Distributional restriction impossible to define

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Abstract

A counterexample is presented showing that it is not possible to define a restriction for distributions.

1. Introduction

For a scalar function $f : \mathbb{R} \to \mathbb{R}$ it is straightforward to define a restriction (or truncation) to an interval $\mathcal{I} \subseteq \mathbb{R}$ as follows

$$f_{\mathcal{I}}(t) := \begin{cases} f(t), & t \in \mathcal{I}, \\ 0, & t \notin \mathcal{I}. \end{cases}$$

In fact, this restriction can be defined for any subset $\mathcal{I} \subseteq \mathbb{R}$ and not just for intervals. In contrast to the domain-changing restriction (usually denoted by $f|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}$) the above defined restriction is still defined on the whole domain. This has the major advantage that the vector space properties of the corresponding function space remains intact, in particular, functions restricted to different intervals can be added in the usual way. For example, with this definition it is very easy to define the set of piecewise-smooth functions simply as the set of all functions which are the (locally finite) sum of smooth functions each of which is restricted to an interval.

An important property of this function restriction is that for an interval $\mathcal{I}$ which is the disjoint union of two smaller intervals, i.e. $\mathcal{I} = \mathcal{I}_1 \dot\cup \mathcal{I}_2$, it holds

$$f_{\mathcal{I}} = f_{\mathcal{I}_1} + f_{\mathcal{I}_2};$$

or more general, if $\mathcal{I}$ is the countable union of pairwise disjoint intervals, i.e., $\mathcal{I} = \bigcup_{i\in\mathbb{N}} \mathcal{I}_i$, then

$$f_{\mathcal{I}} = \sum_{i\in\mathbb{N}} f_{\mathcal{I}_i}.$$

In the context of inconsistent initial values for differential-algebraic equations (DAEs) it was observed that it is necessary to consider solutions in a more general solution space including Dirac impulses (and their derivatives thereof) together with a well defined restriction operator to intervals, for details see e.g. the survey [10].

Since Dirac impulses are elements of Schwartz' distribution space [6], this motivates the general question: *Is it possible to define a restriction of distributions to intervals?*

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[1] The author was supported by the NWO vidi grant 639.032.733.

To be more precise: Let $\mathbb{D}$ denote the space of distribution (see Section 2 for the formal definition and recollection of important properties) and $\mathcal{I} \subseteq \mathbb{R}$ be some interval, is there a restriction map

$$\mathcal{R}_{\mathcal{I}} : \mathbb{D} \to \mathbb{D}, \quad D \mapsto D_{\mathcal{I}}$$

which satisfies some natural properties?

The somewhat surprising answer to the above question is: NO. This can be seen by considering the following *counterexample*, which is a "bad" distribution which cannot be restricted to the interval $(0, \infty)$.

Counterexample. *For $n \in \mathbb{N}_{>0}$ let $d_n := (-1)^n/n$ and*

$$D_n := \sum_{i=1}^{n} d_i \delta_{d_i},$$

where δ_{d_i} is the Dirac impulse with support at d_i, see also Figure 1. Then

$$D := \lim_{n\to\infty} D_n \tag{1}$$

is a distribution for which the restriction $D_{(0,\infty)}$ cannot be well defined.

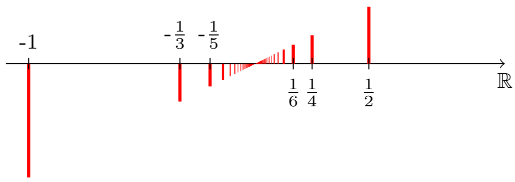

Figure 1: Illustration of 'bad' distribution $D = \sum_{i=1}^{\infty} d_i \delta_{d_i}$, where a Dirac impulse is pictured as a red line whose (directed) length is the corresponding magnitude.

The problem of a distributional restriction was investigated in the authors PhD-thesis [7] in the context of inconsistent initial values for differential-algebraic equations and some of the conclusions without complete proofs have appeared in [8] and in the survey [10].

However, the answer to the question whether there exists a well-defined distributional restriction in the form of a nice counter example (including the full technical details) has not appeared elsewhere, but may be of interest to the general mathematical audience.

The remainder of this note will provide the corresponding details to back up this claim; in particular, 1) that D as defined above is indeed a distribution, 2) formulating the "natural" properties of a restriction and, finally, 3) showing that the restriction of D to the interval $(0, \infty)$ cannot satisfy these natural properties of a restriction and at the same time is a well defined distribution. Afterwards some possibilities to avoid this dilemma are presented.

2. Preliminaries: Distribution theory

Following Schwartz [6] the space of of distribution $\mathbb{D}$ is defined as the dual space of the space of test functions $\mathcal{C}_0^\infty$, i.e.

$$\mathbb{D} := \{D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous}\}$$

where $\mathcal{C}_0^\infty$ is the space of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are smooth (arbitrarily often differentiable) and have compact support, i.e. $\operatorname{supp}\varphi := \overline{\{t \in \mathbb{R} \mid \varphi(t) \neq 0\}}$ is bounded. For a proper definition of continuity it is necessary to specify a topology on $\mathcal{C}_0^\infty$, however, in the following this topology will not be used, instead the following well known characterization of continuity of a linear map $D : \mathcal{C}_0^\infty \to \mathbb{R}$ will be used:

Lemma 1 (see e.g. [4, Sätze 12.7 and 14.5])**.** *A linear map $D : \mathcal{C}_0^\infty \to \mathbb{R}$ is continuous if, and only if, $\lim_{n\to\infty} D(\varphi_n) = 0$ for all sequences $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty$ which converge to zero in the following sense:*

(C_1) $\exists$ compact $K \subseteq \mathbb{R}\ \forall n \in \mathbb{N}$: $\operatorname{supp}\varphi_n \subseteq K$ and

(C_2) $\forall i \in \mathbb{N}$: $\lim_{n\to\mathbb{N}} \left\|\varphi_n^{(i)}\right\|_\infty = 0$ where $\|\cdot\|_\infty$ denotes the supremum norm.

All locally integrable functions $f : \mathbb{R} \to \mathbb{R}$ induce a distribution given by

$$f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \int_\mathbb{R} \varphi f;$$

the space of all such induced distributions is called *regular distributions*. This embedding in the form of an injective homomorphism $f \mapsto f_\mathbb{D}$ of a fairly large function space into the space of distribution is also the reason why distributions are also called *generalized functions*.

The most famous non-regular distribution is the *Dirac impulse* given by

$$\delta : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(0),$$

or, more general, the Dirac impulse at some $t \in \mathbb{R}$:

$$\delta_t : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t).$$

Note that $\operatorname{supp}\delta_t = \{t\}$ where the support of a general distribution $D \in \mathbb{D}$ is defined to be the complement of the union of all open sets on which D vanishes, i.e.

$$\operatorname{supp} D := \mathbb{R}\Big\backslash\bigcup\left\{O \subseteq \mathbb{R} \;\middle|\; \begin{array}{l} O \text{ open and } D(\varphi) = 0 \\ \forall\varphi \in \mathcal{C}_0^\infty \text{ with } \operatorname{supp}\varphi \subset O \end{array}\right\}.$$

By definition, the support of a distribution is always a closed set.

The main advantage of distribution (and the reason they play such an important role in differential equations) is the fact, that they are arbitrarily often differentiable, where the derivative D' of a distribution $D \in \mathbb{D}$ is given by

$$D'(\varphi) := -D(\varphi').$$

This differentiation rule is motivated by the partial integration rule for functions, in fact, for any differentiable function $f : \mathbb{R} \to \mathbb{R}$ it holds that

$$(f_\mathbb{D})' = (f')_\mathbb{D}.$$

It is easily seen that the Dirac impulse is the derivative of the Heaviside step function $\mathbb{1}_{[0,\infty)}$.

Another important property of distribution is the fact that for a sequence of distributions (i.e. a sequence of linear and continuous operators) pointwise converges already implies that the limit operator is again linear and continuous, i.e. a distribution.

Lemma 2 (see e.g. [4, Sätze 28.1, 28.2 and 28.3])**.** *Consider a sequence $(D_n)_{n\in\mathbb{N}}$ of distributions for which the limit $D(\varphi) := \lim_{n\to\infty} D_n(\varphi)$ exists for all $\varphi \in \mathcal{C}_0^\infty$. Then $D \in \mathbb{D}$.*

The section concludes with proving that the Counterexample is indeed a distribution.

Lemma 3. *The limit D given by* (1) *is a distribution.*

Proof. Due to Lemma 2 it suffices to show that for every $\varphi \in \mathcal{C}_0^\infty$ the sequence $D_n(\varphi) = \sum_{i=1}^n d_i\varphi(d_i)$ converges to a finite value in $\mathbb{R}$ as $n \to \infty$. Invoking the Mean-Value Theorem, there exists for any $\varphi \in \mathcal{C}_0^\infty$ a sequence $(\xi_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\varphi(d_i) = \varphi(0) + d_i\varphi'(\xi_i).$$

Consequently,

$$D_n(\varphi) = \varphi(0)\sum_{i=1}^n d_i + \sum_{i=1}^n \varphi'(\xi_i)d_i^2.$$

Due to the Leibniz' alternating series test, $\sum_{i=1}^n d_i = \sum_{i=1}^n (-1)^i/i$ converges to a finite value in $\mathbb{R}$ as $n \to \infty$. Furthermore, it is well known that $\sum_{i=1}^n d_i^2 = \sum_{i=1}^n 1/i^2$ converges absolutely to $\pi^2/6$, which implies that the sum $\sum_{i=1}^n \varphi'(\xi_i)d_i^2$ converges absolutely, because ψ' is bounded and

$$\sum_{i=1}^n \left|\varphi'(\xi_i)d_i^2\right| \leq \|\varphi'\|_\infty \pi^2/6.$$

$\square$

3. Desired properties of distributional restriction

In this section some desired properties of a distributional restriction $\mathcal{R}_\mathcal{I} : \mathbb{D} \to \mathbb{D}, D \mapsto D_\mathcal{I}$ for any interval $\mathcal{I} \subseteq \mathbb{R}$ are formulated:

$\mathbf{R}_1$ The following implications hold for all $D \in \mathbb{D}$ and all $\varphi \in \mathcal{C}_0^\infty$:

$$\begin{aligned} &\text{(i)} \ \operatorname{supp} D \cap \mathcal{I} = \emptyset \implies D_\mathcal{I} = 0,\\ &\text{(ii)} \ \operatorname{supp} \varphi \subseteq \mathcal{I} \implies D_\mathcal{I}(\varphi) = D(\varphi),\\ &\text{(iii)} \ \operatorname{supp} \varphi \cap \mathcal{I} = \emptyset \implies D_\mathcal{I}(\varphi) = 0,\\ &\text{(iv)} \ \operatorname{supp} \varphi \cap \operatorname{supp} D = \emptyset \implies D_\mathcal{I}(\varphi) = 0. \end{aligned}$$

$\mathbf{R}_2$ Let $\mathcal{I} = \bigcup_{i\in\mathbb{N}} \mathcal{I}_i$ the *pairwise disjoint* union of a countable family of intervals, then, for any $D \in \mathbb{D}$,

$$D_\mathcal{I} = \sum_{i=1}^{\infty} D_{\mathcal{I}_i};$$

in particular,

$$D_{\mathcal{I}_1 \dot\cup \mathcal{I}_2} = D_{\mathcal{I}_1} + D_{\mathcal{I}_2}.$$

If such a distributional restriction exists, it is possible to conclude the following important property concerning the restriction of Dirac impulses to an interval.

Lemma 4. *For any interval $\mathcal{I} \subseteq \mathbb{R}$ and any $t \in \mathbb{R}$ it follows that*

$$(\delta_t)_\mathcal{I} = \begin{cases} \delta_t, & t \in \mathcal{I} \\ 0, & t \notin \mathcal{I} \end{cases}$$

Proof. Consider first the case that $t \notin \mathcal{I}$. Then $\operatorname{supp} \delta_t \cap \mathcal{I} = \{t\} \cap \mathcal{I} = \emptyset$ and from $\mathbf{R}_1$(i) it follows that $(\delta_t)_\mathcal{I} = 0$. If $t \in \mathcal{I}$ choose two (unbounded or empty) intervals $\mathcal{I}_l, \mathcal{I}_r \subseteq \mathbb{R}$ such that $\mathbb{R} = \mathcal{I}_l \dot\cup \mathcal{I} \dot\cup \mathcal{I}_r$, then by $\mathbf{R}_1$(ii) $\delta_t = (\delta_t)_\mathbb{R} = (\delta_t)_{\mathcal{I}_l \dot\cup \mathcal{I} \dot\cup \mathcal{I}_r}$, which implies by $\mathbf{R}_2$ that

$$(\delta_t)_\mathcal{I} = \delta_t - (\delta_t)_{\mathcal{I}_l} - (\delta_t)_{\mathcal{I}_r}.$$

Since by construction $t \notin \mathcal{I}_l$ and $t \notin \mathcal{I}_2$ it follows as above that $(\delta_t)_{\mathcal{I}_l} = 0 = (\delta_t)_{\mathcal{I}_r}$ and the proof is complete. ☐

Remark 1. *In [7] some more desired properties for a distributional restriction are formulated, for example the property that $\mathcal{R}_\mathcal{I} : \mathbb{D} \to \mathbb{D}$ is a projector (i.e. linear and idempotent) and that for regular distributions $f_\mathbb{D}$ the distributional restriction generalizes the function restriction, i.e.*

$$(f_\mathcal{I})_\mathbb{D} = (f_\mathbb{D})_\mathcal{I}.$$

However, a careful analysis of the upcoming proof of nonexistence reveals that it is indeed enough to require the properties $\mathbf{R}_1$ and $\mathbf{R}_2$ to arrive at a contradiction. On the other hand, conditions $\mathbf{R}_1$(i) and $\mathbf{R}_1$(iv) are not mentioned in [7], but they seem to be needed to arrive at the non-existence result.

Remark 2. *Property $\mathbf{R}_2$ involves a limiting process, however, it is important to note that this limit is with respect to the domain and does not correspond to a sequence of distributions. In fact, one may be inclined to require the following property for a converging sequence of distributions $(D_n)_{n\in\mathbb{N}}$ with a limit D as in Lemma 2:*

$$D_\mathcal{I} = \lim_{n\to\infty} (D_n)_\mathcal{I}.$$

But this requirement immediately runs into a contradiction, because it is easy to see that both of the two sequences $\big((f_n^r)_\mathbb{D}\big)_{n\in\mathbb{N}}$ and $\big((f_n^l)_\mathbb{D}\big)_{n\in\mathbb{N}}$ given by

$$f_n^r(t) := \begin{cases} n, & t \in (0, 1/n) \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$f_n^l(t) := \begin{cases} n, & t \in (-1/n, 0) \\ 0, & \textit{otherwise}, \end{cases}$$

converge to the Dirac impulse δ. However, $(f_n^r)_{(0,\infty)} = f_n^r$ and $(f_n^l)_{(0,\infty)} = 0$, so the limit of the restriction would in one case be δ and in the other case zero.

4. Restriction for counterexample not possible

Finally, it will now be shown, that a restriction satisfying the properties given in Section 3 does not exist.

Theorem 1. *A distributional restriction to any interval $\mathcal{I} \subseteq \mathbb{R}$ of the form $\mathcal{R}_\mathcal{I} : \mathbb{D} \to \mathbb{D}, D \mapsto D_\mathcal{I}$ satisfying $\mathbf{R}_1$ and $\mathbf{R}_2$ does not exist.*

Proof. Consider the 'bad' distribution D given by the Counterexample and the interval $\mathcal{I} = (0, \infty)$.
Step 1: It is shown, that if $D_\mathcal{I}$ is well defined, than $D_\mathcal{I} = \sum_{k=1}^\infty d_{2k} \delta_{2k}$.
Step 1a: In order to utilize $\mathbf{R}_2$ a suitable family $(\mathcal{I}_k)_{k\in\mathbb{N}}$ is defined.

For $k \in \mathbb{N}_{>0}$ let

$$\mathcal{I}_k := \big[\tfrac{1}{2}(d_{2(k+1)} + d_{2k}), \tfrac{1}{2}(d_{2k} + d_{2(k-1)})\big),$$

with the convention that $d_0 := +\infty$. Then $\mathcal{I} = \bigcup_{k\in\mathbb{N}} \mathcal{I}_k$ and $d_i \in \mathcal{I}_k$ if, and only if, $i = 2k$. In particular, by Lemma 4, $(\delta_{2k})_{\mathcal{I}_k} = \delta_{2k}$ and $(\delta_i)_{\mathcal{I}_k} = 0$ for all $i \neq 2k$ and all $k \in \mathbb{N}$.
Step 1b: It is shown that $D_{\mathcal{I}_k} = d_{2k}\delta_{2k}$.

Consider an arbitrary $\varphi \in \mathcal{C}_0^\infty$; it must be shown that $D_{\mathcal{I}_k}(\varphi) = d_{2k}\varphi(2k)$. Decompose φ as $\varphi = \varphi_\text{in} + \varphi_\text{out} + \varphi_\text{rest}$ where

$$\begin{aligned} \varphi(d_{2k}) &= \varphi_\in(d_{2k}),\\ \operatorname{supp} \varphi_\text{in} &\subseteq \mathcal{I}_k,\\ \operatorname{supp} \varphi_\text{out} \cap \mathcal{I}_k &= \emptyset,\\ \operatorname{supp} \varphi_\text{rest} \cap \operatorname{supp} D &= \emptyset. \end{aligned}$$

These requirements can easily be achieved by choosing $\varphi_\text{in} = \mathbb{1}_\text{in}\varphi$, $\varphi_\text{out} = \mathbb{1}_\text{out}\varphi$, $\varphi_\text{rest} = \mathbb{1}_\text{rest}\varphi$, where the

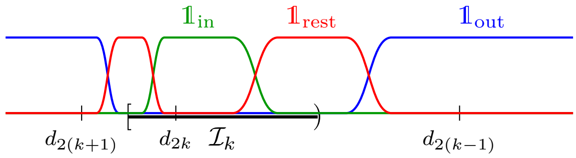

Figure 2: Illustration of $\mathbb{1}_{\text{in}}$ (green), $\mathbb{1}_{\text{out}}$ (blue) and $\mathbb{1}_{\text{rest}}$ (red).

smooth functions $\mathbb{1}_{\text{in}}$, $\mathbb{1}_{\text{out}}$, $\mathbb{1}_{\text{rest}}$ add up to identical one and are chosen as illustrated in Figure 2.

It then follows that

$$D_{\mathcal{I}_k}(\varphi_{\text{in}}) \overset{\mathbf{R}_1(\text{ii})}{=} D(\varphi_{\text{in}}) = d_{2k}\varphi_{\text{in}}(d_{2k}) = d_{2k}\varphi(d_{2k}),$$
$$D_{\mathcal{I}_k}(\varphi_{\text{out}}) \overset{\mathbf{R}_1(\text{iii})}{=} 0,$$
$$D_{\mathcal{I}_k}(\varphi_{\text{rest}}) \overset{\mathbf{R}_1(\text{iv})}{=} 0.$$

Hence, due to linearity, $D_{\mathcal{I}_k}(\varphi) = d_{2k}\varphi(d_{2k})$ which is the claim of Step 1b.

Step 1c: The claim of Step 1 is shown.

Invoking $\mathbf{R}_2$ for the disjoint countable family of intervals $(\mathcal{I}_k)_{k\in\mathbb{N}}$ it now follows that $D_{\mathcal{I}} = \sum_{k=1}^{\infty} d_{2k}\delta_{2k}$.

Step 2: It is shown that $\sum_{k=1}^{\infty} d_{2k}\delta_{2k}$ is not a distribution.

Consider a test function $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi(t) = 1$ for all $t \in [0, 1/2]$. Then

$$D_{\mathcal{I}}(\varphi) = \sum_{k=1}^{\infty} d_{2k} \underbrace{\varphi(d_{2k})}_{=1} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

This shows that $D_{\mathcal{I}}$ cannot be a distribution. □

5. Resolving the dilemma

As mentioned above one motivation for studying a distributional restriction is the problem of inconsistent initial values for differential-algebraic equations (DAEs) of the form

$$E\dot{x} = Ax + f \tag{2}$$

where $E, A \in \mathbb{R}^{m\times n}$ and $f : \mathbb{R} \to \mathbb{R}^n$ is some inhomogeneity. By definition, an inconsistent initial value $x(0^-)$ for (2) can only occur when the past is not governed by the DAE (2), this intuition can be formalized by the following initial trajectory problem (ITP):

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax+f)_{[0,\infty)}, \end{aligned} \tag{3}$$

where $x^0 : \mathbb{R} \to \mathbb{R}^n$ is a given past trajectory. It was observed in the context of electric circuits [11] that an inconsistent initial value should result in a Dirac impulse in the solution; the presence of a Dirac impulse in a solution in response to an inconsistent initial value can also be motivated by considering a limiting process [1]. Hence a rigorous solution framework for (2) needs to consider *distributional* solutions x and a well-defined restriction operator.

The dilemma that it is not possible to define a distributional restriction operator (which is necessary so that the expression used in (3) are actually well defined objects), can be resolved in the context of DAEs in two ways.

5.1. Comparing distributions on intervals

As mentioned in the introduction, instead of considering a restriction operator which results in a distribution again defined on the whole space, one could also consider a restriction which restricts the domain of the operator, i.e. for some interval $\mathcal{I}$ and some $D \in \mathbb{D}$ consider the domain-changing restriction $D\big|_{\mathcal{I}}$ as follows

$$D\big|_{\mathcal{I}} : \{\varphi \in \mathcal{C}_0^\infty \mid \operatorname{supp}\varphi \subseteq \mathcal{I}\} \to \mathbb{R}, \quad \varphi \mapsto D(\varphi).$$

Now the ITP could be reformulated as

$$\begin{aligned} x\big|_{(-\infty,0)} &= x^0\big|_{(-\infty,0)} \\ (E\dot{x})\big|_{[0,\infty)} &= (Ax+f)\big|_{[0,\infty)}. \end{aligned} \tag{4}$$

The problem with this approach is that there is no difference between a restriction to an open or closed interval, in particular, $\delta\big|_{[0,\infty)} = 0$ which is in many situations an undesired result and also prevents a suitable distributional solution theory for DAEs. This problem was resolved in [5] by redefining the inhomogeneity to

$$f_{\text{ITP}} := (E\dot{x}^0 - Ax^0)_{(-\infty,0)} + f_{[0,\infty)}$$

and considering the reformulated ITP

$$\begin{aligned} x\big|_{(-\infty,0)} &= x^0\big|_{(-\infty,0)} \\ E\dot{x} &= Ax + f_{\text{ITP}}. \end{aligned} \tag{5}$$

Under the assumption that x_0 and f are such that f_{ITP} is well defined (as a function), all expressions in the ITP (5) are now well defined. However, in the context of switched DAEs (see e.g. [9]), which can be interpreted as a family of repeated inconsistent initial value problems, the assumption that x^0 is *not* a distribution (so that the restriction to the interval $(-\infty, 0)$ is well defined) is too restrictive in general.

5.2. Considering a subspace of distributions

The underlying problem for the non-existence of a distributional restriction is the fact, that the space of distribution is just too big and contains very 'nasty' objects (including the Counterexample). To resolve this issue, it was suggested in [2] to introduce the space of piecewise-continuous distributions which can be understand as the subspace of distributions which are composes of a piecewise-continuous function and Dirac-impulse (and their derivatives) at isolated time points. In particular, an accumulation of Dirac impulse as in the Counterexample is excluded. A similar idea was proposed in [5] where the space of impulsive-smooth distributions is proposed for studying

DAEs[2]; however, Dirac impulses (and their derivates) are only allowed at $t = 0$, and although the generalization to more location is mentioned, the details are not worked out (in particular, the Counterexample is not formally ruled out). The PhD-thesis [7] combines all the different approaches and proposes the space of piecewise-smooth distributions

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \text{ is piecewise-smooth,} \\ T \subseteq \mathbb{R} \text{ is discrete} \\ \forall t \in T : \operatorname{supp} D_t \subseteq \{t\} \end{array} \right\}$$

for which a distributional restriction can be defined in a straightforward way for $D = f_{\mathbb{D}} + \sum_{t\in T} D_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ as

$$D_{\mathcal{I}} := (f_{\mathcal{I}})_{\mathbb{D}} + \sum_{t\in T\cap\mathcal{I}} D_t.$$

In addition to the desired properties of a distributional restriction discussed in Section 3, it also satisfies the following nice property for all *open* intervals and all $F, G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$:

$$F_{\mathcal{I}} = G_{\mathcal{I}} \quad \Longleftrightarrow \quad F\big|_{\mathcal{I}} = G\big|_{\mathcal{I}}.$$

Furthermore, the space $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is closed under differentiation (similar to the space of impulsive-smooth distributions as in [5], but in contrast to the space of piecewise-continuous distributions as introduced in [2]), hence it inherits a crucial property of the space of distributions, which made them so attractive as a solution space for differential equations in the first place.

6. Conclusion

After formulating some desired properties of a distributional restriction it was shown via a counterexample that it is impossible to define a distributional restriction satisfying these properties. It was also briefly discussed how this dilemma could be resolved in the context of differential-algebraic equations and inconsistent initial values.

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