Impulse-free interval-stabilization of switched differential algebraic equations

Paul Wijnbergen*, Stephan Trenn

Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence
University of Groningen, Groningen, the Netherlands

Abstract

In this paper stabilization of switched differential algebraic equations is considered, where Dirac impulses in both the input and the state trajectory are to be avoided during the stabilization process. First it is shown that stabilizability of a switched DAE and the existence of impulse-free solutions are merely necessary conditions for impulse-free stabilizability. Then necessary and sufficient conditions for the existence of impulse-free solutions are given, which motivate the definition of (impulse-free) interval-stabilization on a finite interval. Under a uniformity assumption, which can be verified for a broad class of switched systems, stabilizability on an infinite interval can be concluded based on interval-stabilizability. As a result a characterization of impulse-free interval stabilizability is given and as a corollary we provide a novel impulse-free null-controllability characterization. Finally, the results are compared to results on interval-stabilizability where Dirac impulses are allowed in the input and state trajectory.

Keywords: Switched Systems, Differential Algebraic Equations, Stabilizability, Controllability, Impulsive behavior

1. Introduction

In this paper we consider *switched differential algebraic equations* (switched DAEs) of the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p, n, m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in [16]. We study impulse-free stabilizability of (1) where impulse-free stabilizabilty means the ability to find for each initial value a control signal such the state converges towards zero and remains impulse free (see the forthcoming Definition 12).

Differential algebraic equations (DAEs) arise naturally when modeling physical systems with certain algebraic constraints on the state variables. These constraints are often eliminated such that the system is described by ordinary differential equations (ODEs). Examples of applications of DAEs in electrical circuits (with distributional solutions) can be found in [15]. However, in the case of switched systems, the elimination process of the constraints is in general different for each individual mode. Therefore there does not exist a description as a switched ODE with a common state variable for every mode in general. This problem can be overcome by studying switched DAEs directly.

Ever since control systems have been considered, the question whether the control objective can be achieved with minimal (quadratic) cost has been of great interest. In the non switched case, optimal control of DAEs has been studies in e.g. [5, 1, 13]. It is proven in both [5] and [1] that stabilizability is a necessary condition for the existence of finite (quadratic) cost regardless of the initial condition, whenever a infinite horizon is considered. For if the state does not tend to zero one can not expect to have finite cost over an infinite time interval. This argument is independent of the underlying system model and hence the state of a switched DAE needs to converge to zero as well in order to achieve finite quadratic cost. Therefore, there is a need for a characterization of all switched DAEs that are stabilizable. Note that here we assume that the switching signal is fixed (i.e. (1) is viewed as a time-varying linear system), in particular, the switching signal is *not* considered to be an (additional) control input.

Several other structural properties of (switched) DAEs have been studied recently. Among those are (impulse-) controllability [8, 20] , stability [11] and observability [9]. However, stabilizability has thus far only been studied in the non-switched case in [6, 10, 3] and in the switched case in [19], where impulse-freeness of solutions was not required.

The aim of this paper is to state necessary and sufficient conditions for impulse-free stabilizability of switched DAE. Stabilizability and impulse controllability are obvious necessary conditions. However, these conditions are

*Corresponding author
Email address: `p.wijnbergen@rug.nl` (Paul Wijnbergen)

not sufficient; this is illustrated by the following example.

$$\text{on } [0,t_1): \quad \dot{x}(t) = \left[\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right] u(t) \qquad \text{on } [t_1,\infty): \quad \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \dot{x}(t) = x(t)$$

This system is impulse controllable since any initial condition can be steered to the impulse controllable space of the second mode. Furthermore, the system is controllable and hence stabilizable. However, for the initial value $x_0 = [1\ 0\ 1]^\top$ there does not exist an input which can steer the state to zero and simultaneously keeps the state impulse-free. Hence this is a system which is both stabilizable and impulse controllable, but not impulse-free stabilizable.

The outline of the paper is as follows: notations and results for non-switched DAEs are presented in Section II. ...

2. Mathematical Preliminaries

2.1. Properties and definitions for regular matrix pairs

In the following, we consider *regular* matrix pairs (E, A), i.e. for which the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form* [2].

Proposition 1. *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.

The matrices S and T can be calculated by using the so-called *Wong sequences* [2, 21]:

$$\begin{array}{lll} \mathcal{V}_0 := \mathbb{R}^n, & \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), & i = 0, 1, ... \\ \mathcal{W}_0 := \{0\}, & \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), & i = 0, 1, ... \end{array} \tag{3}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup \mathcal{W}_i. \tag{4}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (2) holds.

Based on the Wong sequences we define the following projectors and selectors.

Definition 2. *Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form* (2). *The* consistency projector *of (E, A) is given by*

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the differential selector *is given by*

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

and the impulse selector *is given by*

$$\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A = T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B,$$
$$E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E = T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}, \quad B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B.$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). An important feature for DAEs is the so called consistency space, defined as follows:

Definition 3. *Consider the DAE $E\dot{x}(t) = Ax(t) + Bu(t)$, then the* consistency space *is defined as*

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } x \text{ of} \\ E\dot{x} = Ax, \text{ with } x(0) = x_0 \end{array} \right\},$$

and the augmented consistency space *is defined as*

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solutions } (x,u) \text{ of} \\ E\dot{x} = Ax + Bu \text{ and } x(0) = x_0 \end{array} \right\}.$$

In order to express (augmented) consistency spaces in terms of the Wong limits we introduce the following notation for matrices A,B of conformable sizes:

$$\langle A \mid B \rangle := \operatorname{im}[B, AB, \ldots, A^{n-1}B].$$

Proposition 4 ([4]). *Consider the DAE $E\dot{x} = Ax + Bu$ and assume the matrix pair (E, A) is regular, then $\mathcal{V}_{(E,A)} = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\text{diff}}_{(E,A)}$ and $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.*

2.2. Distributional Solutions

The switched DAE (1) usually will not have classical solutions, because each mode of the switched DAE given by the DAE $E_i\dot{x} = A_i x + B_i u$ might have different (augmented) consistency spaces which enforce jumps in the state-variable x. We therefore utilize the piecewise-smooth distributional framework as introduced in [16], i.e. x and u are vectors of piecewise-smooth distributions given by

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, T \subseteq \mathbb{R} \text{ is} \\ \text{discrete}, \forall t \in T: D_t \\ \in \operatorname{span}\{\delta_t, \delta'_t, \delta'''_t, ...\} \end{array} \right\},$$

where $\mathcal{C}^\infty_{\text{pw}}$ denotes the space of piecewise-smooth functions, $f_\mathbb{D}$ denotes the regular distribution induced by f and δ_t denotes the Dirac impulse with support $\{t\}$. For a piecewise smooth distribution $D = f_\mathbb{D} + \sum_{t\in T} D_t \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ three types of "evaluation at time t" are defined: left side evaluation $D(t^-) := f(t^-)$, right side evaluation $D(t^+) := f(t^+)$ and the impulsive part $D[t] := D_t$ if $t \in T$ and $D[t] = 0$ otherwise.

It can be shown (see e.g. [17]) that the space $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ can be equipped with a multiplication, in particular, the multiplication of a piecewise-constant function with a piecewise-smooth distribution is well defined and the switched DAE (1) can be interpreted as an equation within the space of piecewise-smooth distributions. Hence the following solution behavior (depending on σ) is well defined:

$$\mathfrak{B}_\sigma := \{(x,u) \in \mathbb{D}^{n+m}_{\text{pw}\mathcal{C}^\infty} \mid E_\sigma \dot{x} = A_\sigma x + B_\sigma u\},$$

and restrictions of x and u to intervals are well defined as well. In [16] it is shown that the ITP (5)

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{5a}$$

$$(E\dot{x})_{[0,\infty)} = (Ax)_{[0,\infty)} + (Bu)_{[0,\infty)}, \tag{5b}$$

has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. As a direct consequence, the switched DAE (1) with regular matrix pairs is also uniquely solvable (with piecewise-smooth distributional solutions) for any switching signal with locally finitely many switches.

For many applications solutions where impulses are absent are of relevance. This gives rise to the following definition.

Definition 5. *Consider the switched DAE (1) and let (x,u) be a distributional solution. The solution (x,u) is called impulse-free if $x[t] = 0$ and $u[t] = 0$ for all $t \geqslant 0$.*

Note that impulse-free solutions may still contain jumps and hence such solutions are not necessarily solutions in the classical sense.

2.3. Properties of DAE's

For the rest of this section we are considering the DAE

$$E\dot{x} = Ax + Bu. \tag{6}$$

Recall the following definitions and characterization of (impulse) controllability [4].

Proposition 6. *The reachable space of the regular DAE (6) defined as*

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0\ \exists \text{ smooth solution } (x,u) \text{ of } (6) \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\}$$

satisfies $\mathcal{R} = \langle A^{\text{diff}} \mid B^{\text{diff}} \rangle \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

It is easily seen that the reachable space for (6) coincides with the (null-)controllable space, i.e.

$$\mathcal{R} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0\ \exists \text{ smooth solution } (x,u) \text{ of } (6) \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Corollary 7. *The augmented consistency space of (6) satisfies $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}}, B^{\text{imp}} \rangle$.*

Definition 8. *The DAE (6) is impulse controllable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x,u) of the ITP (5) such that $x(0^-) = x_0$ and $(x,u)[0] = 0$, i.e. the state and the input are impulse free at $t = 0$. The space of impulse controllable states of the DAE (6) is given by*

$$\mathcal{C}^{\text{imp}}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \in \mathbb{D}_{pw\mathcal{C}^\infty} \text{ of } (5) \\ \text{s.t. } x(0^-) = x_0 \text{ and } (x,u)[0] = 0. \end{array} \right\}.$$

In particular, the DAE (6) is impulse controllable if and only if $\mathcal{C}^{\text{imp}}_{(E,A,B)} = \mathbb{R}^n$.

The impulse controllable space can be characterized as follows [12].

Proposition 9. *Consider the DAE (6) then*

$$\begin{aligned} \mathcal{C}^{\text{imp}}_{(E,A,B)} &= \mathcal{V}_{(E,A,B)} + \ker E \\ &= \mathcal{V}_{(E,A)} + \mathcal{R} + \ker E \\ &= \mathcal{V}_{(E,A)} + \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle + \ker E. \end{aligned}$$

Definition 10. *The DAE (6) is stabilizable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x,u) of the ITP (5) such that $x(0^-) = x_0$ and $lim_{t\to\infty} x(t) = 0$.*

Stabilizability of a regular DAE can be characterized as follows [7].

Proposition 11. *The DAE (6) is stabilizable if and only if*

$$\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C}^+.$$

According to [18] if the input $u(\cdot)$ is sufficiently smooth, trajectories of (6) are continuous on the open interval (t_0, ∞) and given by

$$\begin{aligned} x(t) = x_u(t,t_0;x_0) &= e^{A^{\text{diff}}(t-t_0)} \Pi_{(E,A)} x_0 \\ &+ \int_{t_0}^t e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s)\, ds - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \end{aligned} \tag{7}$$

In particular, all trajectories can be written as the sum of an autonomous part $x_{\text{aut}}(t,t_0;x_0) = e^{A^{\text{diff}} t} \Pi_{(E,A)} x_0$ and a controllable part $x_u(t,t_0)$ as follows:

$$x_u(t,t_0;x_0) = x_{\text{aut}}(t,t_0;x_0) + x_u(t,t_0).$$

This decomposition remains valid for switched DAEs when evaluated at the initial condition at time t_0^-; the impulsive part of x at the initial time t_0 is then given by

$$x[t_0] = -\sum_{i=0}^{n-1} (E^{\text{imp}})^{i+1} \left(x_0 \delta^{(i)} + \sum_{j=0}^{i} B^{\text{imp}} u^{(i-j)}(t_0^+) \delta^{(j)} \right).$$

3. Stabilizability concepts

The concepts introduced in the previous section are now utilized to investigate impulse free stabilizability of switched DAEs. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals from the following class

$$\Sigma := \left\{ \sigma : \mathbb{R} \to \mathbb{N} \;\middle|\; \begin{array}{l} \sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps} \end{array} \right\},$$

i.e. we exclude an accumulation of switching times (see [16]). By further excluding infinitely many switches in the past and by appropriately relabeling the matrices we can assume that

$$\sigma(t) = k, \qquad \text{for } t_k \leqslant t < t_{k+1}. \tag{8}$$

and that for the first switching instant t_1 it holds that $t_1 > t_0 := 0$. After some results relating interval-wise properties to global properties in the remainder of this section, we will restrict our attention to the bounded interval (t_0, t_f) for some $t_f > 0$. As a consequence there are only finitely many switches in this interval, say $\mathtt{n} \in \mathbb{N}$, and for notation convenience we let $t_{\mathtt{n}+1} = t_f$.

Roughly speaking, in classical literature on non-switched systems, a linear system is called stabilizable if every trajectory can be steered towards zero as time tends to infinity. This definition can readily applied to switched DAEs. Hence we will define impulse free stabilizability for switched DAEs in a similar fashion as follows, based on the definition of stabilizability in [19].

Definition 12 (Impulse-free Stabilizability). *The switched DAE* (1) *with switching signal* (8) *is stabilizable if the corresponding solution behavior* $\mathfrak{B}_\sigma$ *is stabilizable in the behavioral sense on the interval* $[0, \infty)$*, i.e.*

$$\begin{aligned} &\forall (x, u) \in \mathfrak{B}_\sigma \; \exists (x^*, u^*) \in \mathfrak{B}_\sigma : \\ &(x^*, u^*)_{(-\infty,0)} = (x, u)_{(-\infty,0)}, \\ &\textit{and} \; \lim_{t\to\infty} (x^*(t^+), u^*(t^+)) = 0, \end{aligned}$$

and in addition $(x^*, u^*)\,[t] = 0$ *for all* $t \in [0, \infty)$

In the case of switched DAEs, it is reasonable to assume that there are an infinite amount of switching instances as time tends to infinity. This poses a problem when it comes to verifying conditions for stabilizability in a finite amount of steps. To overcome this problem, we investigate stabilizability on a bounded interval. Therefore we introduce the following definition of impulse-free interval stabilizability.

Definition 13 (Interval stabilizabilty). *The switched DAE* (1) *is called* (t_0, t_f)*-stabilizable for a given switching signal* σ*, if there exists a class* $\mathcal{KL}$ *function*[1] $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *with*

$$\beta(r, t_f - t_0) < r, \quad \forall r > 0,$$

and for any initial value $x_0 \in \mathcal{V}_{E_0,A_0,B_0}$ *there exist a local solution* (x, u) *of* (1) *on* (t_0, t_f) *with* $x(t_0^-) = x_0$ *such that*

$$|x(t^+)| \leqslant \beta(|x_0|, t - t_0), \quad \forall t \in (t_0, t_f).$$

If in addition $(x, u)[t] = 0$ *for all* $t \in (t_0, t_f)$*, then the system is called impulse-free* (t_0, t_f)*-stabilizable.*

One should note that a solution on some interval is not necessarily a part of a solution on a larger interval. Consequently, stabilizability does not always imply interval stabilizability. The switched system $0 = x$ on $[0, t_1)$ and $\dot{x} = 0$ on $[t_1, \infty)$ is obviously stabilizable, since the only global solution is the zero solution. However, on the interval $[t_1, s)$ there are nonzero solutions which do not converge towards zero.

Furthermore according to Definition 13 it is required that the norm of the state is smaller at the end of an interval. This means that (impulse-free) interval stability could depend on the length of the interval considered instead of the asymptotic behavior of the system. An unstable oscillating system is thus possibly (impulse-free) interval stable and an asymptotically stable oscillating system is not necessarily (impulse-free) interval stable, depending on the choice of interval. However, under the following uniformity assumption on the switched DAE we can conclude global stabilizability.

Assumption 14 (Uniform interval-stabilizability). *Consider the switched system* (1) *with switching signal* σ*. Let* $\tau_0 := t_0$ *and assume that there exists a strictly increasing sequence* $\tau_i \in (t_0, \infty)$*,* $i \in \mathbb{N}_{>0}$*, of non-switching times such that the system is (impulse-free)* (τ_{i-1}, τ_i)*-stabilizable with* $\mathcal{KL}$ *function* β_i *for which additionally it holds that*

$$\begin{aligned} \beta_i(r, \tau_i - \tau_{i-1}) &\leqslant \alpha r, \quad \forall r > 0, \forall i \in \mathbb{N}_{>0} \\ \beta_i(r, 0) &\leqslant M r, \quad \forall r > 0, \forall i \in \mathbb{N}_{>0}, \end{aligned}$$

for some uniform $\alpha \in (0, 1)$ *and* $M \geqslant 1$*.*

Proposition 15. *If the switched system* (1) *is uniformly (impulse-free) interval-stabilizable in the sense of Assumption 1 then* (1) *is (impulse-free) stabilizable.*

The proof of Proposition 15 is along the same lines as the proof of Proposition 8 in [14].

4. Impulse-free stabilization and controllability

Assumption 14 can be verified for a general class of systems such as systems with periodic switching and systems with a finite amount of modes. Therefore we turn our attention to finding necessary and sufficient conditions for interval stabilizability. As follows from Definition 13, for any initial condition x_0, there needs to exist a solution on

[1]A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is called a class $\mathcal{KL}$ function if 1) for each $t \geqslant 0$, $\beta(\cdot, t)$ is continuous, strictly increasing, with $\beta(0, t) = 0$; 2) for each $r \geqslant 0$, $\beta(r, \cdot)$ is decreasing and converging to zero as $t \to \infty$.

$[t_0, t_f)$ that is impulse-free and satisfies the stability property. Hence we will first discuss necessary and sufficient conditions for a switched DAE to have impulse free solutions for any initial condition x_0 on a bounded interval, i.e. impulse controllability for switched DAEs. Once these conditions are discussed, we will investigate under which conditions these impulse-free solutions are satisfying the stability property.

In the remainder of this section we will use Π_i, A_i^{diff}, E_i^{imp}, B_i^{imp}, B_i^{diff}, $\mathcal{R}_i$, $\mathcal{C}_i$, $\mathcal{C}_i^{\text{imp}}$ to denote the corresponding matrices and subspaces associated to the i-th mode.

4.1. Impulse controllability

As mentioned above, we will first investigate the concept of impulse controllability of a switched DAE, of which the definition is formalized as follows.

Definition 16. *The switched DAE* (1) *with some fixed switching signal σ is called impulse controllable on the interval (t_0, t_f), if for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists a solution $(x, u) \in \mathbb{D}^{n+m}_{pw\mathcal{C}^\infty}$ of* (1) *with $x(t_0^+) = x_0$ which is impulse free.*

Remark 17. *As an alternative for Definition 16, impulse controllability could also be defined in terms of arbitrary initial values $x_0 \in \mathbb{R}^n$. This would result in the immediate necessary condition that the first mode of a switched DAE needs to be impulse controllable. However, given a higher index DAE, Dirac impulses can not be avoided for initial conditions in $(\mathcal{C}_0^{\text{imp}})^\perp$. Therefore it is reasonable to consider initial conditions in $\mathcal{C}_0^{\text{imp}} = \mathcal{V}_{(E_0,A_0,B_0)} + \ker E_0$. Considering the linearity of solutions and the fact that initial conditions in $\ker E_0$ result in trajectories that jump to zero in an impulse free manner, the initial conditions of interest are those contained in $\mathcal{V}_{(E_0,A_0,B_0)}$.*

Remark 18. *If the interval (t_0, t_f) does not contain a switch, then the corresponding switched DAE is* always *impulse controllable on that interval due the definition of the augmented consistency space in terms of smooth (in particular, impulse free) solutions. This seems counter intuitive, because the active mode on that interval is not necessarily impulse controllable; however, recall that impulse controllability for a single mode (see Definition 8) is formulated in terms of the ITP* (5)*, which can be interpreted as a switched system with one switch at $t_1 = 0$. In fact, letting $t_0 = -\varepsilon$, $t_f = \varepsilon$, $(E_0, A_0, B_0) = (I, 0, 0)$ and $(E_1, A_1, B_1) = (E, A, B)$, the DAE* (6) *is impulse controllable if, and only if, the corresponding ITP (reinterpreted as a switched DAE) is impulse controllable on $(-\varepsilon, \varepsilon)$.*

A solution of a switched DAE can only be impulse free, if at each switching instance the solution evaluated at t_i^- is in the impulse controllable space $\mathcal{C}_i^{\text{imp}}$. Therefore we consider the largest set of points from which the impulse controllable space of the next mode can be reached impulse freely from the preceding mode. To that extent we define the following sequence of sets regarding the switched DAE (1) with switching signal (8):

$$\begin{aligned}
\mathcal{K}^b_{\mathtt{n}} &= \mathcal{C}^{\text{imp}}_{\mathtt{n}}, \\
\mathcal{K}^b_{i-1} &= \operatorname{im}\Pi_{i-1} \cap \left(e^{-A^{\text{diff}}_{i-1}(t_{i-1}-t_i)}\mathcal{K}^b_i + \mathcal{R}_{i-1}\right) \\
&\quad + \langle E^{\text{imp}}_{i-1} \mid B^{\text{imp}}_{i-1}\rangle + \ker E_{i-1}, \\
&\quad i = \mathtt{n}, \mathtt{n}-1, \ldots, 1.
\end{aligned}$$

Note that $\operatorname{im}\Pi_{i-1} = \mathcal{V}_{(E_i,A_i)}$ and that $\mathcal{C}_i^{\text{imp}} = \mathcal{V}_{(E_i,A_i)} + \mathcal{R}_i + \ker E_i$. Therefore we have that $\mathcal{K}_i^b \subseteq \mathcal{C}_i^{\text{imp}}$. Note furthermore, that the definition is *backwards* in time; the sequences starts with the last mode $\mathtt{n}$ and ends with the initial mode 0. With these sets, we can prove the following lemma.

Lemma 19. *Consider the (interval restricted) switched DAE $(E_\sigma \dot{x})_{[t_{i-1},t_i)} = (A_\sigma)_{[t_{i-1},t_i)} + (B_\sigma u)_{[t_{i-1},t_i)}$. Then $\mathcal{K}^b_{i-1}$ is the largest set of points at time t^-_{i-1} from which $\mathcal{K}^b_i$ can be reached (at t_i^-) in an impulse free way.*

The proof is similar to the proof of Lemma 19 in [20] and therefore omitted.

Corollary 20. *Consider the switched system* (1) *with switching signal* (8)*. The system is impulse controllable if and only if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}^b_0.$$

Proof. Invoking Lemma 19 inductively, it follows that $\mathcal{K}^b_0$ is the largest set of initial values at t_0 for which an input exists such the the overall solution on (t_0, t_f) is impulse-free. Hence, if the switched DAE (1) is impulse controllable, every consistent initial value must be an element of $\mathcal{K}^b_0$, i.e. $\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}^b_0$. On the other hand, if (1) is not impulse-controllable then there is an initial value $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ for which no impulse-eliminating input exists, i.e. $x_0 \notin \mathcal{K}^b_0$. Which proves the result. ∎

4.2. Impulse-free stabilizabilty

As shown in the introduction, a switched DAE which is impulse controllable and stabilizable is not necessarily impulse-free stabilizable. However, impulse-controllability is an obvious necessary condition for impulse-free stabilizability. In order to stabilize a state on a bouned interval in an impulse-free way, there needs to exists an impulse-free solution in the first place. To that extent, we will make the following standing assumptions throughout the rest of this section:

1. The switched DAE (1) is impulse-controllable.
2. The initial condition is consistent, i.e. $x(t_0^+) = x_0 \in \mathcal{V}_{E_0,A_0,B_0}$.

Under these assumption, we will derive necessary and sufficient conditions for impulse-free stabilizability. The approach taken is as follows. First we consider the space of points that can be reached in an impulse free way from an initial value x_0. It will then be shown that this space is an affine subspace. We then consider an element of this affine subspace with minimal norm; if this norm is smaller than the norm of the corresponding initial value, we can conclude interval stabilizability.

Towards this goal, we consider the following sequence of (affine) subspaces (defined *forward* in time)

$$\begin{aligned} \mathcal{K}_0^f(x_0) &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \mathcal{R}_0, \\ \mathcal{K}_i^f(x_0) &= e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i(\mathcal{K}_{i-1}^f(x_0)\cap \mathcal{C}_i^{\text{imp}}) + \mathcal{R}_i,\ i>0, \end{aligned} \tag{9}$$

For $x_0 = 0$ we drop the dependency on x_0, i.e.

$$\mathcal{K}_i^f := \mathcal{K}_i^f(0).$$

Remark 21. *Note that the above $\mathcal{K}_i^f$ is different from $\mathcal{K}_i^f$ in [20], the latter is defined as the space of all points that can be reached in an impulse-free way, i.e., it is the union of $\mathcal{K}_i^f(x_0)$ over all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$.*

The intuition behind the sequence is as follows: $\mathcal{K}_0^f(x_0)$ are all values for $x_u(t_1^-, x_0)$ which can be reached in an impulse free (in fact, smooth) way during the initial mode 0. Now, inductively, we calculate the set $\mathcal{K}_i^f(x_0)$ of points which can be reached just before the switching time t_{i+1} by first consider the points $\mathcal{K}_{i-1}^f(x_0)$ which can be reached in an impulse free way just before t_i, then pick those which can be continued in mode i impulse-freely by intersecting them with $\mathcal{C}_i^{\text{imp}}$, propagate this set forward according to the evolution operator and finally add the reachable space of mode i. This intuition is verified by the following lemma.

Lemma 22. *Consider the switched system* (1) *on some bounded interval (t_0, t_f) with the switching signal given by* (8)*. Then for all $i = 0, 1, \dots, \mathtt{n}$ and $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$*

$$\mathcal{K}_i^f(x_0) = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ an impulse-free solution } (x,u) \\ \text{of (1) on } (t_0, t_{i+1}) \text{ s.t.} \\ x(t_0^+) = x_0 \wedge x(t_{i+1}^-) = \xi \end{array} \right\}.$$

Proof. First we will show that $x_u(t_i^-, x_0)$ is contained in $\mathcal{K}_i^f(x_0)$ if (x,u) is an impulse free solution on (t_0, t_f). To that extent, consider an impulse-free solutions (x,u) of (1) on (t_0,t_1), which by definition satisfies the solution formula (7), i.e.,

$$x_u(t_1^-, x_0) = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \eta_0,$$

for some $\eta_0 \in \mathcal{R}_0$ and $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. This shows that $x_u(t_1^-, x_0) \in \mathcal{K}_0^f(x_0)$. We proceed inductively by assuming that the statement holds for $i > 0$ and prove the statement for $i+1$.

Let (x,u) be an impulse-free solution on (t_0, t_{i+1}). Then we have that $x_u(t_{i+1}^-, x_0)$ is of the form

$$x_u(t_{i+1}^-, x_0) = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \xi_{i-1} + \eta_i,$$

for some $\eta_i \in \mathcal{R}_i$ and $\xi_{i-1} \in \mathcal{C}_i^{\text{imp}}$. Furthermore, since (x,u) is impulse-free on (t_0, t_{i+1}), it follows that ξ_i can be reached impulse-freely from x_0 and hence $\xi_{i-1} \in \mathcal{K}_{i-1}^f(x_0)$. This proves that $x_u(t_{i+1}^-, x_0) \in \mathcal{K}_i^f(x_0)$.

In the following we will prove that for all elements of $\mathcal{K}_i^f(x_0)$ there exists an impulse-free solution (x,u) with initial condition $x_u(t_0^+, x_0) = x_0$. We will again prove this inductively. Therefore, consider $\xi_0 \in \mathcal{K}_0^f(x_0)$. Then for some $\eta_0 \in \mathcal{R}_0$ we have

$$\xi_0 = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \eta_0.$$

Since $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{C}^{\text{imp}}$, we have that there exists a $\tilde{u}$ such that $x_{\tilde{u}}(t, x_0)$ is impulse-free on $[t_0, t_1)$. Then it follows from the solution formula (7) that

$$x_{\tilde{u}}(t_1^-, x_0) = e^{A^{\text{diff}} t_1}\Pi x_0 + \tilde{\eta}_0,$$

for some $\tilde{\eta}_0 \in \mathcal{R}_0$. Since $\eta_0 \in \mathcal{R}_0$, there exists a smooth input $\hat{u}$ such that $x_{\hat{u}}(t_1^-, 0) = \eta_0 - \tilde{\eta}_0$ and $x_{\hat{u}}(t, 0)$ is impulse-free on $[t_0, t_1)$.

If we define $u = \hat{u} + \tilde{u}$ it then follows from linearity of solutions that $x_u(t_1^-, x_0) = \xi_0$ and is impulse-free on (t_0, t_1). Assuming that the statement holds for $i > 0$ we continue by proving the statement for $i+1$.

Let $\xi_i \in \mathcal{K}_{i+1}^f(x_0)$, then we have for some $\xi_{i-1} \in \mathcal{K}_i^f(x_0) \cap \mathcal{C}_{i-1}^{\text{imp}}$ that

$$\xi_i = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \xi_{i-1} + \eta_i.$$

It follows from the induction assumption that there exists an impulse-free solution (x,u) on (t_0, t_i) with $x_u(t_i^-, x_0) = \xi_{i-1}$, beause $\xi_{i-1} \in \mathcal{K}_i^f(x_0)$. Furthermore, $\xi_{i-1} \in \mathcal{C}_{i-1}^{\text{imp}}$ and $\eta_i \in \mathcal{R}_i$ implies that the impulse-free input u can be altered on the interval $[t_i, t_i+1)$ such that $x_u(t_{i+1}^-, x_0) = \xi_i$ and $x_u(\cdot, x_0)$ is impulse-free. ■

Remark 23. *The assumption that $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ is of crucial importance for Lemma 16. If the zeroth mode is not impulse controllable and we would choose $x_0 \in (\mathcal{V}_{(E_0,A_0,B_0)} + \ker E_0)^\perp$ the occurrence of a dirac impulse would be inevitable. This means that $\mathcal{K}_0^f(x_0)$ should be empty. However, the algorithm* (9) *would state that $\mathcal{K}_0^f(x_0)$ is nonempty, which is not true.*

Remark 24. *If the system is not impulse controllable, then there exist x_0 for which $\mathcal{K}_i^f(x_0) = \varnothing$ as follows from the definition. This also follows from the subspace algorithm because $\mathcal{K}_{i-1}^f(x_0) \cap \mathcal{C}_i^{\text{imp}}$ would be empty for some mode i and the sum of an empty set and a subspace is empty.*

Lemma 22 gives rise to another characterization of impulse controllability, which follows as a corollary.

Corollary 25. *Consider the switched system* (1) *on some interval* (t_0, t_f) *with the switching signal given by* (8) *and the sequence of affine subspaces* $\mathcal{K}_i^f(x_0)$ *given by* (9)*. Then* (1) *is impulse controllable on* (t_0, t_f) *if and only if*

$$\forall x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}: \quad \mathcal{K}_{\mathtt{n}}^f(x_0) \neq \varnothing.$$

Proof. If the system is impulse controllable, then for every initial condition x_0 there exists an impulse free solution (x, u) on (t_0, t_f). Therefore $x(t_f^-) \in \mathcal{K}_{\mathtt{n}+1}^f(x_0)$ (recall the convention that $t_{\mathtt{n}+1} := t_f$) and hence $\mathcal{K}_{\mathtt{n}+1}^f(x_0) \neq \varnothing$. Conversely, if $\mathcal{K}_{\mathtt{n}}(x_0) \neq \varnothing$, then let $\xi \in \mathcal{K}_{\mathtt{n}+1}^f(x_0)$. By definition there exists an impulse free solution (x, u) on (t_0, t_f) with $x(t_0^-) = x_0$ and $x(t_f^-) = \xi$. This holds for every $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ and hence (1) is impulse controllable. ∎

Note that in contrast to Corollary 20 the computations in Corollary 25 run forward in time. Hence this result is useful in the case that not all modes are determined yet and the next mode is to be chosen. If Corollary 20 would be used, all computations would need to be redone, whereas with a forward computation only parts need to be redone.

In the following we will show that $\mathcal{K}_i^f(x_0)$ is an affine shift of $\mathcal{K}_f^f$ and hence $\mathcal{K}_i^f(x_0)$ is an affine subspace. In proving this statement, we will use some general results which can be found in the appendix.

Lemma 26. *Consider the switched system* (1) *with switching signal* (8) *and assume it is impulse-controllable. The impulse-free-reachable space from* x_0 *at* t_i *is an affine shift from the impulse-free reachable space, i.e., there exists a matrix* M_i*, such that*

$$\mathcal{K}_i^f(x_0) = M_i x_0 + \mathcal{K}_i^f. \tag{10}$$

Proof.
First we simplify the notation introducing the following short hand notation $Y_i := e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i$. Then we prove the statement inductively. The statement holds trivially for $n = 0$, for $\mathcal{K}_0^f = Y_0 x_0 + \mathcal{R}_0$ and hence we assume that the statement holds for n. Since we assumed that the system is impulse controllable, we have that $\mathcal{K}_i^f(x_0) \cap \mathcal{C}_{i+1}^{\text{imp}} \neq \varnothing$ for all x_0. Then for $n+1$ we obtain that

$$\begin{aligned}
\mathcal{K}_{i+1}^f(x_0) &= Y_{i+1}(\mathcal{K}_i^f(x_0) \cap \mathcal{C}_{i+1}^{\text{imp}}) + \mathcal{R}_{i+1} \\
&\overset{*}{=} Y_{i+1}((M_i x_0 + \mathcal{K}_i^f) \cap \mathcal{C}_{i+1}^{\text{imp}}) + \mathcal{R}_{i+1}, \\
&\overset{**}{=} Y_{i+1}(N_i M_i x_0 + (\mathcal{K}_i^f \cap \mathcal{C}_{i+1}^{\text{imp}})) + \mathcal{R}_{i+1}, \\
&= Y_{i+1} N_i M_i x_0 + K_{i+1}^f,
\end{aligned}$$

for some matrix N_i, $i \in \{0, 1, ..., \mathtt{n}\}$, where $(*)$ follows from the induction step and $(**)$ follows from Proposition 42 in the appendix. Defining $M_{i+1} = Y_{i+1} N_i M_i$ yields the result. ∎

Note that the matrix M_i in (10) exists only in case that the system is impulse-controllable, otherwise M_i would also need to map to the empty set. In the case M_i does exists, this matrix can be chosen independently of x_0. It is however not necessarily unique, because M_{i+1} is dependent on N_i obtained from Proposition 42 in a nonunique way. It follows from Lemma 43 from the Appendix that N_i can be any matrix for which

$$\begin{aligned}
&1. \quad \operatorname{im}(N_i - I)M_i \subseteq \mathcal{R}_i, \\
&2. \quad \operatorname{im} N_i M_i \subseteq \mathcal{C}_{i+1}^{\text{imp}}.
\end{aligned} \tag{11}$$

Thus, from the proof of Lemma 26 together with Lemma 43 from the Appendix the following constructive result can be obtained.

Corollary 27. *Consider the switched system* (1) *with switching signal* (8) *and assume it is impulse-controllable. Let* $M_0 = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0$*. Then for any choice of* N_i *satisfying* (11)*, a matrix* M_{i+1} *satisfying* (10) *can be calculated sequentially as follows:*

$$M_{i+1} = e^{A_{i+1}^{\text{diff}}(t_{i+2}-t_{i+1})\Pi_{i+1}} N_i M_i.$$

Remark 28. *In order to compute an* N_i *that satisfies* (11) *we can invoke Lemma 44 from the Appendix. This means that given projectors onto* $\mathcal{R}_i$ *and* $\mathcal{C}_{i+1}^{\text{imp}}$*, an* N_i *that satisfies the conditions* (11) *can be constructed by solving*

$$(I - \Pi_{\mathcal{R}_i})\Pi_{\mathcal{C}_{i+1}^{\text{imp}}} Q_i M_i = (I - \Pi_{\mathcal{R}_i}) M_i \tag{12}$$

for Q_i *and defining* $N_i := \Pi_{\mathcal{C}_{i+1}^{\text{imp}}} Q_i$*. Since the existence of a solution of* (12) *is guaranteed by the assumption of impulse-controllability, such a matrix equation can be solved using a linear programming solver.*

Since $\mathcal{K}_i^f(x_0)$ contains all the states that can be reached from x_0 in an impulse free way, it follows that the norm of the state with minimal norm is given by the distance $\operatorname{dist}(\mathcal{K}_i^f(x_0), 0)$. The computation of this distance is straightforward, because $\mathcal{K}_i^f(x_0)$ is an affine subspace. It follows from elementary linear algebra that the distance between an affine subspace and the origin, is equal to the norm of any element projected to the orthogonal complement of the vector space associated to the affine subspace. In the case of $\mathcal{K}_i^f(x_0)$ we would need to project onto $(\mathcal{K}_i^f)^\perp$ with a projector $\Pi_{(\mathcal{K}_i^f)^\perp}$. An important property of these projectors is that their restriction to the corresponding augmented consistency space is well defined.

Lemma 29. *Consider the DAE* (1) *with switching signal* (8)*. For any* $i \in \{0, 1, \ldots, \mathtt{n}\}$ *let* $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$*, then*

$$\Pi_{(\mathcal{K}_i^f)^\perp} \xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Proof. From $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$ and $\Pi_{(\mathcal{K}_i^f)^\perp} + (I - \Pi_{(\mathcal{K}_i^f)^\perp}) = I$, it follows that

$$\Pi_{(\mathcal{K}_i^f)^\perp}\xi + (I - \Pi_{(\mathcal{K}_i^f)^\perp})\xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Since $\operatorname{im}(I - \Pi_{(\mathcal{K}_i^f)^\perp}) = \mathcal{K}_i^f$ and $\mathcal{K}_i^f \subseteq \mathcal{V}_{(E_i,A_i,B_i)}$ we obtain

$$\Pi_{(\mathcal{K}_i^f)^\perp}\xi \in \mathcal{V}_{(E_i,A_i,B_i)} - (I - \Pi_{(\mathcal{K}_i^f)^\perp})\xi \subseteq \mathcal{V}_{(E_i,A_i,B_i)}.$$

as was to be shown. ■

Consequently, the following result follows.

Lemma 30. *Consider the DAE* (1) *with switching signal* (8) *and assume it is impulse-controllable. For any M_i satisfying* (10) *we have that*

$$\min_{x \in \mathcal{K}_i^f(x_0)} |x| = |\Pi_{(\mathcal{K}_i^f)^\perp} M_i x_0|$$

It follows that we can consider $\Pi_{(\mathcal{K}_i^f)^\perp} M_i$ as a linear map from the initial condition x_0 to the state with minimal norm in $\mathcal{K}_i^f(x_0)$. This allows us to formulate the following characterization of impulse-free stabilizability, which is independent of the initial condition x_0 and independent of any coordinate system.

Theorem 31. *Consider the switched DAE* (1) *with switching signal* (8) *and assume it is impulse controllable. Then the system is impulse-free interval-stabilizable on* (t_0, t_f) *if and only if*

$$||\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}||_2 = \sup_{x \neq 0} \frac{|\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n} x|_2}{|x|_2} < 1$$

Proof. It follows from Lemma 30 that $\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}$ is the linear operator that maps x_0 to the element in $\mathcal{K}_\mathtt{n}^f(x_0)$ with minimal norm. Therefore we see that if $||\Pi_{(\mathcal{K}_i^f)^\perp} M_i||_2 < 1$ that for all x_0 there exists an input u such that

$$|x_u(t_f, x_0)| = |\Pi_{(\mathcal{K}_i^f)^\perp} M_i x_0| < |x_0|.$$

From this we can conclude that there exists a class $\mathcal{KL}$ function $\beta(|x_0|, t_f - t_0)$ such that the system is impulse-free interval stabilizable in the sense of Definition 8.

Conversely, if the system is impulse-free interval stabilizable, then there exists a trajectory for each initial condition $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ such that $|x_u(t_f^-, x_0)| \leqslant \beta_i(|x_0|, t_f - t_0) < |x_0|$. This means that for the operator $\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}$ that maps $|x_0|$ to the element with minimal norm that can be reached in an impulse-free way it must hold that

$$||\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n}||_2 = \sup_{x \neq 0} \frac{|\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n} x|_2}{|x|_2} < 1,$$

which proves the result. ■

For many applications it is not sufficient to reduce the norm of the state, but it is necessary to control the state to zero without any Dirac impulses occurring. If a state can be steered to zero in an impulse free way, we call this state *impulse-free null-controllable.* A formal definition of this concept is as follows.

Definition 32. *Consider the system* (1) *with switching signal* (8)*. An initial condition x_0 is called impulse-free null-controllable if there exists an input u such that $x_u(t_f^-, x_0) = 0$ and the trajectory is impulse-free. We call the system impulse-free null-controllable if every $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ is impulse-free null-controllable.*

Using the method from the previous section, the following characterization can readily be stated.

Theorem 33. *Consider the system* (1) *with switching signal* (8)*. An initial value x_0 is impulse-free null-controllable, if and only if for some $i \geqslant 0$*

$$\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f.$$

Proof. If an initial condition is impulse-free null-controllable, there exists an input u such that $x_u(t_f^-, x_0) = 0$ and the trajectory is impulse free. This means that $0 \in \mathcal{K}_{\mathtt{n}+1}^f(x_0)$. As a consequence

$$0 \subseteq M_{\mathtt{n}+1} x_0 + \mathcal{K}_{\mathtt{n}+1}^f,$$

from which it follows that $M_{\mathtt{n}+1} x_0 \in \mathcal{K}_{\mathtt{n}+1}^f$ and therefore $\mathcal{K}_{\mathtt{n}+1}^f(x_0) \subseteq \mathcal{K}_{\mathtt{n}+1}^f$.

Conversely if for some $i = k \geqslant 0$ $\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f$, it follows that $M_i x_0 \in \mathcal{K}_i^f$. As a consequence $0 \in M_i x_0 + \mathcal{K}_i^f = \mathcal{K}_i^f(x_0)$. It follows from the sequence (9) if $\mathcal{K}_i^f \subseteq \mathcal{K}_i^f$ for $i = k \geqslant 0$ that it holds for all $i \geqslant k$. ■

As a direct consequence we can state the following result.

Corollary 34. *Consider the switched system* (1) *with switching signal* (8) *and assume it is impulse controllable. Then the system is impulse-free null-controllable on* (t_0, t_f) *if, and only if, for some* $i \in \{0, 1, \ldots, \mathtt{n}\}$

$$\Pi_{(\mathcal{K}_i^f)^\perp} M_i = 0.$$

Proof. If the system is impulse-null controllable, we have that $\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f$ for all x_0. Then it follows that

$$M_i x_0 + \mathcal{K}_i^f \subseteq \mathcal{K}_i^f,$$

for all x_0 and hence $\operatorname{im} M_i \subseteq \mathcal{K}_i^f$. The result then follows.

Conversely, if $\Pi_{(\mathcal{K}_i^f)^\perp} M_i = 0$, then $\Pi_{(\mathcal{K}_i^f)^\perp} \mathcal{K}_i^f(x_0) = 0$ for all x_0, which implies that $\mathcal{K}_i^f(x_0) \subseteq \mathcal{K}_i^f$ for all x_0. ■

$\mathcal{K}_i^f$ and M_i can both be computed sequentially forward in time. This means that it might not be necessary to have knowledge of all the modes of the switched system. According to Corollary 34 we can conclude impulse-free null-controllability already if the conditions are satisfied for some $i \in \mathbb{N}$.

4.3. Impulsive stabilizability and impulse-controllability

In the case that Dirac impulses are allowed in the trajectory similar results as in the above can be formulated. The crucial condition for impulse-free trajectories is that the state is in the impulse controllable space of the next mode at each switching instance. If this condition is dropped, a similar lemma as Lemma 22 can be formulated after considering the following sequence of sets

$$\begin{aligned}\tilde{\mathcal{K}}_0^f(x_0) &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \mathcal{R}_0,\\ \tilde{\mathcal{K}}_i^f(x_0) &= e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \tilde{\mathcal{K}}_{i-1}^f(x_0) + \mathcal{R}_i,\ i>0,\end{aligned} \tag{13}$$

For $x_0 = 0$ we drop the dependency on x_0, i.e.

$$\tilde{\mathcal{K}}_i^f := \tilde{\mathcal{K}}_i^f(0).$$

Lemma 35. *Consider the switched system* (1) *on some bounded interval* (t_0, t_f) *with the switching signal given by* (8)*. Then for all* $i = 0, 1, \ldots, \mathtt{n}$

$$\tilde{\mathcal{K}}_i^f(x_0) = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ a solution } (x,u) \\ \text{of (1) on } (t_0, t_{i+1}) \text{ s.t.} \\ x(t_0^+) = x_0 \wedge x(t_{i+1}^-) = \xi \end{array} \right\}.$$

Proof. The proof is along similar lines as the proof of Lemma 22 when $\mathcal{C}_i^{\text{imp}}$ is replaced by $\mathbb{R}^n$ for all $i \in \{1, 2, ..., \mathtt{n}\}$. ■

It follows directly that $\tilde{\mathcal{K}}_i^f(x_0)$ is an affine shift from $\tilde{\mathcal{K}}_i^f$, whether the system is impulse controllable or not. This is formalized in the next lemma.

Lemma 36. *Consider the switched system* (1) *with switching signal* (8)*. Then* $\tilde{\mathcal{K}}_i^f(x_0)$ *is an affine shift of* $\tilde{\mathcal{K}}_i^f$*, i.e. for all* i *there exists a matrix* $\tilde{M}_i$ *such that*

$$\tilde{\mathcal{K}}_i^f(x_0) = \tilde{M}_i x_0 + \tilde{\mathcal{K}}_i^f. \tag{14}$$

Proof. Denote $Y_i = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i$ for shorthand notation. Then for $i = 0$ we have $\tilde{M}_0 = Y_0$ satisfies (14). Hence assume the statement holds for i. Then if we define $\tilde{M}_{i+1} = Y_i \tilde{M}_i$ for $i+1$ we have that

$$\begin{aligned}\tilde{\mathcal{K}}_{i+1}^f(x_0) &= Y_{i+1}\tilde{\mathcal{K}}_i^f(x_0) + \mathcal{R}_i,\\ &= Y_i(\tilde{M}_i x_0 + \tilde{\mathcal{K}}_i^f) + \mathcal{R}_i,\\ &= Y_i \tilde{M}_i x_0 + \tilde{\mathcal{K}}_{i+1}^f\\ &= \tilde{M}_{i+1} x_0 + \tilde{\mathcal{K}}_i^f\end{aligned}$$

which proves the statement. ■

Lemma 37. *Consider the DAE* (1) *with switching signal* (8)*. For any* $\tilde{M}_i$ *satisfying* (14) *we have that*

$$\min_{x \in \tilde{\mathcal{K}}_i^f(x_0)} |x| = |\Pi_{(\tilde{\mathcal{K}}_i^f)^\perp} \tilde{M}_i x_0|$$

Theorem 38. *Consider the switched DAE* (1) *with switching signal* (8)*. Then the system is stabilizable if and only if for any* $\tilde{M}_{\mathtt{n}}$ *satisfying* (14)

$$||\Pi_{(\tilde{\mathcal{K}}_{\mathtt{n}}^f)^\perp} \tilde{M}_{\mathtt{n}}||_2 = \sup_{x \neq 0} \frac{|\Pi_{(\tilde{\mathcal{K}}_{\mathtt{n}}^f)^\perp} \tilde{M}_{\mathtt{n}} x|_2}{|x|_2} < 1$$

Proof. The proof is follows the proof of Theorem 31 analogously. ■

As was already shown in the introduction, not every stabilizable system that is also impulse-controllable, is automatically impulse-free stabilizable. This can be explained by viewing $\mathcal{K}_i^f(x_0)$ and $\tilde{\mathcal{K}}_i^f(x_0)$ as affine subspaces. Note that since every state that can be reached impulse-free from x_0 is by definition also an element of $\tilde{\mathcal{K}}_i^f(x_0)$. This leads to the following result.

Lemma 39. *Consider the switched system* (1) *with switching signal* (8) *and assume the system is impulse-controllable. Then*

$$\mathcal{K}_i^f(x_0) \subseteq \tilde{\mathcal{K}}_i^f(x_0).$$

Proof. This follows immediately from Lemma 22 and 35. ■

As a consequence, we can state the following corollary.

Corollary 40. *Consider the system* (1) *with switching signal* (8) *and assume it is impulse-controllable. Then for any* M_i *satisfying* (10) *we have*

$$\tilde{\mathcal{K}}_i^f(x_0) = M_i x_0 + \tilde{\mathcal{K}}_i^f,$$

i.e. M_i *satisfies* (14).

Proof. For any two $x, y \in \tilde{\mathcal{K}}_i^f(x_0)$ we have that $x - y \in \tilde{\mathcal{K}}_i^f$. This means that $x = y + \tilde{\eta}$ for some $\tilde{\eta} \in \tilde{\mathcal{K}}_i^f$. By Lemma (39) we have that $y = M_i x_0 + \eta \in \mathcal{K}_i^f(x_0) \subseteq \tilde{\mathcal{K}}_i^f(x_0)$. This means that for any $x \in \tilde{\mathcal{K}}_i^f(x_0)$ we obtain that $x = M_i x_0 + \eta + \bar{\eta} \subseteq M_i x_0 + \tilde{\mathcal{K}}_i^f(x_0)$. This proves that $\mathcal{K}_i^f(x_0) \subseteq M_i x_0 + \tilde{\mathcal{K}}_i^f$.

Consider $\alpha = M_i x_0 + \tilde{\eta}$ for some $\tilde{\eta} \in \tilde{\mathcal{K}}_i^f$. Then since $\mathcal{K}_i^f \subseteq \tilde{\mathcal{K}}_i^f$ there exits an $\bar{\eta} \in \tilde{\mathcal{K}}_i^f$ and an $\eta \in \mathcal{K}_i^f$ such that $\tilde{\eta} = \bar{\eta} + \eta$. Hence we obtain that $\alpha = M_i x_0 + \bar{\eta} + \eta = \beta + \eta$ for some $\beta \in \mathcal{K}_i^f(x_0) \subseteq \tilde{\mathcal{K}}_i^f(x_0)$. But this means that for some $\tilde{M}_i$ satisfying (14) and $\hat{\eta} \in \tilde{\mathcal{K}}_i^f$ that $\alpha = \tilde{M}_i x_0 + \hat{\eta} + \eta$. Because $\hat{\eta} + \eta \in \tilde{\mathcal{K}}_i^f$ we have that $\alpha \in \tilde{\mathcal{K}}_i^f(x_0)$. Since α was chosen arbitrary, it follows that $M_i x_0 + \tilde{\mathcal{K}}_i^f \subseteq \tilde{\mathcal{K}}_i^f(x_0)$. ■

Given that a system is impulse-controllable and stabilizable, we have that there exist an M_i satisfying (10) and we know that $||\Pi_{(\tilde{\mathcal{K}}_{\mathtt{n}}^f)^\perp} M_{\mathtt{n}}||_2 < 1$. However, the system is impulse-free stabilizable if and only if $||\Pi_{(\mathcal{K}_{\mathtt{n}}^f)^\perp} M_{\mathtt{n}}||_2 < 1$. This is however not implies by the statement that $||\Pi_{(\tilde{\mathcal{K}}_{\mathtt{n}}^f)^\perp} M_{\mathtt{n}}||_2 < 1$. Indeed, since $\mathcal{K}_i^f \subseteq \tilde{\mathcal{K}}_i^f$ we have that

$\operatorname{im}\Pi_{(\tilde{\mathcal{K}}_\mathtt{n}^f)^\perp} \subseteq \operatorname{im}\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp}$, which means that it could happen that there exists an initial condition $x_0 \neq 0$ for which

$$\frac{|\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_\mathtt{n} x_0|}{|x_0|} \geqslant 1, \quad \text{and} \quad \frac{|\Pi_{(\tilde{\mathcal{K}}_\mathtt{n}^f)^\perp} M_\mathtt{n} x_0|}{|x_0|} < 1.$$

As an example, consider the system introduced in the introduction restricted to the interval $[0, t_f)$, i.e. the switched DAE defined by

$$\begin{array}{ll} \text{on } [0,t_1): & \text{on } [t_1,t_f): \\ \dot{x}(t) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u(t) & \begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix} \dot{x}(t) = x(t). \end{array}$$

After some computation it follows that

$$\mathcal{K}_1^f = 0, \quad \tilde{\mathcal{K}}_1^f = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad M_1 = e^{t_f} \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}.$$

Then it is easily verified that

$$\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} = I, \qquad \Pi_{(\tilde{\mathcal{K}}_\mathtt{n}^f)^\perp} = \begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}.$$

Since $||\Pi_{(\mathcal{K}_\mathtt{n}^f)^\perp} M_i||_2 = 1$ and $\Pi_{(\tilde{\mathcal{K}}_\mathtt{n}^f)^\perp} M_i|| = 0$ we can conclude that the system is not impulse-free stabilizable, although it is impulse-controllable and stabilizable. Morover, the system is null-controllable, but not impulse-free null-controllable.

Remark 41. *All the results on stabilizability in this paper can be applied to switched ordinary differential equations (ODEs) without difficulty. In the case of a switched ODE we have $E_i = I$, $\Pi_i = I$, $B_i^{\text{diff}} = B_i$ and $A_i^{\text{diff}} = A_i$. Note that all solutions are trivially impulse-free, hence, impulse-free stabilizability is equivalent to stabilizability.*

5. Conclusion

In this paper stabilization of switched differential algebraic equations was considered, where Dirac impulses in both the input and state-trajectory were to be avoided. Necessary and sufficient conditions for the existence of impulse-free solutions were given, followed by characterizations of (impulse-free) interval stabilizability. The results rely on the fact that the points that can be reached from an initial condition form an affine subspace. It followed that the system is (impulse-free) interval stabilizable if and only if the operator that maps the initial condition to the element of minimal norm (that can be reached in an impulse-free manner) has a norm strictly smaller than one.

A natural future direction of research would be the investigation of controllers achieving interval stabilizability for switched systems. The theory established in this paper could be used as starting point in the search (for feedback) controllers.

References

[1] Douglas J. Bender and Alan J. Laub. The linear-quadratic optimal regulator for descriptor systems. In *Proc. 24th IEEE Conf. Decis. Control, Ft. Lauderdale, FL*, pages 957–962, 1985.

[2] Thomas Berger, Achim Ilchmann, and Stephan Trenn. The quasi-Weierstraß form for regular matrix pencils. *Linear Algebra Appl.*, 436(10):4052–4069, 2012.

[3] Thomas Berger and Timo Reis. Controllability of linear differential-algebraic systems - a survey. In Achim Ilchmann and Timo Reis, editors, *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, pages 1–61. Springer-Verlag, Berlin-Heidelberg, 2013.

[4] Thomas Berger and Stephan Trenn. Kalman controllability decompositions for differential-algebraic systems. *Syst. Control Lett.*, 71:54–61, 2014.

[5] J. Daniel Cobb. Descriptor variable systems and optimal state regulation. *IEEE Trans. Autom. Control*, 28:601–611, 1983.

[6] J. Daniel Cobb. Controllability, observability and duality in singular systems. *IEEE Trans. Autom. Control*, 29:1076–1082, 1984.

[7] Peter Kunkel and Volker Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, Switzerland, 2006.

[8] Ferdinand Küsters, Markus G.-M. Ruppert, and Stephan Trenn. Controllability of switched differential-algebraic equations. *Syst. Control Lett.*, 78(0):32 – 39, 2015.

[9] Ferdinand Küsters, Stephan Trenn, and Andreas Wirsen. Switch observability for homogeneous switched DAEs. In *Proc. of the 20th IFAC World Congress, Toulouse, France*, pages 9355–9360, 2017. IFAC-PapersOnLine 50 (1).

[10] Frank L. Lewis. A tutorial on the geometric analysis of linear time-invariant implicit systems. *Automatica*, 28(1):119–137, 1992.

[11] Daniel Liberzon and Stephan Trenn. On stability of linear switched differential algebraic equations. In *Proc. IEEE 48th Conf. on Decision and Control*, pages 2156–2161, December 2009.

[12] K. Maciej Przyłuski and Andrzej M. Sosnowski. Remarks on the theory of implicit linear continuous-time systems. *Kybernetika*, 30(5):507–515, 1994.

[13] Timo Reis and Matthias Voigt. Linear-quadratic infinite time horizon optimal control for differential-algebraic equations - a new algebraic criterion. In *Proceedings of MTNS-2012*, 2012.

[14] Aneel Tanwani and Stephan Trenn. Detectability and observer design for switched differential algebraic equations. *Automatica*, 99:289–300, 2019.

[15] Javier Tolsa and Miquel Salichs. Analysis of linear networks with inconsistent initial conditions. *IEEE Trans. Circuits Syst.*, 40(12):885 – 894, Dec 1993.

[16] Stephan Trenn. *Distributional differential algebraic equations.* PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany, 2009.

[17] Stephan Trenn. Regularity of distributional differential algebraic equations. *Math. Control Signals Syst.*, 21(3):229–264, 2009.

[18] Stephan Trenn. Switched differential algebraic equations. In Francesco Vasca and Luigi Iannelli, editors, *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, chapter 6, pages 189–216. Springer-Verlag, London, 2012.

[19] Paul Wijnbergen, Mark Jeeninga, and Stephan Trenn. On stabilizability of switched differential algebraic equations. In *Proc. IFAC World Congress 2020, Berlin, Germany*, 2020. to appear.

[20] Paul Wijnbergen and Stephan Trenn. Impulse controllability of switched differential-algebraic equations. In *2020 European Control Conference (ECC)*, pages 1561–1566. IEEE, 2020.

[21] Kai-Tak Wong. The eigenvalue problem $\lambda Tx + Sx$. *J. Diff. Eqns.*, 16:270–280, 1974.

Appendix

Here we recap some general results on (affine) subspaces that result from linear algebra.

Proposition 42. *Let $\mathcal{V}$ and $\mathcal{S}$ be subspaces of $\mathbb{R}^n$ and let $M \in \mathbb{R}^{n\times n}$ be of rank $r \leqslant n$. If $(Mx_0+\mathcal{S})\cap\mathcal{V} \neq \varnothing$ for all $x_0 \in \mathbb{R}^n$, then there exists a matrix $N \in \mathbb{R}^{n\times n}$ such that for all x_0*

$$(Mx_0+\mathcal{S})\cap\mathcal{V} = NMx_0 + \mathcal{S}\cap\mathcal{V}. \tag{15}$$

Proof. Let $m_1, m_2, ..., m_p$ be a basis for the image of M. Then the statement is proven if we can prove that

$$(m_i+\mathcal{S})\cap\mathcal{V} = Nm_i + \mathcal{S}\cap\mathcal{V}, \qquad \forall i \in \{1,2,...,p\}$$

Since we have that $(m_i+\mathcal{S})\cap\mathcal{V} \neq \varnothing$ we have that for all i we have that there exists an $\eta_i \in \mathcal{S}$ such that $m_i + \eta_i \in \mathcal{V}$. Let $\hat{N}$ be a linear map such that

$$\hat{N}m_i = \eta_i.$$

Then if we define $N = I + \hat{N}$ we have that

$$\begin{aligned} Nm_i &= m_i + \hat{N}m_i, \\ &= m_i + \eta_i, \\ &\in \mathcal{V}\cap(m_i+\mathcal{S}) \end{aligned}$$

Since subspaces are closed under addition, it follows that for all $\bar{\eta} \in \mathcal{S}\cap\mathcal{V} \subseteq \mathcal{V}$ we have that

$$Nm_i + \bar{\eta} = m_i + \eta_i + \bar{\eta} \in \mathcal{V}.$$

and

$$m_i + \eta_i + \bar{\eta} = m_i + \hat{\eta} \in m_i + \mathcal{S},$$

for some $\eta_i + \bar{\eta} = \hat{\eta} \in \mathcal{S}$, which proves that there exists an N such that $Nm_i + \mathcal{S}\cap\mathcal{V} \subseteq (m_i+\mathcal{S})\cap\mathcal{V}$.

Conversely, we have for $\xi \in (m_i+\mathcal{S})\cap\mathcal{V}$ and for some $\beta \in \mathcal{S}$ that $\xi = m_i + \beta \in \mathcal{V}$. Let $\beta = \hat{N}m_i + \gamma$, for some $\gamma \in \mathcal{S}$. Then we obtain

$$\begin{aligned} m_i + \beta &= m_i + \hat{N}m_i + \gamma, \\ &= Nm_i + \gamma, \\ &= \xi \in (m_i+\mathcal{S})\cap\mathcal{V}. \end{aligned}$$

It remains to prove that $\gamma \in \mathcal{S}\cap\mathcal{V}$. Since $Nm_i \in (m_i + \mathcal{S})\cap\mathcal{V} \subseteq \mathcal{V}$ by definition, we have that $\xi - Nm_i = \gamma \in \mathcal{V}$. Furthermore, by definition, we had $\gamma \in \mathcal{S}$ and hence $\gamma \in \mathcal{S}\cap\mathcal{V}$. Hence we have proven that $(m_i+\mathcal{S})\cap\mathcal{V} \subseteq Nm_i + \mathcal{S}\cap\mathcal{V}$. With the inclusion in both direction proven, the equality follows. ■

It follows from Proposition 42 that if the intersection $(Mx_0+\mathcal{S})\cap\mathcal{V} \neq \varnothing$ for all x_0, that this matrix N is not unique. In fact, this observation results in the next lemma.

Lemma 43. *With the same notation as in Proposition 42 we have that $N \in \mathbb{R}^{n\times n}$ satisfies* (15) *if and only if*

1. $\operatorname{im}(N-I)M \subseteq \mathcal{S}$,
2. $\operatorname{im} NM \subseteq \mathcal{V}$,

Proof. Assume that N satisfies $\operatorname{im}(N-I) \subseteq \mathcal{S}$ and $\operatorname{im} NY \subseteq \mathcal{V}$. This means that $\operatorname{im}(N-I)Y \subseteq \mathcal{S}$. Hence $NMx_0 \in \mathcal{S} + Mx_0$. Furthermore, by assumption we had that $NMx_0 \in \operatorname{im} N \subseteq \mathcal{V}$ and hence $NMx_0 \in (Mx_0+\mathcal{S})\cap\mathcal{V}$. Hence it follows that $NMx_0 + \mathcal{S}\cap\mathcal{V} \subseteq (Mx_0+\mathcal{S})\cap\mathcal{V}$.

On the otherhand, let $\xi \in (Mx_0+\mathcal{S})\cap\mathcal{V}$. Then $\xi = Mx_0 + \eta$ for some $\eta \in \mathcal{S}$ and $\xi \in \mathcal{V}$. Since $NMx_0 \in \mathcal{V}$ we have that $NMx_0 - \xi \in \mathcal{V}$. From which it follows that $(N-I)Mx_0 \in \mathcal{V}$ and also $(N-I)Mx_0 \in \mathcal{S}$. Thus we have that $NMx_0 - \xi \in \mathcal{S}\cap\mathcal{V}$. From this it follows that $\xi \in NMx_0 + \mathcal{S}\cap\mathcal{V}$ and thus it is proven that under the assumptions (10) holds.

Next assume that (10) holds. Then it follows that

$$\begin{aligned} NMx_0 &\in (Mx_0+\mathcal{S})\cap\mathcal{V} + \mathcal{S}\cap\mathcal{V} \\ &= (Mx_0+\mathcal{S})\cap\mathcal{V}. \end{aligned}$$

Since this holds for all x_0 it follows that $\operatorname{im} NM \subseteq \mathcal{V}$. Furthermore, it follows that $NMx_0 \in Yx_0 + \mathcal{S}$, from which it follows that $(N-I)Mx_0 \in \mathcal{S}$ for all x_0, and thus $\operatorname{im}(N-I)M \subseteq \mathcal{S}$. Which proves the result. ■

Given the subspaces $\mathcal{V}$, $\mathcal{S}$ and the matrix M, a matrix N satisfying the conditions of Lemma 43 can constructively be computed.

Lemma 44. *Let $\Pi_\mathcal{V}$ and $\Pi_\mathcal{S}$ be projectors onto $\mathcal{V}$ and $\mathcal{S}$ respectively. For any Q that solves*

$$(I-\Pi_\mathcal{S})\Pi_\mathcal{V}QM = (I-\Pi_\mathcal{S})M$$

the matrix $N = \Pi_\mathcal{V}Q$ solves (15).

Proof. Since $\operatorname{im} N \subseteq \operatorname{im} \Pi_\mathcal{V} = \mathcal{V}$ the condition $\operatorname{im} NM \subseteq \mathcal{V}$ is satisfied. Furthermore, we have that

$$\begin{aligned} \operatorname{im}(N-I)M &= \operatorname{im}(\Pi_\mathcal{V}Q - I)M, \\ &= \operatorname{im}(\Pi_\mathcal{S} + (I-\Pi_\mathcal{S}))(\Pi_\mathcal{V}Q - I)M \\ &\subseteq \mathcal{S} + \operatorname{im}(I-\Pi_\mathcal{S})(\Pi_\mathcal{V}Q - I)M, \\ &= \mathcal{S} + \operatorname{im}\left((I-\Pi_\mathcal{S})M - (I-\Pi_\mathcal{S})M\right) = \mathcal{S} \end{aligned}$$

Hence N satisfies the conditions of Lemma 43, which proves the result. ■