifold of Ξ^u_{ext}. Some differences of our proposed normal forms and the zero dynamics form in [8] are explained in Remark 3(vi) below.

We also investigate the internal regularizability of DACSs, i.e., given a DACS Ξ^u, when there exists a feedback $u = \alpha(x)$ such that the resulting DAE $E(x)\dot{x} = F(x) + G(x)\alpha(x)$ is *internally regular*. The later notion characterizes the existence and uniqueness of solutions of DAEs, its formal definition will be given in Definition 4.1 below. Regularization problems of nonlinear DAEs and DACSs can be consulted in [10–14], etc, in which both numerical and geometrical methods have appeared. The second aim of this paper is to give a geometric characterization of the internal regularizability of nonlinear DACSs. For linear DACSs, some equivalent characterization of the internal regularizability are given in Theorem 3.5 of [15] using a geometric notion named the augmented Wong sequences (see Remark 1(iv) below). Note that the internal regularizability is called autonomizability in [15], the reason for which we insist to use the word "internal", is to stress the difference between two cases. One case is to consider a DAE "internally" on its maximal invariant submanifold (i.e. on the set where the solutions exist). Another is to consider a DAE "externally" on a whole neighborhood, even although there exist no solutions for any initial point outside the maximal invariant submanifold, it is still meaningful to study how to steer the initial point towards the constraints via e.g., jumps and impulses. The reader may consult [16–19] for the details of the differences between the internal and external analysis of DAEs.

The paper is organized as follows. In Section 2, we recall the notion of maximal controlled invariant submanifold and discuss its relations with the solutions of DACSs. In Section 3, we define the external feedback equivalence of two DACSs and propose two normal forms. In Section 4, we discuss the internal regularization problem. In Section 5, we illustrate our results of Section 3 and Section 4 by two examples of mechanical systems. In Section 6, we give the conclusions of the paper. The Appendix contains an algorithm using which we can construct the maximal controlled invariant submanifold and the feedback which we need to internally regularize a DACS. We use the following notations. We use $\mathbb{R}^{n\times m}$ to denote the set of real valued matrices with n rows and m columns, $GL(n,\mathbb{R})$ to denote the group of nonsingular matrices of $\mathbb{R}^{n\times n}$ and I_n to denote the $n\times n$-identity matrix. We denote by T_xM the tangent space at $x\in M$ of a submanifold M of $\mathbb{R}^n$ and by C^k the class of k-times differentiable functions. For a smooth map $f : X\to\mathbb{R}$, we denote its differentials by $\mathrm{d}f=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\mathrm{d}x_i=[\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X\to\mathbb{R}^m$, where $f=[f_1,\dots,f_m]^T$, we denote its differential by $\mathrm{D}f=\begin{bmatrix}\mathrm{d}f_1\\ \cdots\\ \mathrm{d}f_m\end{bmatrix}$. For two column vectors $v_1\in\mathbb{R}^m$ and $v_2\in\mathbb{R}^n$, we write $(v_1,v_2)=[v_1^T,v_2^T]^T\in\mathbb{R}^{m+n}$. We assume the reader is familiar with some basic notions from differential geometry as smooth manifolds, embedded submanifolds, tangent bundles, distributions, the reader can also consult e.g., the book by Lee [20] for definitions of those notions.

2 | PRELIMINARIES ON THE SOLUTIONS OF DIFFERENTIAL-ALGEBRAIC CONTROL SYSTEMS

We define a solution of a DACS as follows.

Definition 2.1. (Solution) For a DACS $\Xi^u_{l,n,m}=(E,F,G)$, a curve $(x,u) : I\to X\times\mathbb{R}^m$ defined on an open interval $I\subseteq\mathbb{R}$ with $x(\cdot)\in C^1(I)$ and $u(\cdot)\in C^0(I)$ is called a solution of Ξ^u, if for all $t\in I$, $E(x(t))\dot{x}(t)=F(x(t))+G(x(t))u(t)$.

We call a point $x_0\in X$ an *admissible point* of Ξ^u if there exists at least one solution $(x(\cdot),u(\cdot))$ satisfying $x(t_0)=x_0$ for a certain $t_0\in I$. We will denote admissible points by x_a and the set of all admissible points by S_a. Note that for any DACS Ξ^u, there may exist some free variables among the components of x. As a consequence, even for a fixed $u(\cdot)$ defined on $\mathbb{R}$, there is not a unique prolongation of a solution (x,u) defined on I to a maximal solution. For this reason, we will not use the concept of maximal solutions (although they can be defined, see e.g., [8]) except for Section 4, where we can deal with maximal solutions due to an identification of free (algebraic) variables.

Definition 2.2 (controlled invariant submanifold)**.** Consider a DACS $\Xi^u_{l,n,m}=(E,F,G)$. A smooth connected embedded submanifold M is called a controlled invariant submanifold of Ξ^u if for any point $x_0\in M$, there exists a solution $(x,u) : I\to X\times\mathbb{R}^m$ such that $x(t_0)=x_0$ for a certain $t_0\in I$ and $x(t)\in M$ for all $t\in I$.

We fix a point $x_p\in X$, a smooth embedded submanifold M containing x_p is *locally* controlled invariant (around x_p) if $\exists$ a neighborhood U of x_p in X such that $M\cap U$ is controlled invariant (and thus, by definition, connected). Consider a DACS $\Xi^u_{l,n,m}=(E,F,G)$, let $N\subseteq X$ and fix a point $x_p\in N$; we introduce the following constant rank assumption:

(CR) there exists a neighborhood U in X of x_p such that $N\cap U$ is a smooth connected embedded submanifold, and such that $\dim E(x)T_xN=const.$ and $\dim(E(x)T_xN+\operatorname{Im}G(x))=const.$ for $x\in N\cap U$.

The following characterization of local controlled invariance, under the constant rank assumption **(CR)** satisfied for M, was given as Theorem 9 in [9] for DACSs whose ker $E(x)$ is an involutive distribution. The proof proposed in [9] holds actually for any $E(x)$, whose kernal is involutive or not.

Proposition 2.3. *Consider a DACS $\Xi^u=(E,F,G)$ and let M be a smooth embedded submanifold. Assume that M satisfies the above assumption* **(CR)** *around a point $x_p\in M$. Then M is a locally controlled invariant submanifold (around x_p) of Ξ^u if and only if there exists a neighborhood U of x_p in X such that*

$$F(x)\in E(x)T_xM+\operatorname{Im}G(x),\quad \forall x\in M\cap U. \tag{4}$$

A locally controlled invariant submanifold M^*, around a point x_p, is called *maximal* if there exists a neighborhood U of x_p such that for any other locally controlled invariant submanifold M containing x_p, we have $M\cap U\subseteq M^*\cap U$. The following procedure is a geometric method to construct the locally maximal controlled invariant submanifold.

Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix a point $x_p \in X$. Set $M_0 = X$ and suppose that there exist an open neighborhood U_{k-1} of x_p and a sequence of smooth connected embedded submanifolds $M^c_{k-1} \subsetneq \cdots \subsetneq M^c_0$ of U_{k-1} for a certain $k \geq 1$, have been constructed. Define recursively

$$M_k := \left\{ x \in M^c_{k-1} \mid F(x) \in E(x)T_x M^c_{k-1} + \operatorname{Im} G(x) \right\}. \tag{5}$$

Then either $x_p \notin M_k$ or $x_p \in M_k$, and in the latter case assume that $M^c_k = M_k \cap U_k$ is a smooth connected embedded submanifold of U_k for some open neighborhood $U_k \subseteq U_{k-1}$ of x_p.

Proposition 2.4. *In the above recursive procedure, there always exists $k \leq n$ such that either k is the smallest for which $x_p \notin M_k$ (and then there exists a neighborhood of x_p in which there does not exist any controlled invariant submanifold) or k is the smallest such that $x_p \in M^c_k$ and $M^c_{k+1} = M^c_k$ in U_{k+1}. In the latter case, we denote $k^* = k$ and assume that $M^* = M^c_{k^*+1}$ satisfies the constant rank condition* **(CR)** *in a neighborhood $U^* \subseteq U_{k^*+1}$ of x_p in X and then*

(i) x_p is an admissible point and M^* is a locally maximal controlled invariant submanifold;

(ii) M^* coincides locally with the admissible set S_a, i.e., $M^* \cap U^* = S_a \cap U^*$ (by taking a smaller U^*, if necessary).

Proof. If $x_p \notin M_k$, then $M_{i-1} \subsetneq M_i$ for $1 \leq i \leq k$, thus $k \leq n$. Since the set M_k is closed, there always exists an open neighborhood U of x_p such that $M_k \cap U = \emptyset$. So there are no solutions in U and, in particular, no controlled invariant submanifolds in U. If $x_p \in M_k$, then again by $M_{i-1} \subsetneq M_i$ for $1 \leq i \leq k$, there exists the smallest $k \leq n$ such that $\dim M^c_k = \dim M^c_{k+1}$. Since $M^c_{k+1} \subseteq M^c_k$, it follows that $M^c_{k+1} = M^c_k$ in a well chosen $U_{k+1} \subseteq U_k$.

(i) Denoting $k^* = k$, we have $M^c_{k^*+1} = M^c_{k^*}$ in U_{k^*+1} and thus $F(x) \in E(x)T_x M^c_{k^*+1} + \operatorname{Im} G(x)$ for all $x \in M^c_{k^*+1}$. Since $M^* = M^c_{k^*+1}$ satisfies the assumption **(CR)** in $U^* \subseteq U_{k^*+1}$ of x_p, we conclude that $M^* = M^c_{k^*+1}$ is a locally controlled invariant submanifold (around x_p) by Proposition 2.4. It is clear that x_p is admissible since $x_p \in M^*$. Then we show by by an induction argument that any other controlled invariant submanifold $M' \subseteq U^*$ is contained in M^*. First, since $M' \cap U^*$ is controlled invariant, we have that for any $x_0 \in M' \cap U^*$, there exist a solution $(x(\cdot), u(\cdot))$ and $t_0 \in I$ such that $x(t_0) = x_0$ and $x(t) \in M' \cap U^*, \forall t \in I$. Observe that for all $t \in I$,

$$E(x(t))\dot{x}(t) = F(x(t)) + G(x(t))u(t). \tag{6}$$

It follows that $F(x(t)) \in \operatorname{Im} E(x(t)) + \operatorname{Im} G(x(t)), \forall t \in I$. Thus by equation (5), we have $x(t) \in M_1, \forall t \in I$. Suppose that for a certain $k > 1$, we have $x(t) \in M_{k-1}, \forall t \in I$. We then have that $\dot{x}(t) \in T_{x(t)} M_{k-1}, \forall t \in I$ (note that when restricted to U^*, the set M_{k-1} is a submanifold). Thus in $U^* \subseteq U_k$, equation (6) implies $F(x(t)) \in E(x(t))T_{x(t)} M^c_{k-1} + \operatorname{Im} G(x(t))$. It follows that $x(t) \in M_k \cap U^*$, $\forall t \in I$ by equation (5). So we have $x(t) \in M_k \cap U^*, \forall t \in I$, and for every $k \geq 0$. Therefore, $x(t) \in M^* \cap U^*$, $\forall t \in I$ and in particular, we have $x_0 = x(t_0) \in M^* \cap U^*$, which implies that $M' \cap U^* \subseteq M^* \cap U^*$ (actually, $M' \cap U^*$ is contained in the only connected component of $M^* \cap U^*$).

(ii) We now prove that M^* locally coincides with the admissible set S_a on U^*. Since M^* is locally controlled invariant, for any point $x_0 \in M^* \cap U^*$ (take a smaller U^*, if necessary), there exist at least one solution $(x(\cdot), u(\cdot))$ and $t_0 \in I$ such that $x(t_0) = x_0$ (by Definition 2.2), which implies that x_0 is admissible i.e., $x_0 \in S_a$. It follows that $M^* \cap U^* \subseteq S_a \cap U^*$. Conversely, consider any point $x_0 \in S_a \cap U^*$, i.e., x_0 is admissible, then there exists a solution $(x(t), u(t))$ and $t_0 \in I$ such that $x(t_0) = x_0$, so equation (6) holds for the solution $x(t)$ passing through $x_0 \in S_a \cap U^*$. By the same induction argument as that used to prove $M' \cap U^* \subseteq M^* \cap U^*$ of (i) above, we can also show $S_a \cap U^* \subseteq M^* \cap U^*$. Hence we have $S_a \cap U^* = M^* \cap U^*$. □

Remark 1. (i) Proposition 2.4 is a geometric method to construct the locally maximal controlled invariant submanifold M^*. Such an iterative way of identifying the admissible set of a DAE is called the geometric reduction method and has appeared frequently in the geometric analysis of nonlinear DAEs (see e.g., [21–23, 4] and the recent papers [9, 19]). We state a practical implementation of this geometric method as Algorithm 1 of the Appendix, where we also compare our Algorithm 1 with an existing geometric reduction method of Section 3.4 of [4]. A previous version of Algorithm 1 for DAEs (without control u) can be consulted in [24].

(ii) Item (ii) of Proposition 2.4 asserts that in the neighborhood U^* of an admissible point $x_p = x_a$, the solutions of Ξ^u exist on M^* only, which implies that for any point $x_0 \in U^* \backslash M^*$, there are no solutions passing through x_0.

(iii) If for a fixed x_p, we drop the requirements that $x_p \in M_k$ and that M^c_k are connected, then Proposition 2.4 allows to detect all admissible points x_a in U^* that form the union $\bigcup M^*_i$ of all locally maximal controlled invariant submanifolds in U^*. Notice that, first, that union $\bigcup M^*_i$ may have more than one connected components (each of them being a locally maximal controlled invariant submanifold), second, x_p may not be in $\bigcup M_i^*$ (implying that x_p is not accessible) and, third, $\bigcup M_i^*$ can be empty (implying that there are no accessible points in U^*).

(iv) The recursive procedure of Proposition 2.4 leads to the sequence of nested submanifolds

$$M_{k^*+1}^c = M_{k^*}^c \subsetneq M_{k^*-1}^c \subsetneq \cdots \subsetneq M_0^c = U_0.$$

At each step, we construct a submanifold M_{k+1}^c that is of a smaller dimension than M_k^c, except for the last step, where M_{k^*+1}, defined by equation (5), coincides with $M_{k^*}^c$, although not on U_{k^*} but on a smaller neighborhood U_{k^*+1} and $M_{k^*+1}^c$ is actually $M_{k^*}^c$ restricted to U_{k^*+1}. The need to take a smaller neighborhood $U_{k^*+1} \subseteq U_{k^*}$ is a purely nonlinear phenomenon and in the linear case, we have $\mathscr{M}^* = M^* = M_{k^*}^c$, see item (v) below.

(v) If we apply the above procedure of constructing M_k to a linear DACS $\Delta = (E, H, L)$, then we get a sequence of subspaces

$$\mathscr{V}_0 = \mathbb{R}^n, \;\; \mathscr{V}_k = H^{-1}(E\mathscr{V}_{k-1} + \operatorname{Im} L). \tag{7}$$

The sequence $\mathscr{V}_k$ is one of the *augmented Wong sequences* (see [25]), that play an important role in the geometric analysis of linear DACSs (see e.g., [26]). In particular, it is shown in [6] and [15] that the indices of the feedback canonical form of linear DACSs are closely related to these sequences. In the linear case, the submanifold M^* is the largest subspace such that $HM^* \subseteq EM^* + \operatorname{im} L$, which we denote by $\mathscr{M}^*$. Clearly, $\mathscr{M}^* = \mathscr{V}^* = \mathscr{V}_{k^*}$, where k^* is the smallest integer k such that $\mathscr{V}_k = \mathscr{V}_{k+1}$.

3 | TWO NORMAL FORMS UNDER EXTERNAL FEEDBACK EQUIVALENCE

The canonical form of linear DACSs in [6] is under the equivalence relation: $(E, H, L) \sim (QEP^{-1}, Q(H + LF^u)P^{-1}, QLT^{-1})$, where Q, P, T are invertible real matrices and F^u defines a static state feedback. In the following definition, we generalize this equivalence relation to the nonlinear case.

Definition 3.1 (External feedback equivalence). Two DACSs $\Xi^u_{l,n,m} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{l,n,m} = (\tilde{E}, \tilde{F}, \tilde{G})$ defined on X and $\tilde{X}$, respectively, are called external feedback equivalent, shortly ex-fb-equivalent, if there exists a diffeomorphism $\psi : X \to \tilde{X}$ and smooth functions $Q : X \to GL(l, \mathbb{R})$, $\alpha : X \to \mathbb{R}^m$, $\beta : X \to GL(m, \mathbb{R})$ such that

$$\begin{aligned} \tilde{E}(\psi(x)) &= Q(x)E(x)\left(\tfrac{\partial \psi(x)}{\partial x}\right)^{-1}, \\ \tilde{F}(\psi(x)) &= Q(x)\left(F(x) + G(x)\alpha(x)\right), \\ \tilde{G}(\psi(x)) &= Q(x)G(x)\beta(x). \end{aligned} \tag{8}$$

The ex-fb-equivalence of two DACSs is denoted by $\Xi^u \overset{ex-fb}{\sim} \tilde{\Xi}^{\tilde{u}}$. If $\psi : U \to \tilde{U}$ is a local diffeomorphism between neighborhoods U of x_0 and $\tilde{U}$ of $\tilde{x}_0$, and $Q(x)$, $\alpha(x)$, $\beta(x)$ are defined on U, we will talk about local ex-fb-equivalence.

Remark 2. If two DACSs are ex-fb-equivalent, the diffeomorphism $\tilde{x} = \psi(x)$ and the feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$ establish a one-to-one correspondence of solutions $(x(\cdot), u(\cdot))$ and $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ of the DACSs, i.e., $\tilde{x}(\cdot) = \psi(x(\cdot))$ and $u(\cdot) = \alpha(x(\cdot)) + \beta(x(\cdot))\tilde{u}(\cdot)$. On the other hand, if the solutions of two DACSs correspond to each other via a diffeomorphism and a feedback transformation, then the two DACSs are not necessarily ex-fb-equivalent (since the diffeomorphism is defined on the whole neighborhood U but the solutions exist on the maximal controlled invariant submanifold M^* only), which is a main reason we distinguish the "external" and "internal" analysis of DACSs. As a simple example, we consider the following two DAEs $\Xi^u_{2,1,1} = (E, F, G)$ and $\tilde{\Xi}^{\tilde{u}}_{2,1,1} = (\tilde{E}, \tilde{F}, \tilde{G})$, where

$$E(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \;\; F(x) = \begin{bmatrix} (x-1)^2 \\ -x^2 \end{bmatrix}, \;\; G(x) = \begin{bmatrix} 0 \\ e^x \end{bmatrix}, \;\; \tilde{E}(\tilde{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \;\; \tilde{F}(\tilde{x}) = \begin{bmatrix} \tilde{x} \\ 0 \end{bmatrix}, \;\; \tilde{G}(\tilde{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

It is clear that $(x, u) = (1, e^{-1})$ and $(\tilde{x}, \tilde{u}) = (0, 0)$ are the unique solutions of the two DACSs and the diffeomorphism $\tilde{x} = \psi(x) = x - 1$ and the feedback transformation $\tilde{u} = -x^2 + e^x u$ map (x, u) to $(\tilde{x}, \tilde{u})$. However, the two DACSs can not be ex-fb-equivalent since E and $\tilde{E}$ are not of the same rank (two ex-fb-equivalent DACSs should have E-matrices of the same point-wise rank).

Theorem 3.2. (Normal forms) *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$ and fix a point $x_p \in X$. Let $M^* \subseteq X$ be a smooth connected embedded submanifold containing x_p. Assume that M^* is a locally maximal controlled invariant submanifold around x_p and there exists a neighborhood V of x_p such that*

(A1) rank $E(x) = const. = r$ *and* rank $\begin{bmatrix} E(x) & G(x) \end{bmatrix} = const. = r + m_2$, $\forall x \in V$.

(A2) The submanifold M^* *satisfies the constant rank assumption* ***(CR)****, that is,* $\dim E(x)T_xM^* = const. = r_1$ *and* $\dim(E(x)T_xM^* + \operatorname{Im} G(x)) = const. = r_1 + m_1 + m_2$, $\forall x \in M^* \cap V$.

Then there exist a neighborhood $U \subseteq V$ *of* x_p *such that* Ξ^u *is locally ex-fb-equivalent to a DACS represented in the following normal form*

$$(\mathbf{NF}): \begin{bmatrix} I_{r_1} & E_1^2(z) & 0 & E_1^4(z) \\ 0 & E_2^2(z) & I_{r_2} & E_2^4(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1(z) & 0 \\ G_2(z) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{9}$$

where (z_1, z_2) *are local coordinates on* $M^* = \left\{ z \mid z_3 = 0, z_4 = 0 \right\}$ *and* $z = (z_1, z_2, z_3, z_4)$*, where* E_1^2, E_1^4, E_2^2, E_2^4 *are smooth matrix-valued functions defined on* U *with values in* $\mathbb{R}^{r_1 \times (n_1 - r_1)}$, $\mathbb{R}^{r_1 \times (n_2 - r_2)}$, $\mathbb{R}^{r_2 \times (n_1 - r_1)}$, $\mathbb{R}^{r_2 \times (n_2 - r_2)}$*, respectively, where* $r = r_1 + r_2$, $n_1 = \dim M^*$, $n = n_1 + n_2$ *and* $m \geq m_1 + m_2$*. Moreover, for all* $z \in M^*$*, we have that* $E_2^2(z) = 0$, $F_4(z) = 0$ *and* rank $G_2(z) = m_1$.

Furthermore, if the above (A2) is replaced by the condition that there exist a neighborhood V *of* x_p *and an involutive distribution* $\mathcal{D}$ *satisfying that* $\mathcal{D}(x) = T_xM^*$, $\forall x \in M^* \cap V$ *and*

(A3) $\dim E(x)\mathcal{D}(x) = const. = r_1$ *and* $\dim(E(x)\mathcal{D}(x) + \operatorname{Im} G(x)) = const. = r_1 + m_1 + m_2$, $\forall x \in V$,

then there exists a neighborhood $U \subseteq V$ *of* x_p *such that* Ξ^u *is locally ex-fb-equivalent to equation (9), however, with* $E_2^2(z) \equiv 0$ *and* rank $G_2(z) = m_1$, $\forall z \in U$*, which we call the special normal form*

$$(\mathbf{SNF}): \begin{bmatrix} I_{r_1} & E_1^2(z) & 0 & E_1^4(z) \\ 0 & 0 & I_{r_2} & E_2^4(z) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1(z) & 0 \\ G_2(z) & 0 \\ 0 & I_{m_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{10}$$

Proof. Since M^* is a smooth connected embedded submanifold, there exist a neighborhood U_1 of x_p and two vector-valued functions $\zeta_1 : U_1 \to \mathbb{R}^{n_1}$ and $\zeta_2 : U_1 \to \mathbb{R}^{n_2}$ such that $M^* \cap U_1 = \{x \in U_1 \mid \zeta_2(x) = 0\}$, and the differentials $\mathrm{d}\zeta_1$ and $\mathrm{d}\zeta_2$ are linearly independent. In local (ζ_1, ζ_2)-coordinates, defined by the local diffeomorphism $\zeta(x) = (\zeta_1(x), \zeta_2(x))$, Ξ^u is expressed as

$$\begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) \end{bmatrix} \begin{bmatrix} \dot\zeta_1 \\ \dot\zeta_2 \end{bmatrix} = \tilde F(\zeta) + \tilde G(\zeta) u,$$

where $\tilde E_1 : U_1 \to \mathbb{R}^{l \times n_1}$ and $\tilde E_2 : U_1 \to \mathbb{R}^{l \times n_2}$, and where $\begin{bmatrix} \tilde E_1(\zeta(x)) & \tilde E_2(\zeta(x)) \end{bmatrix} = E(x) \left(\frac{\partial \zeta(x)}{\partial x} \right)^{-1}$, $\tilde F(\zeta(x)) = F(x)$, $\tilde G(\zeta(x)) = G(x)$. Then, by assumption (A1), for all $\zeta \in U_2 = U_1 \cap V$, we have

$$\operatorname{rank} \begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) \end{bmatrix} = const. = r, \quad \operatorname{rank} \begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) & \tilde G(\zeta) \end{bmatrix} = const. = r + m_2.$$

Thus, by Dolezal's theorem (see [27]), there exists a smooth map $Q_1 : U_2 \to GL(l, \mathbb{R})$,

$$Q_1(\zeta) \begin{bmatrix} \tilde E_1(\zeta) & \tilde E_2(\zeta) & \tilde G(\zeta) \end{bmatrix} = \begin{bmatrix} \bar E_1(\zeta) & \bar E_2(\zeta) & \bar G_1(\zeta) \\ 0 & 0 & \bar G_2(\zeta) \\ 0 & 0 & 0 \end{bmatrix},$$

where $\bar E_1 : U_2 \to \mathbb{R}^{r \times n_1}$, $\bar E_2 : U_2 \to \mathbb{R}^{r \times n_2}$ and $\bar G_2 : U_2 \to \mathbb{R}^{m_2 \times m}$, such that the matrices $[\bar E_1(\zeta), \bar E_2(\zeta)]$ and $\bar G_2(\zeta)$ above are of full row rank.

By $\dim E(x)T_xM^* = const. = r_1$ of assumption (A2), it is immediate to see that rank $\bar E_1(\zeta) = r_1$ for $\zeta \in M^*$. It follows from the smoothness of $\bar E_1(\zeta)$ that by taking a smaller U_2, if necessary, there exist r_1 columns of $\bar E_1(\zeta)$ that are linearly independent in U_2. Now we write the matrix

$$\left[\bar E_1(\zeta) \,\middle|\, \bar E_2(\zeta) \right] = \left[\begin{array}{cc|cc} E_1^1(\zeta) & E_1^2(\zeta) & E_2^1(\zeta) & E_2^2(\zeta) \\ E_1^3(\zeta) & E_1^4(\zeta) & E_2^3(\zeta) & E_2^4(\zeta) \end{array} \right],$$

where $E_1^1 : U_2 \to \mathbb{R}^{r_1 \times r_1}$ and $E_2^3 : U_2 \to \mathbb{R}^{r_2 \times r_2}$ and where $r_2 = r - r_1$. We can always permute the rows (by a constant Q-transformation) and the columns (by permuting the components of ζ_1) of the above matrix such that $E_1^1(\zeta)$ is invertible. Then by a suitable Q-transformation, $[\bar E_1, \bar E_2]$ admits the form

$$\left[\bar E_1(\zeta) \,\middle|\, \bar E_2(\zeta) \right] = \left[\begin{array}{cc|cc} I_{r_1} & E_1^2(\zeta) & E_2^1(\zeta) & E_2^2(\zeta) \\ 0 & E_1^4(\zeta) & E_2^3(\zeta) & E_2^4(\zeta) \end{array} \right].$$

Since rank $E(x) = \text{rank}\,[\bar{E}_1(\zeta), \bar{E}_2(\zeta)] = r$, the matrix $[E_1^4, E_2^3, E_2^4]$ is of full row rank $r_2 = r - r_1$. Notice that $E_1^4(\zeta) = 0$ for $\zeta \in M^*$ (since rank $\bar{E}_1(\zeta) = r_1$ for $\zeta \in M^*$), so rank $[E_2^3(\zeta), E_2^4(\zeta)] = r_2$ for $\zeta \in M^*$. By the smoothness of $\bar{E}_2(\zeta)$, we have that $[E_2^3(\zeta), E_2^4(\zeta)]$ is of full row rank r_2 for $\zeta \in U_2$. Then we can always permute the columns (by permuting the components of ζ_2) of $\bar{E}_2$ such that E_2^3 is invertible. On the other hand, we write

$$\begin{bmatrix} \bar{G}_1(\zeta) \\ \hline \bar{G}_2(\zeta) \end{bmatrix} = \begin{bmatrix} G_1^1(\zeta) & G_1^2(\zeta) \\ G_1^3(\zeta) & G_1^4(\zeta) \\ \hline G_2^1(\zeta) & G_2^2(\zeta) \end{bmatrix},$$

where $G_2^2(\zeta)$ is a $m_2 \times m_2$ matrix. Since $\bar{G}_2(\zeta)$ is of full row rank m_2 for $\zeta \in U_2$, we can permute the components of u (by a feedback transformation) such that $G_2^2(\zeta)$ is invertible. Since both $E_2^3(\zeta)$ and $G_2^2(\zeta)$ are invertible, we can set

$$Q_2(\zeta) = \begin{bmatrix} I_{r_1} & Q_1^2(\zeta) & Q_1^3(\zeta) & 0 \\ 0 & Q_2^2(\zeta) & Q_2^3(\zeta) & 0 \\ 0 & 0 & Q_3^3(\zeta) & 0 \\ 0 & 0 & 0 & I_{l-m-r} \end{bmatrix},$$

where $Q_1^2 = -E_2^1(E_2^3)^{-1}$, $Q_1^3 = -(G_1^2 - E_2^1(E_2^3)^{-1}G_1^4)(G_2^2)^{-1}$, $Q_2^2 = (E_2^3)^{-1}$, $Q_2^3 = -(E_2^3)^{-1}G_1^4(G_2^2)^{-1}$, $Q_3^3 = (G_2^2)^{-1}$, and then we have

$$Q_2(\zeta)Q_1(\zeta)\left[\tilde{E}_1(\zeta)\ \tilde{E}_2(\zeta)\ \middle|\ \tilde{G}(\zeta)\right] = \left[\begin{array}{cccc|cc} I_{r_1} & \tilde{E}_1^2(\zeta) & 0 & \tilde{E}_2^2(\zeta) & \tilde{G}_1^1(\zeta) & 0 \\ 0 & \tilde{E}_1^4(\zeta) & I_{r_1} & \tilde{E}_2^4(\zeta) & \tilde{G}_1^3(\zeta) & 0 \\ 0 & 0 & 0 & 0 & \tilde{G}_2^1(\zeta) & I_{m_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

where $\tilde{E}_1^2 = E_1^2 + Q_1^2E_1^4$, $\tilde{E}_2^2 = E_2^2 + Q_1^2E_2^4$, $\tilde{G}_1^1 = G_1^1 + Q_1^2E_1^4 + Q_1^3G_2^1$, $\tilde{E}_1^4 = Q_2^2E_1^4$, $\tilde{E}_2^4 = Q_2^2E_2^4$, $\tilde{G}_1^3 = Q_2^2G_1^3 + Q_2^3G_2^1$, $\tilde{G}_2^1 = Q_3^3G_2^1$. Denote $Q_2Q_1\tilde{F} = (F_1, F_2, F_3, F_4)$. Then by the feedback transformation

$$\begin{bmatrix} 0 \\ F_3(\zeta) \end{bmatrix} + \begin{bmatrix} I_{m_1} & 0 \\ \tilde{G}_2^1(\zeta) & I_{m_2} \end{bmatrix} u = \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix},$$

both $\tilde{G}_2^1$ and F_3 become zero. Rewrite $z = \zeta$, $(z_1, z_2) = \zeta_1$, $(z_3, z_4) = \zeta_2$, $(u_1, u_2) = (\tilde{u}_1, \tilde{u}_2)$, $E_1^2 = \tilde{E}_1^2$, $E_2^2 = \tilde{E}_1^4$, $E_1^4 = \tilde{E}_2^2$, $E_2^4 = \tilde{E}_2^4$, $G_1 = \tilde{G}_1^1$, $G_2 = \tilde{G}_1^3$, then it is not hard to see that Ξ^u is locally ex-fb-eq on $U = U_2$ to the normal form **(NF)**, given by (9).

Consider equation (9), by $\dim E(x)T_xM^* = \text{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) \\ 0 & E_2^2(z) \end{bmatrix} = r_1$ for all $z \in M^*$ of the assumption (A2), we have $E_2^2(z) = 0$, for all $z \in M^*$ and then by $\dim(E(x)T_xM^* + \operatorname{Im} G(x)) = \text{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) & G_1(z) & 0 \\ 0 & E_2^2(z) & G_2(z) & 0 \\ 0 & 0 & 0 & I_{m_2} \end{bmatrix} = r_1 + m_1 + m_2$ for all $z \in M^*$ of (A2), we get rank $G_2(z) = m_1$, for all $z \in M^*$. Moreover, by the fact that M^* is a controlled invariant submanifold and due to equation (4) Proposition 2.4, it follows that $F_4(z) = 0$ for all $z \in M^*$.

Now we prove that under the assumptions (A1) and (A3), Ξ^u is locally ex-fb-equivalent to the special normal form **(SNF)**, given by (10). The construction of the **(SNF)** is similar to the above construction of the **(NF)** of equation (9), but we choose the new coordinates $\zeta = (\zeta_1, \zeta_2)$ differently. By the involutivity of $\mathcal{D}$ of the assumption (A3) and Frobenius theorem (see e.g. [20]), there exist a neighborhood U_1 of x^0 and two vector-valued functions $\zeta_1 : U_1 \to \mathbb{R}^{n_1}$ and $\zeta_2 : U_1 \to \mathbb{R}^{n_2}$ such that $\text{span}\{d\zeta_1\} = \text{span}\{d\zeta_1^1, \dots, d\zeta_1^{n_1}\} = \mathcal{D}^\perp$, where $\mathcal{D}^\perp$ denotes the annihilator of the distribution $\mathcal{D}$, and $d\zeta_1$ and $d\zeta_2$ are linearly independent. Since $\mathcal{D}(x) = T_xM^*$ locally for $x \in M^*$, we still have $M^* \cap U_1 = \{x \mid \zeta_2(x) = 0\}$. Observe that the assumption (A3) implies (A2), we may transform Ξ^u into the **(NF)**, given by (9) using the constructions shown above. But now under assumption (A3):

$$\dim E(z)\mathcal{D}(z) = \text{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) \\ 0 & E_2^2(z) \end{bmatrix} = r_1,$$

$$\dim\left(E(z)\mathcal{D}(z) + \operatorname{Im} G(z)\right) = \text{rank}\begin{bmatrix} I_{r_1} & E_1^2(z) & G_1(z) & 0 \\ 0 & E_2^2(z) & G_2(z) & 0 \\ 0 & 0 & 0 & I_{m_2} \end{bmatrix} = r_1 + m_1 + m_2,$$

for all $z \in U_2$, we have that $E_2^2(z) \equiv 0$ and rank $G_2(z) = m_1$, $\forall z \in U_2$. Therefore, under the assumptions (A1) and (A3), the DACS Ξ^u is locally ex-fb-equivalent to the **(SNF)** of (10). $\square$

The following observations are crucial.

Remark 3. (i) If the submanifold M^* exists and Ξ^u satisfies the constant rank assumptions (A1) and (A2), which are regularity assumptions, then Ξ^u is locally ex-fb-equivalent to the **(NF)**, given by (9). If Ξ^u satisfies the constant rank and *involutivity* assumptions (A1) and (A3), then it is locally ex-fb-equivalent the **(SNF)**, given by (10), in which, additionally compared to (9), we have $E_2^2(z) \equiv 0$ and $\operatorname{rank} G_2(z) = m_1$ for all $z \in U$. Note that if M^* is replaced by M being any controlled invariant submanifold (not necessarily maximal) and satisfying (A1) and (A2) or (A1) and(A3), we may still transform Ξ into the form (9) or (10) since we do not use the maximality of M^* to construct the two normal forms as shown in the above of proof. However, if M^* is not locally maximal, we can neither conclude that $M^* = \{z \mid z_3 = z_4 = 0\}$ nor that (z_1, z_2) are the local coordinates on the admissible set $S_a = M^*$.

(ii) By a suitable feedback transformation introducing new controls (u_1^1, u_1^2) (possibly also by a permutation of z_3-variables), the second equation of (10) can be further simplified as

$$\begin{bmatrix} I_{r_2} & E_2^4(z) \end{bmatrix} \begin{bmatrix} \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = F_2(z) + G_2(z)u_1 \Rightarrow \begin{bmatrix} I_{r_2-m_1} & 0 & \tilde{E}_2^4(z) \\ 0 & I_{m_1} & \bar{E}_2^4(z) \end{bmatrix} \begin{bmatrix} \dot{z}_3^1 \\ \dot{z}_3^2 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \bar{F}_2(z) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{G}_2(z) \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix}.$$

where $u_1^1 \in \mathbb{R}^{r_2-m_1}$, $u_1^2 \in \mathbb{R}^{m_1}$ and $\bar{F}_2(z) = 0$ for $z \in M^*$.

(iii) The forms **(NF)** and **(SNF)** are two normal forms under the external feedback equivalence, meaning that both hold locally everywhere around x_p, not just on the maximal controlled invariant manifold M^* passing through x_p. For any point $x_0 \notin M^*$ around x_p, the system does not have solutions passing through x_0 (see item (ii) of Remark 1), but the system admits the above normal forms, which can be useful if we want to steer x_0 towards M^*.

(iv) Note that $M^* = \{z \mid z_3 = 0, z_4 = 0\}$. If we consider Ξ^u "internally", i.e., locally on M^*, by setting z_3 and z_4 to be zero, we get from equation (9) the following system (we may do the same for equation (10)):

$$\begin{bmatrix} I_r & E_1^2(z_1, z_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} F_1(z_1, z_2) \\ F_2(z_1, z_2) \end{bmatrix} + \begin{bmatrix} G_1(z_1, z_2) \\ G_2(z_1, z_2) \end{bmatrix} u_1.$$

Since $\operatorname{rank} G_2(z) = m_1$ for $z \in M^*$, via a suitable feedback transformation with new controls (u_1^1, u_1^2) and a $Q(x)$-transformation (defined on M^* but it can be extended to U^* that is open in X), the above DACS can be transformed into

$$\begin{bmatrix} I_r & E_1^2(z_1, z_2) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_1(z_1, z_2) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G_1^1(z_1, z_2) & 0 \\ 0 & 0 \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix},$$

for some maps $\tilde{F}_1$ and G_1^1. It can be seen from item (i) of Theorem 4.3 below that Ξ^u has solutions isomorphic with those of the first subsystem $\dot{z}_1 + E_1^2(z_1, z_2)\dot{z}_2 = \tilde{F}_1(z_1, z_2) + G_1^1(z_1, z_2)u_1^1$, which we denote by $\Xi^u|_{M^*}$ and call the restriction of Ξ^u to M^*; the latter can be regarded as an ODE control system with controls $w = \dot{z}_2$ and u_1^1:

$$\begin{cases} \dot{z}_1 = \tilde{F}_1(z_1, z_2) + G_1^1(z_1, z_2)u_1^1 - E_1^2(z_1, z_2)w \\ \dot{z}_2 = w. \end{cases}$$

(This is a particular case of a general procedure proposed in [19] under the name of (Q, w)-explicitation). From the above analysis, it is seen that for a fixed control u, the original Ξ^u has a unique maximal solution (see the definition of maximal solution in Section 4) if and only if $n_1 = r_1$ (since in this case, the z_2-variables are absent).

(v) The above two normal forms **(NF)** and **(SNF)** facilitate understanding the actual roles of the variables in the nonlinear DACS Ξ^u. As a result, some generalized states, namely (z_1, z_3), behave like state variables of differential equations and some generalized states, namely (z_2, z_4), enter into the system statically and are algebraic variables. Moreover, some generalized states, namely (z_3, z_4) are constrained and some controls, namely u_1^2 and u_2 are also not free to be chosen (since they are forced to be 0 by the constraints) when the DACS is considered internally on M^*. The generalized state z_2 and the control u_1^1 are the truly free variables of the system.

(vi) Our forms **(NF)** and **(SNF)** are different from the zero dynamics form **(ZDF)** proposed in [8] in many ways. First, the feedback transformations, which play important roles for our normal forms, are not used for the **(ZDF)**. Second, it is assumed for the **(ZDF)** that

$$\dim E(x)T_xM^* + \operatorname{Im} G(x) = \dim M^* + m, \tag{11}$$

while we only assume that $\dim E(x)T_xM^* + \operatorname{Im} G(x) = const.$, which is more general since assumption (11) excludes the existence of free generalized states and control inputs in the internal dynamics. Third, the utilization of the involutive distribution $\mathcal{D}$, not present in **(ZDF)**, shows a possibility to further simplify the structure of the matrix $E(x)$ in **(SNF)**.

4 | INTERNAL REGULARIZATION OF NONLINEAR DACSS

In this section, we first consider the uncontrolled case of (1) i.e., nonlinear DAEs, which are of the form

$$\Xi : E(x)\dot{x} = F(x),$$

and denoted by $\Xi_{l,n} = (E, F)$ or, equivalently, $\Xi^u_{l,n,0} = (E, F, 0)$. If we apply Definition 2.2 to a DAE Ξ, then M^* is called a locally maximal invariant submanifold. It is well known (see e.g., [22], [23], [4], [17] and [19]) that the solutions of a DAE Ξ exist locally on its maximal invariant submanifold M^* only and that the uniqueness of solutions can be characterized by the notion of local *internal regularity*, which is defined below. We will say that a solution $x : I \to M^*$ satisfying $x(t_0) = x_0$, where $t_0 \in I$ and $x_0 \in M^*$, is maximal if for any solution $\tilde{x} : \tilde{I} \to M^*$ such that $t_0 \in \tilde{I}$, $\tilde{x}(t_0) = x_0$ and $x(t) = \tilde{x}(t)$ for all $t \in I \cap \tilde{I}$, we have $\tilde{I} \subseteq I$.

Definition 4.1 (local internal regularity)**.** Consider a DAE $\Xi_{l,n} = (E, F)$ and let M^* be a locally maximal invariant submanifold around a point $x_p \in M^*$. Then Ξ is called locally *internally regular* (around x_p) if there exists a neighborhood $U \subseteq X$ of x_p such that for any point $x_0 \in M^* \cap U$, there exists only one maximal solution $x : I \to M^* \cap U$ satisfying $t_0 \in I$ and $x(t_0) = x_0$.

Remark 4. Consider a DAE $\Xi_{l,n} = (E, F)$ and let M^* be a locally maximal invariant submanifold around a point $x_p \in M^*$. Assume that there exists a neighborhood U of x_p such that $\dim E(x)T_xM^* = const.$, $\forall x \in M^* \cap U$. Then the following conditions are equivalent (see Theorem 4.3.14 of [16] or [19]):

(i) Ξ is internally regular around x_p;

(ii) $\dim E(x)T_xM^* = \dim M^*$, $\forall x \in M^* \cap U$;

(iii) Via a Q-transformation and a local diffeomorphism ψ defined on M^* around x_p, the system $\Xi|_{M^*}$ can be transformed into an ODE $\dot{z}^* = f(z^*)$, where z^* are local coordinates on M^*, given by ψ, and $\Xi|_{M^*}$ denotes Ξ restricted to the submanifold M^* (compare item (iv) of Remark 3).

Definition 4.2 (Local internal regularizability)**.** A DACS $\Xi^u = (E, F, G)$ is called locally *internally regularizable* (around x_p) if there exist a neighborhood U of x_p and a smooth map $\alpha : U \to \mathbb{R}^m$ such that the DAE $\Xi_{l,n} = (E, F + G\alpha)$ is internally regular around x_p.

Now we use Algorithm 1 in the appendix to study the problems that when is a DACS is locally internally regularizable and that how to design internal regularization feedback laws. Note that Algorithm 1 is a practical implementation of the recursive procedure of Proposition 2.4 (see Remark 1(i)) with additional **Assumptions 1** and **2**. At every step of Algorithm 1, we construct a submanifold M^c_k and a local form (equation (24)) under the external feedback equivalence, based on which we give an explicit expression of the restricted/reduced system (see equation (25)). Moreover, at every k-step, we show in details how to construct the coordinate transformations ψ_k and the feedback transformations $(v_k, \bar{v}_k) = a_k + b_k v_{k-1}$, which lead to the local form. In the statement of Theorem 4.3, we refer to the submanifold $M^* = M^c_{k^*+1}$, and the open neighborhood $U^* = U_{k^*+1}$ (in X) of Step $k^* + 1$ of Algorithm 1.

Theorem 4.3. *Consider a DACS $\Xi^u_{l,n,m} = (E, F, G)$, fix a point $x_p \in X$. Suppose that* **Assumptions 1** and **2** *of Algorithm 1 are satisfied. Then M^c_k, for $k = 0, \dots, k^* + 1$, of the recursive procedure of Proposition 2.4 are smooth connected embedded submanifolds and $M^* = M^c_{k^*+1}$ satisfies the constant rank condition* ***(CR)*** *in U^* and thus by that proposition, x_p is an admissible point and M^* is a locally maximal controlled invariant submanifold around x_p, given by*

$$M^* = \left\{ x \mid \bar{z}_1(x) = 0, \dots, \bar{z}_{k^*}(x) = 0 \right\}.$$

Moreover,

(i) *locally, around* x_p *there exist a diffeomorphism* $x = \Psi(\hat{z})$ *and a feedback* $u = a(x) + b(x)\hat{v}$, *where* $\hat{z} = (z^*, \bar{z}) = (z^*, \bar{z}_1, \dots, \bar{z}_{k^*})$ *and* $\hat{v} = (v^*, \bar{v}) = (v^*, \bar{v}_1, \dots, \bar{v}_{k^*+1})$, *transforming the set of all solutions of* $\Xi^u_{l,n,m}$ *into that of* $\hat{\Xi}^{\hat{v}}_{\hat{l},\hat{n},\hat{m}} = (\hat{E}, \hat{F}, \hat{G})$, *where* $\hat{l} = r^* + (n - n^*) + (m - m^*)$, $\hat{n} = n$ *and* $\hat{m} = m$, *given by*

$$\hat{\Xi}^{\hat{v}} : \begin{cases} E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)v^*, \\ \bar{z}_1 = 0, \dots, \bar{z}_{k^*} = 0, \\ \bar{v}_1 = 0, \dots, \bar{v}_{k^*} = 0, \bar{v}_{k^*+1} = 0, \end{cases} \tag{12}$$

where $E^* = E_{k^*+1} : M^* \to \mathbb{R}^{r^* \times n^*}$, $F^* = F_{k^*+1} : M^* \to \mathbb{R}^{r^*}$, $G^* = G_{k^*+1} : M^* \to \mathbb{R}^{r^* \times m^*}$ *and* $n^* = n_{k^*} = n_{k^*+1}$, $r^* = r_{k^*+1}$, $m^* = m_{k^*+1}$ *come from Step* k^*+1 *of Algorithm 1, and where* z^* *are local coordinates on* M^*, *and* $\operatorname{rank} E^*(z^*) = r^*$, $\forall z^* \in M^*$, *i.e.,* $E^*(z^*)$ *is of full row rank.*

(ii) *The DACS* Ξ^u *is internally regularizable around* x_p *if and only if* $r^* + \bar{m} \geq n^*$, *where* $\bar{m} = m - m^*$, *i.e., for any point* $x \in M^* \cap U$, *where* $U \subseteq X$ *is an open neighborhood of* x_p,

$$\dim(E(x)T_x M^* + \operatorname{Im} G(x)) \geq \dim M^*. \tag{13}$$

(iii) *Since* $E^*(z^*)$ *of equation (12) is of full row rank* r^*, *we assume that the first* r^* *columns of* $E^*(z^*)$ *are linearly independent (if not, we can always permute the components of* z^**). Rewrite* $E^*(z^*)\dot{z}^* = [E_1^*(z^*) \ E_2^*(z^*)] \begin{bmatrix} \dot{z}_1^* \\ \dot{z}_2^* \end{bmatrix}$ *such that* $E_1^*(z^*)$ *is invertible. If (13) holds, then define the following feedback law for (12):*

$$\hat{v} = \begin{bmatrix} v^* \\ \bar{v}_1 \\ \vdots \\ \bar{v}_{k^*+1} \end{bmatrix} = \begin{bmatrix} \alpha^*(z_1^*, z_2^*) \\ z_2^* \\ 0 \end{bmatrix} = \hat{\alpha}(z^*). \tag{14}$$

where $v^* = \alpha^*(z_1^*, z_2^*)$ *and* $\alpha^* : M^* \to \mathbb{R}^{m^*}$ *is an arbitrary smooth map. Finally, the feedback law* $u = \alpha(x) = a(x) + b(x)\hat{\alpha}(\psi^{-1}(x))$, *where the diffeomorphism* $x = \Psi(z^*, \bar{z})$ *and invertible feedback* $u = a(x) + b(x)\hat{v}$ *are those of item (i) and* $\psi(z^*) = \Psi(z^*, 0)$, *internally regularizes the original system* Ξ^u.

Proof. (i) At every Step k of Algorithm 1, consider the DACSs $\tilde{\Xi}^{\tilde{v}}_k = \Xi^v_{k-1} = (E_{k-1}, F_{k-1}, G_{k-1})$ and $\hat{\Xi}^{\hat{v}}_k = (\hat{E}_k, \hat{F}_k, \hat{G}_k)$, the latter given by (24). Then we show that the following items are equivalent.

(a) $(z_{k-1}(\cdot), v_{k-1}(\cdot))$, where $z_{k-1}(\cdot) = \psi_k^{-1}(z_k(\cdot), \bar{z}_k(\cdot))$ and $v_{k-1}(\cdot) = \alpha_k(z_{k-1}(\cdot)) + \beta_k((z_{k-1}(\cdot))(v_k(\cdot), \bar{v}_k(\cdot))$, is a solution of Ξ^v_{k-1};

(b) $(z_k(\cdot), \bar{z}_k(\cdot), v_k(\cdot), \bar{v}_k(\cdot))$ is a solution of $\hat{\Xi}^{\hat{v}}_k$;

(c) $\bar{z}_k(\cdot) = 0$, $\bar{v}_k(\cdot) = 0$ and $(z_k(\cdot), v_k(\cdot))$ is a solution of

$$\Xi^v_k : E_k(z_k)\dot{z}_k = F_k(z_k) + G_k(z_k)v_k,$$

where $E_k = \hat{E}^1_k$, $F_k = \hat{F}^1_k$, $G_k = \hat{G}^1_k$, and where $\hat{E}^1_k$, $\hat{F}^1_k$, $\hat{G}^1_k$ are defined by formula (25).

Since $\tilde{\Xi}^{\tilde{v}}_k = \Xi^v_{k-1}$ is locally ex-fb-equivalent to $\hat{\Xi}^{\hat{v}}_k$ via Q_k, ψ_k, α_k and β_k, we have that item (a) and item (b) above are equivalent (see Remark 2). The equivalence of item (b) and item (c) follows from the fact that the solution exists on M_k only and should respect the constraints ($\bar{z}_k = 0$ and $\bar{v}_k = 0$).

Then by the equivalence of (c) and (a), at the first step of Algorithm 1, we have that $(z_1(\cdot), 0, v_1(\cdot), 0)$ is a solution of $E_1(z_1)\dot{z}_1 = F_1(z_1) + G_1(z_1)v_1, \bar{z}_1 = 0, \bar{v}_1 = 0$, if and only if $(z_0(\cdot), v_0(\cdot))$ is a solution of $\Xi^v_0 = \Xi^u = (E, F, G)$, where $z_0(\cdot) = \psi_1^{-1}(z_1(\cdot), \bar{z}_1(\cdot))$ and $v_0(\cdot) = \alpha_1(z_0(\cdot)) + \beta_1(z_0(\cdot))(v_1(\cdot), \bar{v}_1(\cdot))$. In general, by an induction argument, we can prove that $(z_k(\cdot), 0, \dots, 0, v_k(\cdot), 0, \dots, 0)$ is a solution of

$$E_k(z_k)\dot{z}_k = F_k(z_k) + G_k(z_k)v_k, \quad \bar{z}_1 = 0, \dots, \bar{z}_k = 0, \quad \bar{v}_1 = 0, \dots, \bar{v}_k = 0,$$

if and only if $(x(\cdot), u(\cdot))$ is a solution of Ξ^u, where $x(\cdot)$ and $u(\cdot)$ are given by the following iterative formula

$$x(\cdot) = z_0(\cdot) = \psi_1^{-1}(z_1(\cdot), 0), \ z_1(\cdot) = \psi_2^{-1}(z_2(\cdot), 0), \ \dots, \ z_{k-1}(\cdot) = \psi_k^{-1}(z_k(\cdot), 0),$$

and

$$\begin{cases} u(\cdot) &= \alpha_1(x(\cdot)) + \beta_1(x(\cdot))(v_1(\cdot),0),\\ v_1(\cdot) &= \alpha_2(z_1(\cdot)) + \beta_2(z_1(\cdot))(v_2(\cdot),0),\\ \vdots \\ v_{k-1}(\cdot) &= \alpha_k(z_{k-1}(\cdot)) + \beta_k(z_{k-1}(\cdot))(v_k(\cdot),0). \end{cases}$$

We may write $x(\cdot) = \Psi(z_k(\cdot),0,\dots,0)$ and $u(\cdot) = a(z_k(\cdot),0,\dots,0) + b(z_k(\cdot),0,\dots,0)(v_k(\cdot),0,\dots,0)$, for some maps $\Psi : U_k \to \mathbb{R}^n$, $a : U_k \to \mathbb{R}^{m\times n}$, $b : U_k \to \mathbb{R}^{m\times m}$. Since for each k, ψ_k is a local diffeomorphism, it can be deduced that the Jacobian matrix of Ψ at x_p is invertible, which implies that Ψ (and thus Ψ^{-1}) is a local diffeomorphism. By the invertibility of all β_k at x_p, we may deduce that b is invertible at x_p as well, which means that a and b define a feedback transformation. In particular, set $k = k^*+1$, we can see that any solution of Ξ^u is mapped via the diffeomorphism Ψ and the feedback transformation defined by a and b into that of equation (12). Consider Step k^*+1 of Algorithm 1, note that the Q_{k^*+1}-transformation ensures that $\tilde{E}^1_{k^*+1}(z_{k^*})$ is of full row rank. By $M^c_{k^*+1} = \left\{z_{k^*} \in M^c_{k^*} \cap U_{k^*+1} \mid \tilde{F}^3_{k^*+1}(z_{k^*}) = 0\right\}$ and the fact that $\dim M^c_{k^*} = n_{k^*} = n_{k^*+1} = \dim M^c_{k^*+1}$, we have $\tilde{F}^3_{k^*+1}(z_{k^*}) \equiv 0$, $\forall z_{k^*} \in M^c_{k^*} \cap U_{k^*+1}$. As a consequence, the $\bar{z}_{k^*+1}$-coordinates are not present, so there is no equation $\bar{z}_{k^*+1} = 0$ in (12). Moreover, we have $M^c_{k^*+1} = M^c_{k^*}$ in U_{k^*+1}, it follows that $z_{k^*+1} = z_{k^*}$. Finally, it is seen from $E^*(z^*) = E_{k^*+1}(z_{k^*+1}) = \hat{E}^1_{k^*+1}(z_{k^*}) = \tilde{E}^1_{k^*+1}(z_{k^*})$ that $E^*(z^*)$ is of full row rank for all $z^* = z_{k^*+1} \in M^* = M^c_{k^*+1}$.

(ii) To begin with, we prove that Ξ^u is internally regularizable if and only if $\hat{\Xi}^{\hat{v}}$, given by (12), is internally regularizable. Observe that Ξ^u is internally regularizable, i.e., there exists a feedback $u = \alpha(x)$ such that $\Xi = (E, F + G\alpha)$ is internally regular if and only if the algebraic constraint $0 = u - \alpha(x)$ is such that the DAE $\Xi^\alpha : \begin{cases} E(x)\dot{x} = F(x) + G(x)u \\ 0 \quad = u - \alpha(x) \end{cases}$ has a unique maximal solution $(x(\cdot), u(\cdot))$ satisfying $x(t_0) = x_0$ and $u(t_0) = \alpha(x_0)$ for any $x_0 \in M^*_\alpha \cap U$, where M^*_α is a locally maximal invariant submanifold of Ξ^α and U is a neighborhood of x_p. By item (i) of Theorem 4.3, there is a one-to-one correspondence, given by a local diffeomorphism $\hat{z} = \Psi^{-1}(x)$ and a feedback transformation $u = a(\hat{z}) + b(\hat{z})\hat{v}$, between the solutions of Ξ^u and those of $\hat{\Xi}^{\hat{v}}$. As a consequence, Ξ^u is internally regularizable if and only if there exists $\alpha : M^* \to \mathbb{R}^n$ such that the DAE

$$\begin{cases} \hat{E}(\hat{z})\dot{\hat{z}} = \hat{F}(\hat{z}) + \hat{G}(\hat{z})\hat{v} \\ 0 \quad = \hat{v} - \hat{\alpha}(\hat{z}), \end{cases}$$

where $\hat{\alpha}(\hat{z}) = b^{-1}(\alpha(\Psi(\hat{z})) - a(\Psi(\hat{z})))$, has a unique maximal solution $(\hat{z}(\cdot), \hat{v}(\cdot))$ satisfying $\hat{z}(t_0) = \hat{z}_0$, where $\hat{z}_0 = \Psi^{-1}(x_0)$, and $\hat{v}(t_0) = b^{-1}(\alpha(x_0) - a(x_0))$, for any $x_0 \in M^*_\alpha \cap U$, i.e., $\hat{\Xi}^{\hat{v}}$ is internally regularizable. Now we will show that $\hat{\Xi}^{\hat{v}}$ is internally regularizable if and only if (13) holds. Since $E^*(z^*)$ of (12) is of full row rank, we may view the first equation of (12) as an ODE control system with extra free variables. More precisely, assume that the first r^* columns of $E^*(z^*)$ are linearly independent (if not, we can always permute the components of z^*), then we can rewrite $E^*(z^*)\dot{z}^*$ as $\begin{bmatrix} E^*_1(z^*) & E^*_2(z^*) \end{bmatrix} \begin{bmatrix} \dot{z}^*_1 \\ \dot{z}^*_2 \end{bmatrix}$, where $z^* = (z^*_1, z^*_2)$ and $E^*_1 : M^* \to R^{r^*\times r^*}$ is invertible. Thus we can rewrite the first equation of (12) as

$$\begin{cases} \dot{z}^*_1 = (E^*_1)^{-1}F^*(z^*) + (E^*_1)^{-1}G^*(z^*)v^* - (E^*_1)^{-1}E^*_2(z^*)w \\ \dot{z}^*_2 = w. \end{cases} \tag{15}$$

It follows that $\hat{\Xi}^{\hat{v}}$ is internally regularizable if and only if the free variables z^*_2 can be fixed via the constraints $\bar{v} = 0$, which is equivalent to the number of constrained inputs $\bar{v}$ (there are $\bar{m} = m - m^*$ of them) being not less than the number of components of z^*_2 (which is $n^* - r^*$) and thus to (13).

(iii) If $\bar{m} = m - m^* \geq n^* - r^*$, then there are enough components of constrained inputs $\bar{v} = 0$ that can be used to fix the free variables z^*_2. Namely, set $\hat{v} = (v^*, \bar{v}', \bar{v}'') \in \mathbb{R}^{m^*} \times \mathbb{R}^{\bar{m}'} \times \mathbb{R}^{\bar{m}''}$, where $\bar{m}' = n^* - r^*$ and $\bar{m}'' = \bar{m} - (n^* - r^*)$ then we impose $z^*_2 = 0$ by setting $\bar{v}' = z^*_2 = 0$ and the remaining components $\bar{v}'' = 0$ to construct a controlled invariant submanifold. We can choose $v^* = \alpha^*(z^*)$ arbitrarily and then $M^*_\alpha = \left\{z^* \in M^* \mid z^*_2 = 0\right\}$ is an invariant submanifold of the closed loop system $\hat{\Xi}^{\hat{\alpha}}$, obtained from $\hat{\Xi}^{\hat{v}}$ via $\hat{v} = \hat{\alpha}(z^*)$, where

$$\hat{v} = (v^*, \bar{v}', \bar{v}'') = (\alpha^*(z^*), z^*_2, 0) = \hat{\alpha}(z^*). \tag{16}$$

Now using the diffeomorphism $x = \Psi(z^*, \bar{z})$ and invertible feedback $u = a(x) + b(x)\hat{v}$ that transform solutions of Ξ^u into that of $\hat{\Xi}^{\hat{v}}$ (see item (i)), we conclude that the feedback laws $u = \alpha(x) = a(x) + b(x)\hat{\alpha}(\psi^{-1}(x))$, where $\hat{\alpha}$ is given by (16) and $\psi(z^*) = \Psi(z^*, 0)$, internally regularizes the original system Ξ^u. The obtained feedback law gives (14). □

Remark 5. (i) Note that we perform k^*+1 steps of Algorithm 1. Actually, we get M^* already at the step k^*, however, we need to perform one step more to know that Algorithm 1 stops (because $n_{k^*+1} = n_{k^*}$) but also in order to normalize the system $\Xi^v_{k^*}$ and obtain $\Xi^{v*} = \Xi^v_{k^*+1} = (E^*, F^*, G^*)$.

(ii) The first equation of (12), i.e., $E^*(z^*)\dot{z}^* = F^*(z^*) + G^*(z^*)v^*$, which we denote by $\Xi^u|_{M^*}$, has isomorphic solutions with Ξ^u and can be seen as the "internal" dynamics of Ξ^u. Since $E^*(z^*)$ is of full row rank, we may view $\Xi^u|_{M^*}$ as an ODE control system with two kinds of inputs, namely v^* and w (see equation (15) or item (iv) of Remark 3).

(iii) The procedure of internal regularization, leading to Theorem 4.3(iii), that we propose is not unique at two stages. First, by setting $\bar{v}' = \bar{\alpha}(z^*)$ for any $\bar{\alpha}$ such that $\frac{\partial \bar{\alpha}(z^*)}{\partial z_2^*}$ is invertible, we can find $z_2^* = \gamma(z_1^*)$ satisfying $\bar{\alpha}(z_1^*, \gamma(z_1^*)) = 0$ and thus we constrain the z_2^*-variables via $\hat{v}' = \bar{\alpha}(z^*) = 0$. Second, we can choose $v^* = \alpha^*(z^*)$ arbitrarily and that choice does not affect internal regularity of Ξ^u (nor the invariant submanifold M^*_α) since the feedback law $v^* = \alpha^*(z^*)$ does not influence the constraints $\bar{v}' = \bar{\alpha}(z^*) = 0$.

(iv) A linear DACS $\Xi = (E, H, L)$, given by (2), is internally regularizable/autonomizable (see Theorem 3.5 of [15]) if and only if

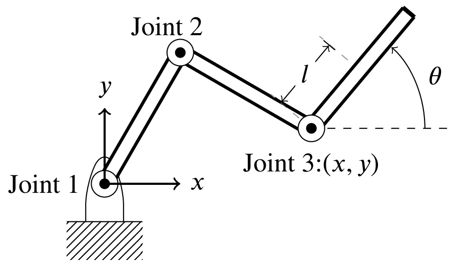
$$\dim(E\mathscr{V}^* + \mathrm{Im} L) \geq \dim \mathscr{V}^*,$$

where $\mathscr{V}^*$ is the limit of the augmented Wong sequence $\mathscr{V}_k$ of (7), which is, clearly, a linear counterpart of M^*. Thus item (ii) of Theorem 4.3 is a nonlinear generalization of the above result for linear DACSs.

(v) Combining the results of Theorem 3.2 and Theorem 4.3, it is seen that if a DACS Ξ^u is ex-fb-equivalent to the **(NF)** or the **(SNF)**, then Ξ^u is internally regularizable if and only if $r_1 + m_1 + m_2 \geq n_1$.

5 | EXAMPLES

Example 5.1. Consider the model of a 3-link manipulator taken from [28] as shown in the following figure, where Joint 1 and Joint 2 are active, and Joint-3 is passive and called the free joint.



The dynamic equations of the system are given by:

$$\begin{cases} m\ddot{x} - ml\sin\theta\ddot{\theta} - ml\dot{\theta}^2\cos\theta = f_x \\ m\ddot{y} + ml\cos\theta\ddot{\theta} - ml\dot{\theta}^2\sin\theta = f_y \\ -ml\sin\theta\ddot{x} + ml\cos\theta\ddot{y} + (I + ml^2)\ddot{\theta} = f_\theta, \end{cases} \tag{17}$$

where m, I, l are constants representing the mass, the moment of inertia and the half length of the free-link, respectively, x and y are the position variables of the free joint, and θ is the angle between the base frame and the link frame, f_x and f_y are the translational force at the free joint, f_θ is the torque around the free joint. We regard f_x and f_y as the control inputs to the system and

$$f_\theta = 0,$$

due to the passivity of the free joint. We require the trajectories of system (17) to respect the following constraint:

$$x - y = 0. \tag{18}$$

Denote $J = \frac{I}{m}$, $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \dot{y}$, $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$ and choose the generalized state $z_0 = (x_1, x_2, y_1, y_2, \theta_1, \theta_2)$. Rewrite (17) and (18) together as a DACS $\Xi^u_{6,6,2} = (E_1, F_1, G_1)$, given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & -ml\sin\theta_1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & ml\cos\theta_1 \\ 0 & -ml\sin\theta_1 & 0 & ml\cos\theta_1 & 0 & m(J+l^2) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ 0 \\ x_1 - y_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

Consider Ξ^u around a point $z_{0p} = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p})$, where

$$x_{1p} = x_{2p} = y_{1p} = y_{2p} = \theta_{1p} = \theta_{2p} = 0.$$

We now apply Algorithm 1 to Ξ^u. Step 0: Set $X = S^1 \times \mathbb{R}^5$.

Step 1: We have that rank $E_1(z_0) = \operatorname{rank} \begin{bmatrix} E_1(z_0) & G_1(z_0) \end{bmatrix} = 5$ around z_{0p}. Since $E(z_0)$ and $G(z_0)$ are already in the desired form, set $Q_1 = I_6$ and we get

$$M_1 = \left\{ z_0 \mid F_1(z_0) \in \operatorname{Im} E_1(z_0) + \operatorname{Im} G_1(z_0) \right\} = \left\{ z_0 \mid x_1 - y_1 = 0 \right\}.$$

Clearly, $z_{0p} \in M_1$, choose a new coordinate $\bar{z}_1 = x_1 - y_1$, keep the remaining coordinates $z_1 = (x_2, y_1, y_2, \theta_1, \theta_2)$ unchanged and we can see that the transformed system is different from Ξ^u by the first and the last equations only, i.e.,

$$\dot{x}_1 = x_2 \Rightarrow \dot{y}_1 + \dot{\bar{z}}_1 = x_2 \ \text{ and } \ 0 = x_1 - y_1 \Rightarrow 0 = \bar{z}_1.$$

Thus Ξ^u restricted to $M_1^c = \left\{ z_0 \in X \mid \bar{z}_1 = 0 \right\}$ is $\Xi^u|_{M_1^c} = \Xi^u_2 = (E_2, F_2, G_2)$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & -ml\sin\theta_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & ml\cos\theta_1 \\ -ml\sin\theta_1 & 0 & ml\cos\theta_1 & 0 & m(J+l^2) \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}. \tag{19}$$

Step 2: We have that rank $E_2(z_1) = \operatorname{rank} \begin{bmatrix} E_2(z_1) & G_2(z_1) \end{bmatrix} = 4$, so via a suitable $Q_2(z_1)$, we transform the first equation $\dot{y}_1 = x_2$ of (19) into $0 = x_2 - y_2$ and get

$$M_2 = \left\{ z_1 \in M_1^c \mid F_2(z_1) \in \operatorname{Im} E_2(z_1) + \operatorname{Im} G_2(z_1) \right\} = \left\{ z_1 \mid x_2 - y_2 = 0 \right\}.$$

Clearly, $z_{0p} \in M_1$, we then choose a new coordinate $\bar{z}_2 = x_2 - y_2$ and keep the remaining coordinates $z_2 = (y_1, y_2, \theta_1, \theta_2)$ unchanged. The system Ξ^u represented in new coordinates and restricted to $M_2^c = \left\{ z_1 \in M_1^c \mid \bar{z}_2 = 0 \right\}$ is $\hat{\Xi}^u|_{M_2^c} = \Xi^u_3 = (E_3, F_3, G_3)$, given by

$$\begin{bmatrix} 0 & m & 0 & -ml\sin\theta_1 \\ 1 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 \\ 0 & ml(\cos\theta_1 - \sin\theta_1) & 0 & m(J+l^2) \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} ml\theta_2^2\cos\theta_1 \\ y_2 \\ ml\theta_2^2\sin\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

Step 3: we have locally that rank $E_3(z_2) = 3$ and rank $\begin{bmatrix} E_3(z_2) & G_3(z_2) \end{bmatrix} = 4$. Then choose

$$Q_3(z_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ c(\theta_1) & 0 & d(\theta_1) & l(\cos\theta_1 + \sin\theta_1) \end{bmatrix},$$

where $c(\theta_1) = J + l^2 \sin\theta_1(\cos\theta_1 + \sin\theta_1)$ and $d(\theta_1) = -J - l^2\cos\theta_1(\cos\theta_1 + \sin\theta_1)$. Thus applying the left-multiplication by Q_3 on Ξ^u_3, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 \\ 0 & m & 0 & m(J+l^2) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2\sin\theta_1 \\ 0 \\ ml\theta_2^2 J(\cos\theta_1 - \sin\theta_1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ c(\theta_1) & d(\theta_1) \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

It follows that $k^* = 2$ and $M^* = M_2^c$ since

$$M_3 = \left\{ z_2 \mid Q_3 F_3(z_2) \in \operatorname{Im} Q_3 E_3(z_2) + \operatorname{Im} Q_3 G_3(z_2) \right\} = M_2^c.$$

Thus $z_{0p} \in M^*$ is an admissible point. Now set

$$\begin{bmatrix} v \\ \tilde{v} \end{bmatrix} = a(z_2) + b(z_2) \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 \\ ml\theta_2^2 J(\cos\theta_1 - \sin\theta_1) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ c(\theta_1) & d(\theta_1) \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

Via Q_3 and the above feedback transformation, Ξ^u_3 is ex-fb-equivalent (on M^*) to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 \\ 0 & m & 0 & m(J+l^2) \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot y_1 \\ \dot y_2 \\ \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ ml\theta_2^2\sin\theta_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ \bar v \end{bmatrix}.$$

Hence we get $z^* = z_2 = (y_1, y_2, \theta_1, \theta_2)$ and

$$E^*(z^*) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & ml\cos\theta_1 \\ 0 & m & 0 & m(J+l^2) \end{bmatrix}, \; F^*(z^*) = \begin{bmatrix} y_2 \\ ml\theta_2^2\sin\theta_1 \\ 0 \end{bmatrix}, \; G^*(z^*) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

By item (ii) of Theorem 4.3, $r^* + (m - m^*) = 3 + 1 = n^* = 4$ implies that our system Ξ^u is internally regularizable. A feedback that internally regularizes Ξ^u can be deduced, by item (iii) of Theorem 4.3, from

$$\bar v = \theta_1 - \theta_{ref} \Rightarrow ml\theta_2^2 J(\cos\theta_1 - \sin\theta_1) + c(\theta_1)f_x + d(\theta_1)f_y = \theta_1 - \theta_{ref}. \tag{20}$$

where θ_{ref} is a constant and represents a desired value of θ. The above equation has a infinite number of solutions. However, if we have designed $v = K(z^*)$ for some function $K(z^*)$ to, e.g., stabilize the internal system $\Xi^{u*} = (E^*, F^*, G^*)$ (which can be viewed as an ODE since E^* is of full row rank), then the feedback which internal regularizes and stabilizes Ξ^u can be uniquely solved via (20) together with $v = f_y = K(z^*)$. The solution is

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c(\theta_1) & d(\theta_1) \end{bmatrix}^{-1} \begin{bmatrix} K(z^*) \\ \theta_1 - \theta_{ref} - ml\theta_2^2 J(\cos\theta_1 - \sin\theta_1) \end{bmatrix}.$$

Note that our system Ξ^u satisfies the assumptions (A1) and (A3) of Theorem 3.2 since rank $E(z_0) = 5$, rank $\begin{bmatrix} E(z_0) & G(z_0) \end{bmatrix} = 5$, and the distribution

$$\mathcal{D}(z_0) = \operatorname{span}\left\{ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2} \right\}$$

satisfies $\mathcal{D}(z_0) = T_{z_0}M^*$ locally for all $z_0 \in M^*$ and $\dim E(z_0)\mathcal{D}(z_0) = 3$, $\dim(E(z_0)\mathcal{D}(z_0) + \operatorname{Im} G(z_0)) = 4$. In fact, Ξ^u is locally ex-fb-equivalent to

$$\textbf{(SNF)}: \begin{bmatrix} I_3 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot\zeta_1 \\ \dot\zeta_2 \\ \dot\zeta_3 \end{bmatrix} = \begin{bmatrix} F_1(\zeta) \\ F_2(\zeta) \\ F_4(\zeta) \end{bmatrix} + \begin{bmatrix} G_1(\zeta) \\ G_2(\zeta) \\ 0 \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix},$$

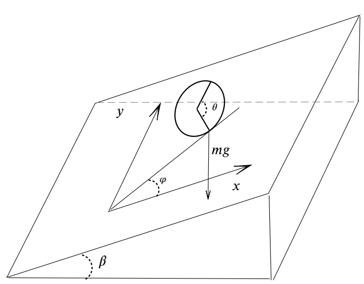
where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ (we use ζ for **(SNF)** since z are already used as coordinates of the system obtained via Algorithm 1), $\zeta_1 = (y_1, y_2, \theta_2)$, $\zeta_2 = \theta_1$, $\zeta_3 = (\bar z_2, \bar z_1) = (x_2 - y_2, x_1 - y_1)$,

$$F_1(\zeta) = \begin{bmatrix} y_2 \\ l\theta_2^2\sin\theta_1 - \frac{l^4\theta_2^2\cos\theta_1\sin\theta_1(\cos\theta_1 - \sin\theta_1)}{c(\theta_1)} \\ \frac{-l^3\theta_2^2\sin\theta_1(\cos\theta_1 - \sin\theta_1)}{c(\theta_1)} \end{bmatrix}, \; G_1(\zeta) = \begin{bmatrix} 0 & 0 \\ \frac{(J+l)}{mc(\theta_1)} & \frac{-l^2\cos\theta_1\sin\theta_1}{c(\theta_1)} \\ \frac{l(\cos\theta_1 - \sin\theta_1)}{mc(\theta_1)} & \frac{l\sin\theta_1}{c(\theta_1)} \end{bmatrix},$$

$$F_2(\zeta) = \begin{bmatrix} 0 \\ \bar x_2 \end{bmatrix}, \; F_4(\zeta) = \bar x_1, \; G_2(\zeta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Note that, for the new coordinated system **(SNF)**, $M^* = \{\zeta \mid \zeta_3 = 0\}$. The variables $\zeta_1 = (y_1, y_2, \theta_2)$ and $\zeta_3 = (\bar z_2, \bar z_1)$ perform as the states of differential equations (there are differential equations for $\dot\zeta_1$ and $\dot\zeta_3$), but ζ_3 are constrained and equal to 0. The variable $\zeta_2 = \theta_1$ is a free algebraic variable and u_1^2 is a constrained control input.

Example 5.2. Consider a rolling disk on an inclined plane as shown in the following figure. We denote the position of the disk by (x, y), the angles θ and φ describe the attitude of the disk with respect to the inclined plane, β is the angle between the horizontal surface and the inclined plane.



If there are no external forces acting on the system, the Lagrangian of the system is given by

$$\mathcal{L} = -mgx\sin\beta + \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2,$$

where m is the mass, J is the moment of inertia and throughout we assume $m = 1$ and $J = 1$, for simplicity. The following nonholonomic constraints represent the kinematic equations of the system

$$\begin{cases} 0 = -\sin\varphi \mathrm{d}x + \cos\varphi \mathrm{d}y \\ 0 = \cos\varphi \mathrm{d}x + \sin\varphi \mathrm{d}y - \mathrm{d}\theta \end{cases} \tag{21}$$

and we can derive the dynamic equations of the system as

$$\begin{cases} \ddot{x} = -g\sin\beta - \lambda_1\sin\varphi + \lambda_2\cos\varphi \\ \ddot{y} = \lambda_1\cos\varphi + \lambda_2\sin\varphi \\ \ddot{\theta} = -\lambda_2 \\ \ddot{\varphi} = 0. \end{cases} \tag{22}$$

We have $X = \mathbb{R}^8 \times T^2$ and choose control inputs as (τ_1, τ_2, τ_2), where $\tau_1 = \sin\beta$, and τ_2 and τ_3 are external torques in the directions of $\dot{\theta}$ and $\dot{\varphi}$, respectively. We study the problem whether we can find an input force such that the trajectories of the system respect the following constraint:

$$\frac{\pi}{2} = \varphi + \beta. \tag{23}$$

The constraint (23) is equivalent to $0 = \sin\beta - \cos\varphi$. Now consider (21), (22) and (23) together with the controls (τ_1, τ_2, τ_3), we get a DACS $\Xi^u_{10,11,3} = (E, F, G)$, given by

$$\Xi^u : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\varphi_1 & 0 & \cos\varphi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\varphi_1 & 0 & \sin\varphi_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\lambda_1\sin\varphi_1+\lambda_2\cos\varphi_1 \\ y_2 \\ \lambda_1\cos\varphi_1+\lambda_2\sin\varphi_1 \\ \theta_2 \\ -\lambda_2 \\ \varphi_2 \\ 0 \\ 0 \\ 0 \\ -\cos\varphi_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix},$$

where $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \dot{y}$, $\theta_1 = \theta$, $\theta_2 = \dot{\theta}$, $\varphi_1 = \varphi$, $\varphi_2 = \dot{\varphi}$. The generalized state is $\xi = (x, \dot{x}, y, \dot{y}, \varphi, \dot{\varphi}, \theta, \dot{\theta}, \lambda_1, \lambda_2)$. We consider Ξ^u around a point $\xi_p = (x_{1p}, x_{2p}, y_{1p}, y_{2p}, \theta_{1p}, \theta_{2p}, \varphi_{1p}, \varphi_{2p}, \lambda_{1p}, \lambda_{2p})$, where

$$x_{1p} = x_{2p} = y_{1p} = y_{2p} = \theta_{1p} = \theta_{2p} = \varphi_{1p} = \varphi_{2p} = \lambda_{1p} = \lambda_{2p} = 0.$$

Consider $\varphi \in (-\pi/2, \pi/2)$ and $\beta \in (\pi/2, \pi/2)$. Applying Algorithm 1 to Ξ^u, we get

$$M_0^c = \mathbb{R}^8 \times (\pi/2, \pi/2) \times (\pi/2, \pi/2), \quad M_1^c = \left\{\xi \in M_0^c \mid x_2\sin\varphi_1 - y_2\cos\varphi_1 = 0, -x_2\cos\varphi_1 - y_2\sin\varphi_1 + \theta_2 = 0\right\},$$
$$M_2^c = \left\{\xi \in M_1^c \mid \varphi_2\theta_1 - \lambda_1 - \tfrac{g}{2}\sin 2\varphi_1 = 0\right\}, \quad M^* = M_3^c = M_2^c.$$

It follows that $\xi_p \in M^*$ and that locally around x_p

$$\operatorname{rank} E(\xi) = r = 8, \quad \operatorname{rank}\left[E(\xi)\ \ G(\xi)\right] = r + m_2 = 9.$$

The distribution $\mathcal{D}(\xi) = \operatorname{span}\left\{g_1, g_2, g_3, g_4\right\} + \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \lambda_2}\right\}$, where

$$g_1 = \cos\varphi_1\frac{\partial}{\partial x_2} + \sin\varphi_1\frac{\partial}{\partial y_2} + \frac{\partial}{\partial \theta_2}, \quad g_2 = -y_2\varphi_2\frac{\partial}{\partial x_2} + x_2\varphi_2\frac{\partial}{\partial y_2} + g\cos 2\varphi_1\frac{\partial}{\partial \theta_1},$$
$$g_3 = -\theta_1\frac{\partial}{\partial \theta_1} + \varphi_2\frac{\partial}{\partial \varphi_2}, \quad g_4 = \varphi_2\frac{\partial}{\partial \lambda_1} + \frac{\partial}{\partial \theta_1},$$

satisfies $\mathcal{D}(\xi) = T_\xi M^*$ locally for all $\xi \in M^*$ and that

$$\dim E(\xi)\mathcal{D}(\xi) = r_1 = 6, \quad \dim(E(\xi)\mathcal{D}(\xi) + \operatorname{Im} G(\xi)) = r_1 + m_1 + m_2 = 8.$$

Thus the assumptions (A1) and (A3) of Theorem 3.2 are satisfied. Now set

$$Q(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \sin\varphi_1 & 0 & -\cos\varphi_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\cos\varphi_1 & 0 & -\sin\varphi_1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$z_1 = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ \theta_1 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}, \; z_2 = \lambda_2, \; z_3 = \begin{bmatrix} \tilde{y}_2 \\ \tilde{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_2 \sin\varphi_1 - y_2 \cos\varphi_1 \\ -x_2 \cos\varphi_1 - y_2 \sin\varphi_1 + \theta_2 \end{bmatrix}, \; z_4 = \tilde{\lambda}_1 = \theta_2\varphi_2 - \lambda_1 - \frac{g}{2}\sin 2\varphi,$$

$$\alpha(\xi) = \begin{bmatrix} \cos\varphi_1 \\ \varphi_2\tilde{y}_2 - 2\lambda_2 \\ 0 \end{bmatrix}, \; \beta(\xi) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then, via $Q(\xi)$, $\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \alpha(\xi) + \beta(\xi) \begin{bmatrix} u_1^1 \\ u_1^2 \\ u_2 \end{bmatrix}$ and $z = (z_1, z_2, z_3, z_4)$, Ξ^u is locally (around ξ_p) ex-fb-equivalent to

$$\textbf{(SNF)}: \begin{bmatrix} I_6 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ 0 \\ F_4(z) \end{bmatrix} + \begin{bmatrix} G_1^1(z) & 0 & 0 \\ 0 & G_2^2(z) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_1^2 \\ u_2 \end{bmatrix},$$

where

$$F_1(z) = \begin{bmatrix} x_2 \\ (\tilde{\lambda}_1 + \frac{g}{2}\sin 2\varphi_1 - \theta_2\varphi_2)\sin\varphi_1 + (\lambda_2 - g)\cos\varphi_1 \\ x_2 \tan\varphi_1 - \frac{\tilde{y}_2}{\cos\varphi_1} \\ \tilde{\theta}_2 - \tilde{y}_2 \tan\varphi_1 + \frac{x_2}{\cos\varphi_1} \\ \varphi_2 \\ 0 \end{bmatrix}, \; F_2(z) = \begin{bmatrix} \tilde{\lambda}_1 \\ 0 \end{bmatrix}, \; F_4(z) = \begin{bmatrix} \tilde{y}_2 \\ \tilde{\theta}_2 \end{bmatrix}, \; G_1^1(z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; G_2^2(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, by item (iv) of Remark 3, the algebraic variables are $z_2 = \lambda_2$ and $z_4 = \tilde{\lambda}_1$. The variables z_2 and u_1^1 are free to choose. The generalized states $z_3 = (\tilde{y}_2, \tilde{\theta}_2)$, $z_4 = \tilde{\lambda}_1$ and the controls (u_1^2, u_2) are constrained and required to be 0 by the algebraic constraints. Moreover, by item (iv) of Remark 3, we have

$$\Xi^u|_{M^*}: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{\theta}_1 \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ (\frac{g}{2}\sin 2\varphi_1 - \theta_2\varphi_2)\sin\varphi_1 + (\lambda_2 - g)\cos\varphi_1 \\ x_2 \tan\varphi_1 \\ \frac{x_2}{\cos\varphi_1} \\ \varphi_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1^1,$$

which is an ODE control system with one free variable λ_2 and one control $u_1^1 = \tau_3$. We can see from item (i) of Theorem 4.3 that $\Xi^u|_{M^*}$ has isomorphic trajectories with that of Ξ^u. Moreover, since $\dim(E(\xi)T_\xi M^* + \operatorname{Im} G(\xi)) = 8 > \dim M^* = 7$, by item (ii) of Theorem 4.3, the system Ξ^u is internally regularizable, e.g., a feedback which internally regularizes the system is given by $u_2 = \lambda_2 \Rightarrow \tau_1 = \cos\varphi_1 + \lambda_2$, $u_2^1 = 0 \Rightarrow \tau_2 = 0$ and $\tau_3 = u_1^1 = \gamma(x)$ for any smooth function $\gamma(x)$. Note that $\gamma(x)$ could be a designed function as a feedback to control the system $\Xi^u|_{M^*}$ (note that by substituting $\lambda_2 = u_2 = 0$, the system $\Xi^u|_{M^*}$ becomes a single-input ODE control system).

6 | CONCLUSION

In this paper, we propose two normal forms for nonlinear DACSs under external feedback equivalence to simplify the structure of systems and to clarify different roles of variables, which is our first main result. One normal form requires only the existence of a maximal controlled invariant submanifold and some constant rank assumptions of system matrices while another requires additionally the involutivity of some distribution. Moreover, we give a necessary and sufficient geometric condition for a nonlinear DACS to be internally regularizable (second main result), we also formulate an algorithm to calculate the maximal invariant submanifold and a feedback which internally regularizes the system. Two examples of mechanical systems are given to illustrate the proposed normal forms and the internal regularization algorithm.

7 | APPENDIX

We give some remarks on Algorithm 1.

Remark 6.

(i) The **Assumption 1** that rank $\tilde{E}_k(z_{k-1}) = const.$ and rank $[\tilde{E}_k(z_{k-1}), \tilde{G}(z_{k-1})] = const.$ is made to produce the full row rank matrices $\tilde{E}_k^1$ and $\tilde{G}_k^2$ and the zero-level set $M_k = \left\{z_{k-1} \in W_k \mid \tilde{F}_k^3(z_{k-1}) = 0\right\}$. The **Assumption 2** that rank $\mathrm{D}\tilde{F}_k^3(z_{k-1}) = const.$ makes it possible to use the components of $\tilde{F}_k^3$ with linearly independent differentials as a part of new local coordinates. Those two assumptions are somewhat related to but different from the two assumptions in [4], e.g., in order to produce a smooth embedded submanifold, the author of [4] assumes that $H_k(x, \dot{x}) = \tilde{E}_k(x)\dot{x} - \tilde{F}_k(x)$ is a submersion.

(ii) The dimensions r_k, n_k, m_k satisfy

$$\begin{cases} r_0 \geq ... \geq r_k \geq ... \geq 0, & n_0 \geq ... \geq n_k \geq ... \geq 0, \quad m_0 \geq ... \geq m_k \geq ... \geq 0, \\ n_{k-1} \geq r_k, & r_{k-1} - r_k - (m_{k-1} - m_k) \geq n_{k-1} - n_k. \end{cases}$$

The integers r_k, n_k, m_k indicate the values of dim $E(x)T_x M_k$, dim M_k, and that the vector v_k is m_k-dimensional, respectively, and illustrate well the evolution of the reduction procedure.

(iii) Our Algorithm 1 is related to the geometric reduction method used in Section 3.4 of [4]. In both, one constructs a sequence of submanifolds recursively and then reduces/restricts the DACS to the constructed submanifolds. The main difference is that Algorithm 1 deals with DAEs with an extra control u, i.e., DACSs, while in [4] only DAEs are discussed and no feedback transformations are involved. Moreover, we relate our Algorithm 1 with the recursive procedure given before Proposition 2.4. Actually, Step k of Algorithm 1 provides an explicit construction of the manifolds M_k^c of the procedure.

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□

Algorithm 1 Internal regularization algorithm for nonlinear DACSs

Initiatlization: Consider $\Xi^{u}_{l,n.m} = (E, F, G)$, fix $x_p \in X$ and let $U_0 \subseteq X$ be an open connected subset containing x_p. Below all sets U_k are open in X and W_k are open in M^c_{k-1}.

Step 0: Set $z_0 = x$, $v_0 = u$, $E_0(z_0) = E(x)$, $F_0(z_0) = F(x)$, $G_0(z_0) = G(x)$, $M_0 = X$, $M^c_0 = U_0$, $r_0 = l$, $n_0 = n$, $m_0 = m$.

Step k**:**

1: Suppose that we have defined at Step $k-1$: an open neighborhood $U_{k-1} \subseteq X$ of x_p, a smooth embedded connected submanifold M^c_{k-1} of U_{k-1} and a DACS $\Xi^{v}_{k-1} = (E_{k-1}, F_{k-1}, G_{k-1})$ given by smooth matrix-valued maps

$$E_{k-1} : M^c_{k-1} \to \mathbb{R}^{r_{k-1}\times n_{k-1}},\ \ F_{k-1} : M^c_{k-1} \to \mathbb{R}^{r_{k-1}},\ \ G_{k-1} : M^c_{k-1} \to \mathbb{R}^{r_{k-1}\times m_{k-1}},$$

whose arguments are denoted $z_{k-1} \in M^c_{k-1}$.

2: Rename the maps as $\tilde{E}_k = E_{k-1}$, $\tilde{F}_k = F_{k-1}$, $\tilde{G}_k = G_{k-1}$ and define $\tilde{\Xi}^{\tilde{v}}_k := (\tilde{E}_k, \tilde{F}_k, \tilde{G}_k)$.

Assumption 1: There exists an open neighborhood $U_k \subseteq U_{k-1} \subseteq X$ of x_p such that rank $\tilde{E}_k(z_{k-1}) = const. = r_k \le n_{k-1}$ and rank $[\tilde{E}_k(z_{k-1}), \tilde{G}_k(z_{k-1})] = const. = r_k + m_{k-1} - m_k$, $\forall z_{k-1} \in W_k = U_k \cap M^c_{k-1}$.

3: Find a smooth map $Q_k : W_k \to GL(r_{k-1}, \mathbb{R})$, such that $\tilde{E}^1_k$ and $\tilde{G}^2_k$ of

$$Q_k\tilde{E}_k = \begin{bmatrix} \tilde{E}^1_k \\ 0 \\ 0 \end{bmatrix},\ \ Q_k\tilde{F}_k = \begin{bmatrix} \tilde{F}^1_k \\ \tilde{F}^2_k \\ \tilde{F}^3_k \end{bmatrix},\ \ Q_k\tilde{G}_k = \begin{bmatrix} \tilde{G}^1_k \\ \tilde{G}^2_k \\ 0 \end{bmatrix}$$

are of full row rank, where $\tilde{E}^1_k : W_k \to \mathbb{R}^{r_k\times n_{k-1}}$, $\tilde{G}^2_k : W_k \to \mathbb{R}^{(m_{k-1}-m_k)\times m_{k-1}}$, $\tilde{F}^3_k : W_k \to \mathbb{R}^{r_{k-1}-r_k-m_{k-1}+m_k}$ (so all the matrices depend on z_{k-1}).

4: Following (5), define $M_k = \left\{ z_{k-1} \in W_k \mid \tilde{F}^3_k(z_{k-1}) = 0 \right\}$.

Assumption 2: $x_p \in M_k$ and rank $\mathrm{D}\tilde{F}^3_k(z_{k-1}) = const. = n_{k-1} - n_k$ for $z_{k-1} \in M_k \cap U_k$, by taking a smaller U_k (if necessary).

5: By the above assumption, $M_k \cap U_k$ is a smooth embedded submanifold and by taking again a smaller U_k, we may assume that $M^c_k = M_k \cap U_k$ is connected and choose new coordinates $(z_k, \bar{z}_k) = \psi_k(z_{k-1})$ on W_k, where $\bar{z}_k = (\varphi^1_k(z_{k-1}), ..., \varphi^{n_{k-1}-n_k}_k(z_{k-1}))$, with $\mathrm{d}\varphi^1_k(z_{k-1}), ..., \mathrm{d}\varphi^{n_{k-1}-n_k}_k(z_{k-1})$ being all independent rows of $\mathrm{D}\tilde{F}^3_k(z_{k-1})$, and z_k are any complementary coordinates such that ψ_k is a local diffeomorphism.

6: Choose new control inputs $\begin{bmatrix} v_k \\ \bar{v}_k \end{bmatrix} = a_k(z_{k-1}) + b_k(z_{k-1})v_{k-1}$, where $a_k = \begin{bmatrix} 0 \\ \tilde{F}^2_k \end{bmatrix}$, $b_k = \begin{bmatrix} \tilde{b}_k \\ \tilde{G}^2_k \end{bmatrix}$, and where $\tilde{b}_k : W_k \to \mathbb{R}^{m_{k-1}\times m_k}$ is chosen such that $b_k(z_{k-1})$ is invertible $\forall z_{k-1} \in W_k$ (by taking again a smaller U_k, if necessary).

7: Set $\hat{E}_k = Q_k\tilde{E}_k\left(\frac{\partial\psi_k}{\partial z_{k-1}}\right)^{-1}$, $\hat{F}_k = Q_k(\tilde{F}_k + G_k\alpha_k)$, $\hat{G}_k = Q_k\tilde{G}_k\beta_k$, $\alpha_k = -b^{-1}_k a_k$ and $\beta_k = b^{-1}_k$.

8: By Definition 3.1, $\tilde{\Xi}^{\tilde{v}}_k \overset{ex-fb}{\sim} \hat{\Xi}^{\hat{v}}_k = (\hat{E}_k, \hat{F}_k, \hat{G}_k)$ via Q_k, ψ_k, and (α_k, β_k), where

$$\hat{\Xi}^{\hat{v}}_k : \begin{bmatrix} \hat{E}^1_k(z_k,\bar{z}_k) & \bar{E}^1_k(z_k,\bar{z}_k) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{z}_k \\ \dot{\bar{z}}_k \end{bmatrix} = \begin{bmatrix} \hat{F}^1_k(z_k,\bar{z}_k) \\ 0 \\ \tilde{F}^3_k(z_k,\bar{z}_k) \end{bmatrix} + \begin{bmatrix} \hat{G}^1_k(z_k,\bar{z}_k) & \bar{G}^1_k(z_k,\bar{z}_k) \\ 0 & I_{m_{k-1}-m_k} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_k \\ \bar{v}_k \end{bmatrix}, \tag{24}$$

with $\hat{E}^1_k : W_k \to \mathbb{R}^{r_k\times n_k}$, $\hat{F}^1_k : W_k \to \mathbb{R}^{r_k}$, $\hat{G}^1_k : W_k \to \mathbb{R}^{r_k\times m_k}$ and $\left[\hat{E}^1_k \circ\psi_k\ \ \bar{E}^1_k \circ\psi_k\right] = \tilde{E}^1_k\left(\frac{\partial\psi_k}{\partial z_{k-1}}\right)^{-1}$, $\hat{F}^1_k \circ\psi_k = \tilde{F}^1_k\alpha_k$ and $\left[\hat{G}^1_k \circ\psi_k\ \ \bar{G}^1_k \circ\psi_k\right] = \tilde{G}^1_k\beta_k$.

9: Set $\bar{z}_k = 0$ and $\bar{v}_k = 0$ to define the restricted DACS on $M^c_k = \left\{ z_{k-1} \in W_k \mid \bar{z}_k = 0 \right\}$ as

$$\hat{\Xi}^{\hat{v}}_k|_{M^c_k} : \hat{E}^1_k(z_k, 0)\dot{z}_k = \hat{F}^1_k(z_k, 0) + \hat{G}^1_k(z_k, 0)v_k. \tag{25}$$

10: On M^c_k, define a system

$$\Xi^{v}_k : E_k(z_k)\dot{z}_k = F_k(z_k) + G_k(z_k)v_k,$$

where $E_k(z_k) = \hat{E}^1_k(z_k, 0)$, $F_k(z_k) = \hat{F}^1_k(z_k, 0)$, $G_k(z_k) = \hat{G}^1_k(z_k, 0)$ are matrix-valued maps and $E_k : M^c_k \to \mathbb{R}^{r_k\times n_k}$, $F_k : M^c_k \to \mathbb{R}^{r_k}$, $G_k : M^c_k \to \mathbb{R}^{r_k\times m_k}$.

Repeat: Step k for $k = 1, 2, 3, \dots$, **until** $n_k = n_{k-1}$, set $k^* = k - 1$ and perform Step $k = k^* + 1$.

Result: Set $n^* = n_{k^*} = n_{k^*+1}$, $r^* = r_{k^*+1}$, $m^* = m_{k^*+1}$, $M^* = M^c_{k^*+1}$, $U^* = U_{k^*+1}$, $z^* = z_{k^*+1} = z_{k^*}$, $v^* = v_{k^*+1}$ and $\Xi^{v*} = (E^*, F^*, G^*)$ with $E^* = E_{k^*+1}$, $F^* = F_{k^*+1}$, $G^* = G_{k^*+1}$.