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Synchronization via funnel coupling with application for decentralized optimization

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Research Seminar, KTH Stockholm, Schweden, 23 October 2020, 14:00 - 14:30

Decentralized optimization

Problem statement

Given

> N agents with individual n-dimensional dynamics:

 $\dot{x}_i = f_i(t, x_i) + u_i$

- > undirected connected coupling-graph G = (V, E)
- > local feedback $u_i = \gamma_i(x_i, x_{\mathcal{N}_i})$

Desired

Control design for practical synchronization

$$x_1 \approx x_2 \approx \ldots \approx x_n$$



 $u_{1} = \gamma_{1}(x_{1}, x_{2}, x_{3})$ $u_{2} = \gamma_{2}(x_{2}, x_{1}, x_{3})$ $u_{3} = \gamma_{3}(x_{3}, x_{1}, x_{2})$ $u_{4} = \gamma_{4}(x_{4}, x_{3})$

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A "high-gain" result

Let $\mathcal{N}_i := \{j \in V \mid (j,i) \in E\}$ and $d_i := |\mathcal{N}_i|$ and \mathcal{L} be the Laplacian of G.

Diffusive coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j)$$
 or, equivalently, $u = -k \mathrel{\mathcal{L}} x$

Theorem (Practical synchronization¹)

Assumptions: G undirected, connected, all solutions of average dynamics

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))$$

remain bounded. Then $\forall \varepsilon > 0 \ \exists K_{\varepsilon} > 0 \ \forall k \geq K_{\varepsilon}$: Diffusive coupling results in

$$\limsup_{t \to \infty} |x_i(t) - x_j(t)| < \varepsilon \quad \forall i, j \in V$$

¹ J. Kim, J. Yang, H. Shim, J.-S. Kim, and J.H. Seo. IEEE Transactions on Automatic Control 2016.

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Remarks on high-gain result

Common trajectory

It even holds that

$$\limsup_{t\to\infty}|x_i(t)-s(t)|<\varepsilon,$$

where
$$s(\cdot)$$
 solves $\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t)), \quad s(0) = \frac{1}{N} \sum_{i=1}^{N} x_i.$

Independent of coupling structure and amplification k.

Error feedback

With $e_i := x_i - \overline{x}_i$ and $\overline{x}_i := \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} x_j$ diffusive coupling has the form

$$u_i(t) = -k \, d_i \, e_i(t)$$

Attention: $e_i \neq x_i - s$, in particular, agents do not know "limit trajectory" $s(\cdot)$

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Example²



Simulations in the following for ${\cal N}=5$ agents with dynamics

 $f_i(t, x_i) = (-1 + \delta_i) x_i + 10 \sin t + 10 m_i^1 \sin(0.1t + \theta_i^1) + 10 m_i^2 \sin(10t + \theta_i^2),$

with randomly chosen parameters $\delta_i, m_i^1, m_i^1 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$.

Parameters identical in all following simulations, in particular $\delta_2 > 1$, hence agent 2 has unstable dynamics (without coupling).

² Shim, H. and Trenn, S. Proc. 54th IEEE Conf. Decision Control (CDC) 2015.

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Decentralized optimization













² Shim, H. and Trenn, S. Proc. 54th IEEE Conf. Decision Control (CDC) 2015.

1

1.5

2

2.5

3

0.5

-20

0

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4

4.5

3.5

Remaining problems

Problems

- > necessary gain depends on global network structure
- > all agents must use the same gain
- > no direct control over convergence rate
- > only practical synchronization, no asymptotic convergence

High gain adaptive control

Problems very similar to high gain control problems!

 $\hookrightarrow \mathsf{Funnel} \ \mathsf{Control}$

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Funnel coupling

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Reminder Funnel Controller



Theorem (Practical tracking³)

Funnel Control

$$k(t) = \frac{1}{\psi(t) - |e(t)|}$$

works, in particular, errors remains within funnel for all times.

Basic idea for funnel coupling

$$u = -k \mathcal{L} x \longrightarrow u = -k(t) \mathcal{L} x$$

³ A. Ilchmann, E.P. Ryan and C.J. Sangwin. ESAIM: Control, Opt. Calc. Variations, 2002

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Funnel coupling rule

Problem

 $u = -k(t)\mathcal{L}x$ is not decentralized and no obvious error signal

$\hookrightarrow \mathsf{node}\mathsf{-wise}\ \mathsf{viewpoint}$

Reminder diffusive coupling

$$u_i = -k_i e_i$$
 with $e_i = x_i - \overline{x}_i$

Combine diffusive coupling with Funnel Controller

$$u_i(t) = -k_i(t) \, e_i(t)$$
 with $k_i(t) = rac{1}{\psi(t) - |e_i(t)|}$

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Example revisited





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Observations from simulations

Funnel synchronization seems to work

- > errors remain within funnel
- > practical synchronizations is achieved
- $\,\,$ $\,$ limit trajectory does not coincide with solution $s(\cdot)$ of

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t)), \qquad s(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0).$$

What determines the new limiting trajectory?

- Coupling graph?
- > Funnel shape?
- > Gain function?

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Weakly centralized funnel synchronization

Weakly centralized Funnel synchronization

Analogously as for diffusive coupling, all agents use the same gain:

 $u_i(t) = -k_{\max}(t) d_i e_i(t)$ with $k_{\max}(t) := \max_{i \in V} \frac{1}{\psi(t) - |e_i(t)|}$

Theorem (SHIM & T. 2015 (CDC))

Assumption:

- > No "finite escape time" of x_i
- > The graph is connected, undirected and d-regular with $d > rac{N}{2} 1$
- > Funnel boundary $\psi:[0,\infty)\to [\psi,\overline{\psi}]$ is differentiable, non-increasing and

 $|e_i(0)| < \psi(0), \quad \forall i = 1, 2, \dots, N.$

Then weakly centralized funnel synchronization works.

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Node wise funnel synchronization: general case

Theorem (LEE, T. & SHIM 2020, submitted)

Multiagent system with symmetric, connected coupling graph under funnel coupling:

$$\dot{x}_i = f_i(t, x_i) - \mu\left(\frac{e_i(t)}{\psi(t)}\right) \qquad e_i(t) := x_i - \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} x_j, \quad \mu(\eta) := \frac{\eta}{1 - |\eta|}$$

Assume that $\dot{\overline{\chi}}(t) = \max_{i \in \mathcal{N}} f_i(t, \overline{\chi}(t))$ and $\underline{\dot{\chi}}(t) = \min_{i \in \mathcal{N}} f_i(t, \underline{\chi}(t))$ do not exhibit finite escape time, then

 $|e_i(t)| < \psi(t) \quad \forall t \in [0,\infty).$

Furthermore, the emergent behavior is given by

$$\dot{\xi} = h_{\mu}(f_1(t,\xi), f_2(t,\xi), \dots, f_N(t,\xi)),$$

where h_{μ} is unique function implicitely given by $\sum_{i=1}^{N} \mu^{-1} (h_{\mu}(f_1, \dots, f_N) - f_i) = 0.$

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Why not average dynamics?

Laplacian feedback

Diffusive coupling

$$u = -k \mathcal{L} x$$

has Laplacian feedback matrix $k\mathcal{L}$

Non-Laplacian feedback Funnel synchronization $u = -K(t) \mathcal{L} x = - \begin{bmatrix} k_1(t) & & \\ & k_2(t) & \\ & & \ddots & \\ & & & k_N(t) \end{bmatrix} \mathcal{L} x$

has non-Laplacian feedback matrix $K(t)\mathcal{L}$, in particular $[1, 1, \dots, 1]^{\top}$ is not a left-eigenvector of $K(t)\mathcal{L}$.

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Diffusive coupling revisited

Diffusive coupling with edgewise gain

$$u_i = -\mathbf{k_i} \sum_{i}^{N} (x_i - x_j) \longrightarrow u_i = -\sum_{i}^{N} \mathbf{k_{ij}} \cdot (x_i - x_j)$$

 \rightsquigarrow $u = -\mathcal{L}_K x$ with symmetric $\mathcal{L}_K!$

Conjecture

If $k_{ij} = k_{ji}$ are all sufficiently large, then practical synchronization occurs with desired limit trajectory s of average dynamics.

Proof technique from $\rm K{\scriptstyle IM}$ et al. 2016 should still work in this setup.

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Edgewise funnel coupling

Edgewise funnel gains \rightarrow edgewise funnel coupling

$$u_i = -\sum_{j \in \mathcal{N}_i} \mathbf{k}_{ij} \cdot (x_i - x_j) \longrightarrow u_i = -\sum_{j \in \mathcal{N}_i} \mathbf{k}_{ij}(t) \cdot (x_i - x_j)$$

Edgewise funnel gain

$$k_{ij}(t) = \frac{1}{\psi(t) - |e_{ij}|}, \quad \text{with} \quad e_{ij} := x_i - x_j$$

Properties:

- > Decentralized, i.e. u_i only depends on state of neighbors
- > Symmetry, $k_{ij}(t) = k_{ji}(t)$
-) Laplacian feedback, $u = -\mathcal{L}_K(t, x)x$

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Decentralized optimization

Simulation



Properties

- + Synchronization occurs
- + Predictable limit trajectory (given by average dynamics)
- + Local feedback law
- No asymptotic convergence towards limit trajectory because $\psi(t) \geq \psi > 0$

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Synchronization via funnel coupling with application for decentralized optimization (17 / 31)



Decentralized optimization

Edgwise-wise funnel control

Theorem (LEE, BERGER, SHIM, T. 2020, in preparation)

Multiagent system with symmetric, connected coupling graph under funnel coupling:

$$\dot{x}_i = f_i(t, x_i) - \sum_{j \in \mathcal{N}_i} \frac{x_i - x_j}{\psi(t) - |x_i - x_j|}$$

If f_i is globally Lischitz in x_i then

 $|x_i(t) - x_j(t)| < \psi(t) \quad \forall t \in [0, \infty)$

Furthermore, the emergent behaviour is given by $s(\cdot)$ where

$$\dot{s} = \frac{1}{N} \sum_{i=1}^{N} f_i(t,s), \quad s(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0).$$

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Funnel control and asymptotic tracking



Asymptotic tracking only with unbounded gain

 $\begin{array}{lllllllllllllllllllllllllll} \mbox{Asymptotic tracking} & \Leftrightarrow & \psi(t) \xrightarrow[t \to \infty]{} 0 & \Rightarrow & \psi(t) - \|e(t)\| \xrightarrow[t \to \infty]{} 0 \\ \Leftrightarrow & k(t) = \frac{1}{\psi(t) - \|e(t)\|} \xrightarrow[t \to \infty]{} \infty \end{array}$

Conclusion: Funnel control and asymptotic tracking not compatible ?

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Rewrite rule for funnel control

Important observation: $\psi(t) \to 0 \implies ||e(t)|| \to 0$ Hence $u(t) = -k(t)e(t) \to \infty \cdot 0$ not necessarily unbounded!

$$u(t) = -\frac{1}{\psi(t) - \|e(t)\|} e(t) = -\frac{1}{1 - \left\|\frac{e(t)}{\psi(t)}\right\|} \frac{e(t)}{\psi(t)} =: -\alpha(\eta(t)) \cdot \eta(t) \text{ where } \eta(t) := \frac{e(t)}{\psi(t)}$$

Asymptotic funnel control possible⁴ Rewriting classical funnel control rule $\hookrightarrow \psi(t) \to 0$ allowed!

 4 Lee, J.G. and Trenn, S.: Asymptotic tracking via funnel control. Proc. 58th IEEE Conf. Decision Control $_{-\psi(t)}$ (CDC) 2019, pp. 4228–4233, Nice, France, 2019.



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Synchronization via funnel coupling with application for decentralized optimization (22 / 31)

The problem

Minimization problem

For some differentiable and convex $F : \mathbb{R}^n \to \mathbb{R}$ find x^* which solves

 $\min_{x\in \mathbb{R}^n} F(x)$

Challenge

$$F(x) = \sum_{i=1}^{N} F_i(x)$$

Find x^* in a decentralized way with N agents who only know their $F_i : \mathbb{R}^n \to \mathbb{R}$, which are also not assumed to be convex.

Some approaches available already, new approach: Utilize ideas from funnel control

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Local gradient descent and coupling

Lemma (Global gradient descent)

$$\dot{x}(t) = -\eta \nabla F(x(t)), \quad x(0) = x_0 \in \mathbb{R}^n$$

Then $x(t) \rightarrow x^*$ for all x_0 and all $\eta > 0$,

Local gradient descent + coupling

$$\dot{x}_i = -\nabla F_i(x_i) + \frac{u_i}{x_i} \qquad x_i(0) = x_{0,i} \in \mathbb{R}^n$$

Find suitable control law

$$x_i(t) \xrightarrow{u_i} x^*$$
 in particular, synchronization / consensus

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Funnel coupling and decentralized optimization

Limit trajectory under edgewise funnel coupling

$$\lim_{t \to \infty} |x_i(t) - s(t)| = 0 \quad \forall i \in V$$

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))$$

Average dynamics for local gradient descent

For $f_i(t,s) = -\nabla F_i(s)$ we have

$$\dot{s}(t) = -\frac{1}{N} \sum_{i=1}^{N} \nabla F_i(s(t)) = -\frac{1}{N} \nabla F(s(t))$$

 \hookrightarrow Average dynamics = global gradient descent

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Decentralized optimization via funnel coupling

$$\underbrace{\min_{x \in \mathbb{R}^n} F(x)}_{ki \in \mathbb{R}^n} F(x) = \sum_{i=1}^N F_i(x) \quad \text{convex with unique minimizer } x^* \in \mathbb{R}^n$$

$$\dot{x}_i = -\nabla F_i(x) - \sum_{j \in \mathcal{N}_i} \left(\frac{e_{ij}^1}{\psi - |e_{ij}^1|}, \frac{e_{ij}^2}{\psi - |e_{ij}^2|}, \dots, \frac{e_{ij}^n}{\psi - |e_{ij}^n|} \right)^\top, \quad x_i(0) = x_{i,0} \in \mathbb{R}^n, \ i \in \mathcal{V}$$

Theorem (Decentralized optimization via funnel coupling⁵)

Assume ∇F_i is globally Lipschitz and coupling graph is undirected and connected, funnel boundary $\psi \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0})$ with $\psi(t) \to 0$ as $t \to \infty$ and $||x_{i,0} - x_{j,0}|| < \psi(0) \ \forall (i,j) \in \mathcal{E}$. Then edge-wise funnel coupling yields

 $\lim x_i(t) = x^*$

⁵ J.G. Lee, T. Berger, S. Trenn and H. Shim. Proc. European Control Conference (ECC) 2020.

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Main steps of proof

➡ Skip Proof

Step 1: Show that solution of closed loop exists on $[0,\infty)$, in particular

$$\|x_i(t) - x_j(t)\|_{\infty} < \psi(t) \quad \forall t \ge 0$$

Step 2: Write $x_i = x_{avg} + (t_i^{\top} \otimes I_n)\tilde{x}$, where t_i^{\top} is the *i*-th row of $T(T^{\top}T)^{-1}$, T is the incidence matrix of a spanning tree and

$$x_{\mathsf{avg}} = \frac{1}{N} \sum_{i=1}^{N} x_i$$
 and $\widetilde{x} = (T^{\top} \otimes I_n) (x_1^{\top}, \dots, x_N^{\top})^{\top}$

Then $\|\tilde{x}(t)\|_{\infty} < \psi(t)$, in particular, $x_i(t) \to x_{avg}(t)$

Step 3: Use Lyapunov function $V(x_{avg}) := F(x_{avg}) - F(x^*)$ to show convergence of x_{avg} towards x^* .

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Convergence rate

Average dynamics = gradient descent

$$\dot{s} = -\frac{1}{N} \sum_{i=1}^{N} \nabla F_i(s)$$

Convergence towards \boldsymbol{x}^* not influenced by choice of coupling rule

Edge-wise funnel coupling

$$x_i(t) - s(t) o 0$$
 as $t o \infty$

Convergence rate (and transient behavior) directly influenceable by choice of ψ

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Example: Distributed Least-square solver

Problem

Find least-square solution of Ax = b, where $A \in \mathbb{R}^{M \times n}$ with $M = \sum_{i=1}^{N} m_i$. Then

$$\frac{1}{2} \|Ax - b\|_2^2 = \sum_{i=1}^N \frac{1}{2} \|A_i x - b_i\|_2^2$$

where $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$ are the corresponding block rows of A and b.

Local gradient descent with funnel coupling

$$\dot{x}_{i} = -A_{i}^{\top}(A_{i}x_{i} - b_{i}) - \sum_{j \in \mathcal{N}_{i}} \left(\frac{e_{ij}^{1}}{\psi - |e_{ij}^{1}|}, \frac{e_{ij}^{2}}{\psi - |e_{ij}^{2}|}, \dots, \frac{e_{ij}^{n}}{\psi - |e_{ij}^{n}|}\right)^{\top}$$

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Decentralized optimization

Simulation

For
$$A = [1, 1, 2, 2, 1]^{\top}$$
 and $b = [1, 10, 20, 18, 100]^{\top}$,
 $x_1(0) = 0, x_2(0) = -x_3(0) = 0.1, x_4(0) = -x_5(0) = 0.2, \psi(t) = \exp(-0.8t)$ and a line coupling we obtain

Average dynamics = global gradient descent







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Synchronization via funnel coupling with application for decentralized optimization (30 / 31)



Decentralized optimization

Simulation

For
$$A = [1, 1, 2, 2, 1]^{\top}$$
 and $b = [1, 10, 20, 18, 100]^{\top}$,
 $x_1(0) = 0, x_2(0) = -x_3(0) = 0.1, x_4(0) = -x_5(0) = 0.2,$
 $\psi(t) = \exp(-0.8t)$ and a line coupling we obtain

Average dynamics = global gradient descent



Edgewise Funnel coupling



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Conclusion

From high gain coupling to node-wise funnel coupling

- > High gain coupling leads to practical synchronization
- > Replacing constant gain by funnel gain also leads to practical synchronization
- > Limit trajectory not simple average anymore
- > Choice of funnel gain function \leadsto design of limit trajectory

Edge-wise funnel coupling

- > Edge-wise funnel coupling --- Average limit trajectory
- > Many possible applications

Outlook

- > Decentralized observers
- > Formation control with obstacle avoidence