The differentiation index of nonlinear differential-algebraic equations versus the relative degree of nonlinear control systems*

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It is claimed in [1] that the notion of the relative degree in nonlinear control theory is closely related to that of the differentiation index for nonlinear differential-algebraic equations (DAEs). In this paper, we give more insights on this claim via a recent proposed concept (see [2]) called the explicitation of DAEs. The explicitation attaches a class of control systems to a given DAE, we show that the relative degree of the systems in the explicitation class is invariant in some sense and that the differentiation index of the original DAE coincides with the maximum of the relative degree of the explicitation systems.

1 Introduction

In the present paper, we study nonlinear differential-algebraic equations (DAEs) of the following form

$$\Xi : E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X \subseteq \mathbb{R}^n$ is called the generalized state, $E : X \to \mathbb{R}^{n\times n}$ and $F : X \to \mathbb{R}^n$ are $\mathcal{C}^\infty$-smooth maps, we denote DAE (1) by $\Xi_{n,n} = (E, F)$, or simply Ξ. Various notions of DAE index serving as measures for different purposes were proposed in the DAE literatures, see the surveys in [1,3,4]. In particular, the notion of the differentiation index plays a significant role in the numerical solution theory of nonlinear DAEs [1,5,6], which is the least number of differentiations such that the differential array of (3) could recover x' as a function of x and t only (see its formal definition in Section 3 below). On the other hand, we study nonlinear control systems of form (2), the relative degree (see Definition 3.1 below) of (2) is, roughly speaking, the number of differentiations of the outputs y such that the inputs u can be recovered. The relative degree is a widely used notion by the system and control community for some problems as input-output decoupling and linearization, see e.g., [7, 8].

Apart from the similarities in the definitions and initiatives of the differentiation index and the relative degree, we show one simple case that the two notions are related: For a SISO control system Σ of form (2), by setting the output $y = 0$, we get a DAE with the generalized state $(x, v) \in \mathbb{R}^{n+1}$. Then it is easy to understand that the differentiation index ν_d of the defined DAE is the relative degree ρ of Σ plus one since after ρ times of differentiations of y, we need one more to let v' show up. A general result of the former case is stated as Proposition 1 of [5] and some discussions on comparing the two notions could be consulted in [10]. In the present paper, we use a recent proposed method called the explicitation which attaches a class of control systems to a DAE, then we study the relations of the relative degree of the control systems in the explicitation class and the differentiation index of the DAE.

2 Explicitation of nonlinear differential-algebraic equations

To make connections between DAEs and control systems such that the results from classic control theory could applied to DAEs, a method called the explicitation of DAEs is studied in the thesis [2]. Note the explicitation is firstly proposed in [9] for linear DAEs and used in [11] for the linearization problems of semi-explicit DAEs, we now recall its formal definition for nonlinear DAEs of form (1). We assume throughout that $\operatorname{rank} E(x) = const. = r$ for all x around a nominal point x_0 and note that x_0 could be an admissible/consistent point or not, and here being admissible means that there always exists a solution of Ξ passing through x_0.

Definition 2.1 For a DAE $\Xi_{n,n} = (E, F)$, by a (Q, v)-explicitation, we will call a control system $\Sigma = \Sigma_{n,m,m} = (f, g, h)$, where $m = n - r$, given by

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)v \\ y = h(x), \end{cases} \tag{2}$$

with $f(x) = E_1^\dagger(x)F_1(x)$, $\operatorname{Im} g(x) = \ker E(x)$, $h(x) = F_2(x)$, where $Q(x)E(x) = \begin{bmatrix} E_1(x) \\ 0 \end{bmatrix}$, $Q(x)F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}$ and $E_1^\dagger$ denotes the right inverse of E_1. The class of all (Q, v)-explicitations will be called the explicitation class. If a particular control system Σ belongs to the explicitation class of Ξ, we will write $\Sigma \in \mathbf{Expl}(\Xi)$.

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Remark 2.2 The explicitation preserves $\mathcal{C}^1$-solutions of DAEs in the following way: for a DAE Ξ and any control system $\Sigma \in \mathbf{Expl}(\Xi)$, a $\mathcal{C}^1$-curve $x(\cdot)$ is a solution of Ξ if and only if there exists a $\mathcal{C}^0$-control $v(\cdot)$ such that $x(\cdot)$ is a solution of Σ satisfying $y = h(x(\cdot)) = 0$.

Notice that a given DAE Ξ has many (Q, v)-explicitations since the construction of $\Sigma \in \mathbf{Expl}(\Xi)$ is not unique: there is a freedom in choosing $Q(x)$, $E_1^{\dagger}(x)$, and $g(x)$. The following proposition illustrates that as a consequence of this non-uniqueness of construction, the explicitation Σ of Ξ is a system defined up to a feedback transformation $v = \alpha(x) + \beta(x)\tilde{v}$, an output multiplication $\tilde{y} = \eta(x)y$ and a generalized output injection given by $\gamma(x)y = \gamma(x)h(x)$, or in other words, a class of control systems.

Proposition 2.3 *Consider a DAE $\Xi_{l,n} = (E, F)$, then two control systems $\Sigma_{n,m,m} = (f, g, h)$ and $\tilde{\Sigma}_{n,m,m} = (\tilde{f}, \tilde{g}, \tilde{h})$ both belong to the same explicitation class $\mathbf{Expl}(\Xi)$ if and only if $\exists$ smooth matrix-valued functions α, γ and invertible smooth matrix-valued functions β, η, mapping $f \mapsto \tilde{f} = f + \gamma h + g\alpha$, $g \mapsto \tilde{g} = g\beta$ and $h \mapsto \tilde{h} = \eta h$.*

3 The differentiation index and the relative degree

The differential index is originally proposed for DAEs of the general form $\Xi^{gen} : H(t, x, x') = 0$: define the differential array of Ξ^{gen} by

$$H_k(t, x, x', w) = \begin{bmatrix} H \\ \mathrm{D}_t H + \mathrm{D}_x H x' + \mathrm{D}_{x'} H x'' \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k} H \end{bmatrix} (t, x, x', w) = 0, \tag{3}$$

where $w = \left[x^{(2)}, \ldots, x^{(k+1)}\right]$, the differentiation index ν_d is the least integer k such that equation (3) uniquely determines x' as a function of (x, t), i.e., $x' = a(x, t)$. We may also define the differential index for DAE Ξ of form (1) by writing $H(x, x') = E(x)\dot{x} - F(x)$. Now recall the following definition of the (vector) relative degree for nonlinear control systems.

Definition 3.1 ([7]) A control system $\Sigma_{n,m,m} = (f, g, h)$ has a (vector) relative degree $\rho = (\rho_1, \ldots, \rho_m)$ at a point x_0 if (i) $L_g L_f^k h(x) = 0$, for all $1 \le j \le m$, for all $k < \rho_i - 1$, for all $1 \le i \le m$, and for all x in a neighborhood of x_0; (ii) The $m \times m$ decoupling matrix: $D(x) = \left(L_{g_j} L_f^{\rho_i - 1} h_i(x)\right)_{i,j=1,\ldots,m}$ is invertible at x_0.

Now we are ready to make a connection for the two defined notions.

Theorem 3.2 *Consider a DAE $\Xi_{m,m} = (E, F)$ and an admissible point $x_0 \in X$. Assume that there exists a control system $\Sigma \in \mathbf{Expl}(\Xi)$ such that Σ has a well-defined relative degree $\rho = (\rho_1, \ldots, \rho_m)$ at x_0, then*

(i) *any other control system $\tilde{\Sigma} \in \mathbf{Expl}(\Xi)$ has either the same relative degree ρ with Σ or no well-defined relative degree at x_0;*

(ii) *the differentiation index ν_d of Ξ exists and satisfies that $\nu_d = \max\{\rho_1, \ldots, \rho_m\}$.*

Remark 3.3 (i) Recall from Proposition 2.3 that the two control systems Σ and $\tilde{\Sigma}$ in $\mathbf{Expl}(\Xi)$ can be transformed into each other by the transformations defined by α, β, γ, η. It is known that the relative degree is invariant under feedback transformations defined by α, β but may change under general output injections $\eta(x)y$ and output multiplications $\gamma(x)y$. However, item (i) of Theorem 3.2 shows that if ρ and $\tilde{\rho}$ are the relative degrees of Σ and $\tilde{\Sigma}$, respectively, then $\rho = \tilde{\rho}$, i.e., for those systems in the same explicitation class and have a well-defined relative degree, the later notion is invariant.

(ii) It is sometimes difficult to check the differentiation index of a given DAE since higher order terms as x'', $x^{(3)}, \ldots$ of the differential array H_k could also show up when we try to express x' in terms of (x, t). But now by the result of Theorem 3.2, if we can find one control system $\Sigma \in \mathbf{Expl}(\Xi)$ such that its relative degree is well-defined, then we can immediately conclude that the differentiation index of Ξ exists and equals to the maximum of the relative degree of Σ.

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