Synchronization with prescribed transient behavior: Heterogeneous multi-agent systems under funnel coupling ⋆

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Abstract

In this paper, we introduce a nonlinear time-varying coupling law, which can be designed in a fully decentralized manner and achieves approximate synchronization with arbitrary precision, under only mild assumptions on the individual vector fields and the underlying graph structure. Meanwhile, we consider undirected graphs and scalar input affine systems for simplicity. The proposed coupling law is motivated by the so called funnel control method studied in adaptive control under the observation that arbitrary precision synchronization can be achieved for heterogeneous multi-agent systems by the high-gain coupling, and thus, we follow to call our coupling law as '(node-wise) funnel coupling.' By getting out of the conventional proof technique in the funnel control study, we now can obtain even asymptotic synchronization with the same funnel coupling law. Moreover, the emergent collective behavior that arises for a heterogeneous multi-agent system when enforcing arbitrary precision synchronization by the proposed funnel coupling law, has been analyzed in this paper. In particular, we introduce a single scalar dynamics called 'emergent dynamics' that is capable of illustrating the emergent synchronized behavior by its solution trajectory. Characterization of the emergent dynamics is important because, for instance, one can design the emergent dynamics first such that the solution trajectory behaves as desired, and then, provide a design guideline to each agent so that the constructed vector fields yield the desired emergent dynamics. A particular example illustrating the utility of the emergent dynamics is given also in the paper as a distributed median solver.

Key words: synchronization, heterogeneous multi-agents, emergent dynamics, funnel control

1 Introduction

1.1 Synchronization of multi-agent systems

During the last decade, synchronization and collective behavior of a multi-agent system have attracted increasing attention because of numerous applications in diverse areas, e.g., biology, physics, and engineering. An initial study was about identical multi-agents (Olfati-Saber & Murray, 2004; Moreau, 2004; Ren & Beard, 2005; Seo, Shim, & Back, 2009), but the interest soon transferred to the heterogeneous case because, uncertainty, disturbance, and noise are prevalent in practice. Therefore, it is a natural follow-up to study synchronization of a heterogeneous multi-agent system. Earlier results in this direction such as (Wieland, Wu, & Allgöwer, 2013) have found that for synchronization to happen in a heterogeneous network, each agent must contain a common internal model that is the same for all the agents. However, recalling that heterogeneity can be given, for instance, by noise, the assumption that a common internal model exists may be too ideal, and approximate (practical) synchronization has been studied as an alternative (Montenbruck, Bürger, & Allgöwer, 2015; Ha, Noh, & Park, 2015). We want to note that it is only recent that some attempts are made to analyze the emergent collective behavior of heterogeneous multi-agent systems that achieve approximate synchronization (Kim, Yang, Shim, Kim, & Seo, 2016; Panteley & Loría, 2017; Lee & Shim, 2020).

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In this respect, a number of papers have considered the construction of a local controller to achieve arbitrary precision approximate synchronization (or asymptotic synchronization) for heterogeneous multi-agent systems. In particular, output regulation theory, backstepping method, high-gain observer, adaptive control, and optimal control have been utilized. Meanwhile, to the best of our knowledge, these works either have a common internal model assumption (De Persis & Jayawardhana, 2012; Isidori, Marconi, & Casadei, 2014; Modares, Lewis, Kang, & Davoudi, 2017; Casadei & Astolfi, 2017), use sufficiently small (or large) parameters which depend on the global information such as the network topology (Su & Huang, 2014; Montenbruck et al., 2015; Zhang, Saberi, Stoorvogel, & Grip, 2016; Kim et al., 2016; Panteley & Loría, 2017), need additional communication channel (Lee, Yun, & Shim, 2018; Su, 2019), or assume individual stability in the broad sense such as output feedback passivity (DeLellis, Di Bernardo, & Liuzza, 2015).

1.2 Novel funnel coupling law and system class

In this paper, we introduce a novel nonlinear time-varying coupling law, which overcomes the above mentioned restrictions, in particular, which

- can be designed in a fully decentralized manner, especially without the need of any global information such as the vector field of other agents or the number of agents in the network,
- does not require any additional assumptions on the individual vector fields such as stability in the broad sense or the common internal model assumption,
- does not need additional communication and uses only the given diffusive coupling terms,
- and achieves prescribed performance, in particular, uniform approximate synchronization with arbitrary precision.

For the set $\mathcal{N} := \{1, \ldots, N\}$ of agents, the individual dynamics for agent $i \in \mathcal{N}$ are assumed to be given by [1]

$$\dot{x}_i(t) = f_i(t, x_i(t)) + u_i(t, \nu_i(t)), \tag{1a}$$
$$\nu_i(t) = \sum_{j \in \mathcal{N}_i} \alpha_{ij} \cdot (x_j(t) - x_i(t)), \tag{1b}$$

where $\mathcal{N}_i$ is a subset of $\mathcal{N}$ whose elements are the indices of the agents that send the information to agent i. The coefficient α_{ij} is the ij-th element of the adjacency matrix that represents the given fixed undirected interconnection graph. In the description, the internal state at time $t \in \mathbb{R}$ is represented by $x_i(t) \in \mathbb{R}$, and u_i : $[t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is the nonlinear time-varying coupling law to be presented later on, which is a continuous mapping from the diffusive coupling term ν_i to the control input and is possibly time-varying. It is a *heterogeneous* multi-agent system in the sense that the vector field f_i is possibly different from each other. We only assume the following properties on the open loop dynamics of each agent.

[1] If the input gain is not unity as illustrated in (1) and possibly state-dependent such as $g_i(t, x_i(t))$ but still invertible and known, then, by re-defining the input $\hat{u}_i := u_i/g_i$, it can be seen that the system satisfies the assumption under consideration.

Assumption 1 *For each $i \in \mathcal{N}$, the function f_i : $[t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is measurable in t, locally Lipschitz with respect to x_i, and bounded on each compact subset of $\mathbb{R}$ uniformly in $t \in [t_0, \infty)$.* $\diamond$

Note that the time-varying f_i can include an external input, a disturbance, and/or noise as well.

Our novel coupling law to ensure synchronization with prescribed performance is inspired by the so called funnel controller (Ilchmann, Ryan, & Sangwin, 2002); in fact, given the desired time-varying error bound ψ_i for each agent i our goal is to ensure that the diffusive error term ν_i given by (1b) evolves within the funnel

$$\mathcal{F}_{\psi_i} := \{(t, \nu_i) \mid |\nu_i| < \psi_i(t)\},$$

see also Figure 1.

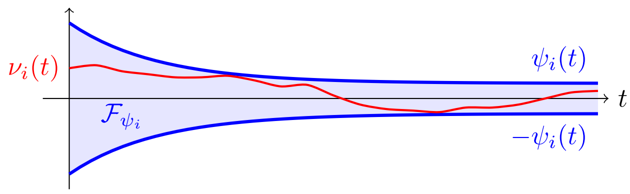

Fig. 1. Prescribed performance: the synchronization error ν_i evolves within the funnel $\mathcal{F}_{\psi_i}$.

In fact, we can achieve uniform arbitrary precision approximate synchronization in the sense that for any arbitrarily small $\eta > 0$ and any given bounded set of initial values we can easily choose funnel boundaries ψ_i such that $|\nu_i(t_0)| < \psi_i(t_0)$ and $\psi_i(\infty) \le \eta$ which then, utilizing our proposed decentralized control method, results in $\limsup_{t\to\infty} |\nu_i(t)| \le \eta$ for each agent $i \in \mathcal{N}$ with a uniform convergence rate (given by the shape of the funnels).

Note that the property $\limsup_{t\to\infty} |\nu_i(t)| \le \eta$ implies $\limsup_{t\to\infty} \|\mathcal{L}x(t)\|_\infty \le \eta$ which then due to connectivity of $\mathcal{G}$ implies that, for all $i, j \in \mathcal{N}$,

$$\limsup_{t\to\infty} \|x_i(t) - x_j(t)\| < \frac{2\sqrt{N}}{\lambda_2}\eta, \tag{2}$$

where λ_2 is the algebraic connectivity of the graph (see (A.6) in the Appendix).

To achieve this control objective, we propose for each agent $i \in \mathcal{N}$ the following *(node-wise) funnel coupling* law

$$u_i(t, \nu_i) := \mu_i\left(\frac{\nu_i}{\psi_i(t)}\right) := \gamma_i\left(\frac{|\nu_i|}{\psi_i(t)}\right)\frac{\nu_i}{\psi_i(t)} \in \mathbb{R} \quad (3)$$

where ν_i is given by (1b) and ψ_i, γ_i satisfies the following assumption.

Assumption 2 *Each function $\psi_i : [t_0, \infty) \to \mathbb{R}_{>0}$ is positive, bounded, and differentiable with bounded derivative, i.e., there exists $\overline{\psi} > 0$ and $\theta_\psi > 0$ such that $0 < \psi_i(t) \leq \overline{\psi}$ and $\left|\dot{\psi}_i(t)\right| \leq \theta_\psi$ for all $t \in [t_0, \infty)$ and $i \in \mathcal{N}$. Moreover, there exists $r_\psi > 0$ such that $\psi_i(t) \leq r_\psi \min_j \psi_j(t)$ for all $t \in [t_0, \infty)$. The gain functions $\gamma_i : [0, 1) \to \mathbb{R}_{\geq 0}$, $i \in \mathcal{N}$, are strictly increasing and satisfies $\lim_{s\to 1} \gamma_i(s) = \infty$.* ⋄

Note that the above assumption on the performance functions ψ_i regarding the constant r_ψ only becomes relevant when (some of) the performance functions asymptotically converge to zero. A possible (agent independent) choice for γ_i and ψ_i is

$$\gamma_i(s) = \frac{1}{1-s} \text{ and } \psi_i(t) = (\overline{\psi} - \eta)e^{-\lambda(t-t_0)} + \eta,$$

where $\overline{\psi}, \lambda > 0$ and $\eta \geq 0$.

This funnel coupling law is motivated by the observation that approximate synchronization with arbitrary precision can be obtained by the high-gain linear coupling law $u_i(t, \nu_i) = k\nu_i$ (Lee & Shim, 2020) (which corresponds to the high-gain property in the funnel control study). In fact, as for the funnel controller, it will be proven that the funnel coupling law achieves synchronization with respect to the given performance functions ψ_i, i.e., we have $|\nu_i(t)| < \psi_i(t)$ for all $t \geq t_0$ and $i \in \mathcal{N}$, under only mild assumptions. We emphasize that now transient performance can also be guaranteed as done by the funnel control.

1.3 Related approaches

This idea has been first proposed in (Shim & Trenn, 2015), however, due to some technical reasons, the analysis was conducted only for the weakly centralized funnel coupling, i.e., $u_i(t) = \max_j \gamma_j(|\nu_j|/\psi(t))\nu_i/\psi(t)$, and only when the underlying graph is d-regular with $d > N/2 - 1$, where d is the degree of every node. By getting out of the conventional proof technique in the funnel control study, these technical limitations have been resolved in this paper, and we can now consider fully decentralized coupling law (3) with an arbitrarily given graph which is undirected and connected. This new approach also allows the performance functions ψ_i to converge asymptotically to zero, i.e., $\lim_{t\to\infty} \psi_i(t) = 0$, by which we obtain *asymptotic* synchronization for heterogeneous multi-agent systems. In particular, we have $\lim_{t\to\infty} \nu_i(t) = 0$, $i \in \mathcal{N}$ by the fact that $|\nu_i(t)| < \psi_i(t)$ for all $t \geq t_0$ and $i \in \mathcal{N}$. This, in fact, seems to violate the common presumption, in the synchronization community, that heterogeneous multi-agent systems can not asymptotically synchronize without a common internal model. This violation is resolved by observing that we use a time-varying coupling law, which is not considered in the framework of the internal model principle for multi-agent systems (Wieland et al., 2013). In fact, unlike the internal model principle results, it is observed in this paper, that as the performance function approaches zero, the coupling term approaches to possibly non-zero time-varying signal, which compensates the heterogeneity of the individual agents. Specific use of this idea to solve distributed consensus optimization can be found in (Lee, Berger, Trenn, & Shim, 2020a). We want to emphasize that even when asymptotic synchronization is achieved, the input $u_i(t, \nu_i(t))$ can still be bounded. In fact, even though the performance functions ψ_i are asymptotically converging to zero, the diffusive term ν_i which also converges asymptotically to zero makes the fraction $\nu_i(t)/\psi_i(t)$ be strictly contained inside the interval $(-1, 1)$ uniformly, making the input $\mu_i(\nu_i(t)/\psi_i(t))$ to be uniformly bounded. We refer to Section 2 for sufficient conditions that guarantee the boundedness of input.

Relying also on the observation that arbitrary precision synchronization can be achieved by the high-gain linear coupling law, a dynamic coupling law motivated by the λ-tracking studied in adaptive controls (Ilchmann & Ryan, 1994) given, for instance, as

$$u_i(t, \nu_i(t)) = k_i(t)\nu_i(t),$$
$$\dot{k}_i(t) = \begin{cases} |\nu_i(t)|(|\nu_i(t)| - \eta_i) & \text{if } |\nu_i(t)| > \eta_i, \\ 0 & \text{otherwise,} \end{cases}$$

has been introduced in (Shafi & Arcak, 2014; Li, Ren, Liu, & Fu, 2013; Lv, Li, Duan, & Feng, 2017; Kim & De Persis, 2017; Lee et al., 2018). But, most of them considered homogeneous networks, and for a heterogeneous network, additional communication between the coupling gains k_i has been introduced to ensure that the collective behavior of the network is as desired. In fact, funnel control has advantages compared to λ-tracker, where one is that the transient behavior can be directly controlled, i.e., the time to reach the desired accuracy can be managed, and the other is that the gain is not monotonically increasing, so that when the error is small, the gain also becomes small, and thus, does not amplify the measurement noise unnecessarily.

1.4 Emergent dynamics

To estimate the behavior of the network when synchronization is achieved in this way, as in (Kim et al., 2016;

Panteley & Loría, 2017; Lee & Shim, 2020), the emergent collective behavior that arises from the closed loop system (1) with (3) is analyzed in this paper. In particular, we introduce a single scalar dynamics which we call 'emergent dynamics' (which depends on the individual vector fields f_i and the functions μ_i, ψ_i for all $i \in \mathcal{N}$) that is capable of illustrating the emergent synchronized behavior of the whole network by its solution trajectory. Characterization of the emergent collective behavior or the emergent dynamics is important, for instance, when synthesizing a heterogeneous network for some specific purposes. In particular, one can design the emergent dynamics first such that the solution trajectory behaves as desired, and then, provide a design guideline to each agent (which allows fully decentralized design) so that the constructed vector field and μ_i function yields the desired emergent dynamics. This scheme of constructing a heterogeneous network with the desired collective behavior is first introduced in (Lee & Shim, 2020) and has many interesting applications, e.g., distributed state estimation, estimation of the number of agents, and economic dispatch problem. For instance, in (Lee & Shim, 2020), it is analyzed that the emergent behavior of a heterogeneous network (1) under the high-gain coupling $u_i(t, \nu_i) = k\nu_i$, follows the 'blended dynamics' given by

$$\dot{s} = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s). \tag{4}$$

Under this observation, in (Lee & Shim, 2020), for example, a network that estimates the number of agents is designed as $\dot{x}_1 = -x_1 + 1 + k\nu_1$ and $\dot{x}_i = 1 + k\nu_i$ for $i \neq 1$, which has $\dot{s} = -(1/N)s + 1$ as its blended dynamics, i.e., the emergent collective behavior asymptotically converges to the number of agents N. Now, the emergent dynamics to be introduced later on takes clearly a different form compared to the blended dynamics due to the difference in the coupling law, and by this difference, a new application might occur, which needs further inspection. A particular example illustrating the utility of the emergent dynamics is given in Section 4.5 as a distributed median solver. We emphasize that each agent may be unstable (or malfunctioning, or even malicious), as long as their combination, i.e., the emergent dynamics is stable, hence the stability of the group is maintained. This may provide a way to keep public good against malicious agents by a majority of good neighbors. Synchronization achieved in this way is also robust against external disturbance, noise, and/or uncertainty in the agent dynamics due to the stability of the emergent dynamics.

1.5 Paper organization and notation

The paper is organized as follows. In Section 2, it is proven that the proposed node-wise funnel coupling law achieves synchronization with respect to the given performance function. Some sufficient conditions that ensure boundedness of the inputs are also given at the end of that section. Section 3 analyzes the emergent collective behavior that arises when enforcing synchronization by the proposed funnel coupling law. Then, in Section 4, we discover the properties of the emergent dynamicsand also discuss the possible application related to these properties.

Notation: Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of a graph is defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$, where $\mathcal{A} = [\alpha_{ij}]$ is the adjacency matrix of the graph and $\mathcal{D}$ is the diagonal matrix whose diagonal entries are determined such that each row sum of $\mathcal{L}$ is zero. By its construction, it contains at least one eigenvalue of zero, whose corresponding eigenvector is $1_N := [1, \ldots, 1]^T \in \mathbb{R}^N$, and all the other eigenvalues have non-negative real parts. For undirected graphs, the zero eigenvalue is simple if and only if the corresponding graph is connected. For vectors or matrices a and b, $\mathrm{col}(a, b) := [a^T, b^T]^T$. For matrices $A_1, \ldots, A_k$, we denote by $\mathrm{diag}(A_1, \ldots, A_k)$ the corresponding block diagonal matrix. For a set $\Xi \subseteq \mathbb{R}$, $|x|_\Xi$ denotes the distance between the value $x \in \mathbb{R}$ and Ξ, i.e., $|x|_\Xi := \inf_{y\in\Xi} |x-y|$.

2 Heterogeneous multi-agent systems under node-wise funnel coupling

The intuition of the funnel coupling law (3) is simple, following that of funnel control, which is to enlarge the gain infinitely large as the error (diffusive term) approaches the funnel boundary. Then, the high-gain coupling (or the high-gain property in the funnel control study) precludes boundary contact. For instance, if agent i has only one neighbor denoted as agent j, and if the difference between two agents, $\nu_i(t) = \alpha_{ij}(x_j(t) - x_i(t))$, approaches the funnel boundary $\pm\psi_i(t)$ so that $\psi_i(t) - |\nu_i(t)|$ becomes closer to zero, then the gain $\gamma_i(|\nu_i(t)|/\psi_i(t))$ gets larger towards infinity, and the state x_i will tend to its neighbor x_j since the large coupling term dominates the vector field $f_i(t, x_i)$, and the error $\nu_i(t)$ will remain inside the funnel. However, with more than one neighbor, this intuition becomes no longer straightforward because two neighbors may attract x_i in the opposite direction with almost infinite power. In the following, we will prove that all the errors $\nu_i(t)$ remain inside the funnel (hence approximate or asymptotic synchronization follows), which is however far more complicated and also requires the following technical assumption, which guarantees that if the diffusive term is contained in the funnel, i.e., $|\nu_i(t)| < \psi_i(t)$, then finite time escape cannot occur.

Assumption 3 *The dynamical systems defined by*

$$\dot{\overline{\chi}}(t) = \max_{i\in\mathcal{N}} f_i(t, \overline{\chi}(t)), \quad \dot{\underline{\chi}}(t) = \min_{i\in\mathcal{N}} f_i(t, \underline{\chi}(t)),$$

have complete solutions $\overline{\chi}, \underline{\chi} : [t_0, \infty) \to \mathbb{R}$ for any initial values $\overline{\chi}(t_0), \underline{\chi}(t_0) \in \mathbb{R}$ and for any initial time t_0. ⋄

We stress that if the functions f_i are globally Lipschitz in x_i, then Assumption 3 is satisfied.

Lemma 1 *In addition to Assumptions 2 and 3, let us assume that the solution of the system* (1) *coupled via* (3) *exists on* $[t_0, \omega)$ *for some* $\omega \in (t_0, \infty)$ *and satisfies* $|\nu_i(t)| < \psi_i(t)$ *for all* $t \in [t_0, \omega)$ *and* $i \in \mathcal{N}$. *Then, there exists* $M > 0$ *(depending on* $t_0, \omega, x_1(t_0), \ldots x_N(t_0)$*) such that* $|x_i(t)| < M$ *for all* $i \in \mathcal{N}$ *and* $t \in [t_0, \omega)$. ◇

Before giving the proof of Lemma 1 we want to highlight the importance of Lemma 1 by showing that Assumption 3 (which is independent of the input) guarantees that there is no finite escape time of the state-variables when the funnel coupling is used. This is remarkable because the inputs can grow unbounded on a finite interval when the state-difference approaches the funnel boundary.

PROOF. Choose a time-varying index $J(t) \in \mathcal{N}$ such that $x_{J(t)}(t) = \max_i x_i(t)$ and $\dot{x}_{J(t)}(t) \geq \dot{x}_k(t)$ for all those $k \in \mathcal{N}$ with $x_k(t) = \max_i x_i(t)$. Then, the upper right Dini derivative of $x_{J(t)}(t)$ denoted as $D^+ x_{J(t)}(t)$ satisfies

$$\begin{aligned} D^+ x_{J(t)}(t) &\leq \dot{x}_{J(t)}(t) \\ &= f_{J(t)}(t, x_{J(t)}(t)) + \gamma_{J(t)} \left(\frac{|\nu_{J(t)}(t)|}{\psi_{J(t)}(t)} \right) \frac{\nu_{J(t)}(t)}{\psi_{J(t)}(t)} \\ &\leq f_{J(t)}(t, x_{J(t)}(t)) \leq \max_i f_i(t, x_{J(t)}(t)) \end{aligned}$$

where the second inequality follows from the fact that $\gamma_{J(t)}$ and $\psi_{J(t)}$ are non-negative and $\nu_{J(t)}(t)$ is non-positive, because $x_{J(t)}(t)$ is a maximum. Hence, by Assumption 3, there exists $M_+ > 0$ (depending on $t_0, \omega, x_1(t_0), \ldots x_N(t_0)$) such that $x_{J(t)}(t)$ is upper bounded by M_+ for $t \in [t_0, \omega)$. Similarly, we can find $M_- > 0$ such that $\min_i x_i(t) \geq -M_-$ for all $t \in [t_0, \omega)$, which concludes our claim. □

Assumption 4 *The communication graph induced by the adjacency element* α_{ij} *is undirected and connected, and thus, the Laplacian matrix* $\mathcal{L}$ *is symmetric, having one simple eigenvalue of zero.* ◇

Theorem 2 *Consider the system* (1) *coupled via node-wise funnel coupling* (3). *Under Assumptions 1–4, and the assumption that* $|\nu_i(t_0)| < \psi_i(t_0)$ *for all* $i \in \mathcal{N}$, *funnel coupling leads to a solution defined on the whole time interval* $[t_0, \infty)$. *In particular, we have* $|\nu_i(t)| < \psi_i(t)$ *for all* $t \in [t_0, \infty)$ *and* $i \in \mathcal{N}$ *and in view of* (2), *approximate/asymptotic synchronization occurs.* ◇

The proof of the main theorem relies on the following crucial technical result, which will be proven first.

Lemma 3 *In addition to the assumptions of Theorem 2, let us assume that the solution of the system* (1) *coupled via* (3) *exists on* $[t_0, \omega)$ *for some* $\omega \in (t_0, \infty]$ *and satisfies* $|\nu_i(t)| < \psi_i(t)$ *for all* $t \in [t_0, \omega)$ *and* $i \in \mathcal{N}$. *If there exists* $\overline{M}_f$ *(which may depend on* t_0, ω, *and* $x_i(t_0)$, $i \in \mathcal{N}$*) such that*

$$|f_j(t, x_j(t)) - f_i(t, x_i(t))| \leq \overline{M}_f, \quad \forall i \in \mathcal{N},\, j \in \mathcal{N}_i, \tag{5}$$

for all $t \in [t_0, \omega)$, *then the index sets defined as*

$$\mathcal{I}_+ := \left\{ i \in \mathcal{N} : \exists \{\tau_k\} \to \omega, \ s.t. \ \lim_{k\to\infty} \frac{\nu_i(\tau_k)}{\psi_i(\tau_k)} = 1 \right\}$$

$$\mathcal{I}_- := \left\{ i \in \mathcal{N} : \exists \{\tau_k\} \to \omega, \ s.t. \ \lim_{k\to\infty} \frac{\nu_i(\tau_k)}{\psi_i(\tau_k)} = -1 \right\}$$

are empty. ◇

PROOF. Assume first that the index set $\mathcal{I}_+$ is nonempty. In order to arrive at a contradiction, we will construct a sequence of strictly increasing index sets $\mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \mathcal{J}_3 \subsetneq \ldots$ that are all contained in $\mathcal{I}_+$; which of course is impossible due to the finiteness of $\mathcal{I}_+$. Hence $\mathcal{I}_+$ must be empty, and analogous argument yield that $\mathcal{I}_-$ must also be empty.

So, first, take any element of $\mathcal{I}_+$, say j_1. Then, by the definition of $\mathcal{I}_+$ there exists a strictly increasing time sequence $\{\tau_k^1\}$ such that $\lim_{k\to\infty} \tau_k^1 = \omega$ and satisfies $\lim_{k\to\infty} \nu_{j_1}(\tau_k^1)/\psi_{j_1}(\tau_k^1) = 1$. Then

$$\mathcal{J}_1 := \left\{ i \in \mathcal{I}_+ : \lim_{k\to\infty} \frac{\nu_i(\tau_k^1)}{\psi_i(\tau_k^1)} = 1 \right\},$$

is nonempty (because it contains j_1) and is a subset of $\mathcal{I}_+$. Inductively, assume now that for $n \geq 1$ a nonempty index set is given by

$$\mathcal{J}_n := \left\{ i \in \mathcal{I}_+ : \lim_{k\to\infty} \frac{\nu_i(\tau_k^n)}{\psi_i(\tau_k^n)} = 1 \right\},$$

where $\{\tau_k^n\}_{k\in\mathbb{N}}$ is a strictly increasing sequence converging to ω. We now construct a strictly increasing sequence $\{\tau_k^{n+1}\}$ converging to ω as $k \to \infty$ such that the corresponding set $\mathcal{J}_{n+1}$ contains $\mathcal{J}_n$ and there is $j_{n+1} \in \mathcal{I}_+$ which is in $\mathcal{J}_{n+1}$ but not in $\mathcal{J}_n$. Therefore we will first construct a sequence $\{s_p^{n+1}\}_{p\in\mathbb{N}}$ such that

$$\forall i \in \mathcal{J}_n : \lim_{p\to\infty} \frac{\nu_i(s_p^{n+1})}{\psi_i(s_p^{n+1})} = 1 \tag{6}$$

and such that for each $p \in \mathbb{N}$ there is an index $j_p \in \mathcal{I}_+ \setminus \mathcal{J}_n$ with

$$\frac{\nu_{j_p}(s_p^{n+1})}{\psi_{j_p}(s_p^{n+1})} > 1 - \delta_p^n; \tag{7}$$

here $\{\delta_p^n\}_{p\in\mathbb{N}}$ is some strictly decreasing sequence converging to zero with $\delta_0^n > 0$ such that for all $i \in \mathcal{N} \setminus \mathcal{I}_+$ and all $t \in [t_0, \omega)$ we have $\nu_i(t)/\psi_i(t) \le 1 - \delta_0^n$. Since $\mathcal{I}_+ \setminus \mathcal{J}_n$ is finite we find a subsequence $\tau_k^{n+1} := s_{p_k}^{n+1}$ and an index $j_{n+1} \in \mathcal{I}_+ \setminus \mathcal{J}_n$ such that

$$\lim_{k\to\infty} \frac{\nu_{j_{n+1}}(\tau_k^{n+1})}{\psi_{j_{n+1}}(\tau_k^{n+1})} = 1;$$

in other words, $j_{n+1} \in \mathcal{J}_{n+1}$, where $\mathcal{J}_{n+1}$ is defined analogously as $\mathcal{J}_n$ via the sequence $\{\tau_k^{n+1}\}$. Since (6) also holds for any subsequence, it follows that $\mathcal{J}_n \subsetneq \mathcal{J}_{n+1}$. Therefore, it remains to construct the sequence s_p^{n+1} such that (6) and (7) hold.

Towards this goal, we will first choose a strictly decreasing sequence $\{\varepsilon_p^n\}$ with $\varepsilon_p^n \to 0$ as $p \to \infty$ and such that each $\varepsilon_p^n > 0$ is so small that

$$\begin{aligned} 1-\varepsilon_p^n &\ge \max_{i,j\in\mathcal{N}} \mu_i^{-1}\left(\frac{\mu_j(1-\delta_p^n)}{1-\delta_p^n}\right) \quad \text{and} \\ 1-\varepsilon_p^n &\ge \max_{i\in\mathcal{N}} \mu_i^{-1}\left(\frac{M_0 + N\theta_\psi(1+r_\psi)}{\underline{\alpha}\delta_p^n}\right) \end{aligned} \tag{8}$$

where $\underline{\alpha} := \min_{i\in\mathcal{N},\, j\in\mathcal{N}_i} \alpha_{ij} > 0$ and

$$M_0 := \theta_\psi \sum_{i\in\mathcal{I}^+} r_\psi + \sum_{i\in\mathcal{I}^+}\sum_{j\in\mathcal{N}} \alpha_{ij}\overline{M}_f.$$

This choice is possible because $\mu_i : (-1,1) \to \mathbb{R}$ is strictly increasing and bijective due to Assumption 2. Furthermore, let

$$W_n(t) := \sum_{i\in\mathcal{J}_n} \frac{\nu_i(t)}{\psi_1(t)}, \quad t \in [t_0, \omega).$$

Then, by the definition of $\mathcal{J}_n$, $\lim_{k\to\infty} W_n(\tau_k^n) = \sum_{i\in\mathcal{J}_n} \psi_i(\omega)/\psi_1(\omega)$ and $W_n(t) < \sum_{i\in\mathcal{J}_n} \psi_i(t)/\psi_1(t)$ for all $t \in [t_0, \omega)$. Choose a subsequence $\{\tau_{k_p}^n\}_{p\in\mathbb{N}}$ of $\{\tau_k^n\}_{k\in\mathbb{N}}$ such that

$$W_n(\tau_{k_p}^n) \ge \sum_{i\in\mathcal{J}_n} \frac{\psi_i(\tau_{k_p}^n)}{\psi_1(\tau_{k_p}^n)} - \frac{1}{r_\psi}\frac{\varepsilon_p^n}{2}, \quad \forall p \in \mathbb{N}.$$

Based on this sequence, we now define a sequence $\{s_p^{n+1}\}_{p\in\mathbb{N}}$ as follows, see also Figure 2, [2]

$$s_p^{n+1} := \max\left\{ s \in [t_0, \tau_{k_p}^n] \;\middle|\; W_n(s) = \sum_{i\in\mathcal{J}_n} \frac{\psi_i(s)}{\psi_1(s)} - \frac{1}{r_\psi}\varepsilon_p^n \right\}.$$

[2] Without loss of generality we assume $W_n(t_0) < \sum_{i\in\mathcal{J}_n} \psi_i(t_0)/\psi_1(t_0) - \varepsilon_0^n/r_\psi$.

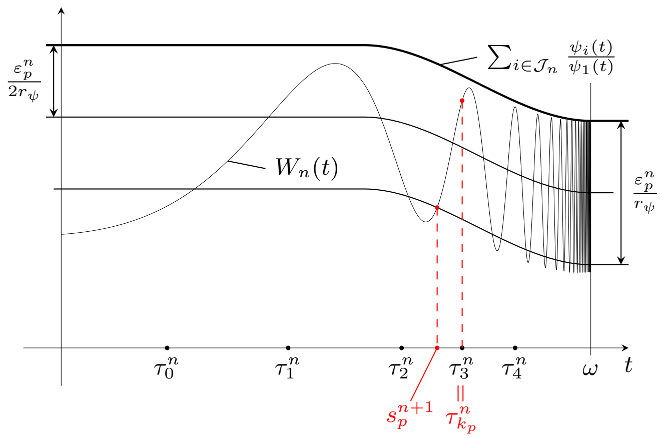

Fig. 2. Illustration of the choice of the sequence $\{s_p^{n+1}\}_{p\in\mathbb{N}}$ based on $\{\tau_k^n\}_{k\in\mathbb{N}}$.

By the choice of s_p^{n+1} we have that $W_n(s_p^{n+1}) \to \sum_{i\in\mathcal{J}_n} \psi_i(\omega)/\psi_1(\omega)$ as $p \to \infty$ and therefore (6) holds. Furthermore, the choice of s_p^{n+1} also implies

$$\dot{W}_n(s_p^{n+1}) \ge \sum_{i\in\mathcal{J}_n} \frac{\dot{\psi}_i(s_p^{n+1})\psi_1(s_p^{n+1}) - \psi_i(s_p^{n+1})\dot{\psi}_1(s_p^{n+1})}{\psi_1(s_p^{n+1})^2}.$$

By assuming now that (7) does *not* hold; we will show in the following that $\psi_1(s_p^{n+1})\dot{W}_n(s_p^{n+1}) < -N\theta_\psi(1+r_\psi)$ and hence arrive at the contradiction [3]

$$\begin{aligned} &\sum_{i\in\mathcal{J}_n}\left[\dot{\psi}_i(s_p^{n+1}) - \frac{\psi_i(s_p^{n+1})}{\psi_1(s_p^{n+1})}\dot{\psi}_1(s_p^{n+1})\right] \\ &\quad \le \psi_1(s_p^{n+1})\dot{W}_n(s_p^{n+1}) < -N\theta_\psi(1+r_\psi) \\ &\quad \le -\sum_{i\in\mathcal{J}_n}\sup_{t\in[t_0,\omega)}\left[\left|\dot{\psi}_i(t)\right| + \frac{\psi_i(t)}{\psi_1(t)}\left|\dot{\psi}_1(t)\right|\right]. \end{aligned}$$

The derivative of W_n can be bounded as follows

$$\begin{aligned} \dot{W}_n(t) &= -\frac{\dot{\psi}_1(t)\psi_i(t)}{\psi_1(t)^2}\sum_{i\in\mathcal{J}_n}\frac{\nu_i(t)}{\psi_i(t)} \\ &\quad + \frac{1}{\psi_1(t)}\sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}}\alpha_{ij}(f_j(t,x_j(t)) - f_i(t,x_i(t))) \\ &\quad + \frac{1}{\psi_1(t)}\sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}}\alpha_{ij}\left[\mu_j\left(\tfrac{\nu_j(t)}{\psi_j(t)}\right) - \mu_i\left(\tfrac{\nu_i(t)}{\psi_i(t)}\right)\right] \\ &\le \frac{M_0}{\psi_1(t)} + \frac{1}{\psi_1(t)}\sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}}\alpha_{ij}\left[\mu_j\left(\tfrac{\nu_j(t)}{\psi_j(t)}\right) - \mu_i\left(\tfrac{\nu_i(t)}{\psi_i(t)}\right)\right] \end{aligned}$$

where we used (5) and $\mathcal{J}_n \subseteq \mathcal{I}^+$. Invoking now that the

[3] Note that $\psi_i(t)/\psi_1(t) \le \psi_i(t)/\min_j \psi_j(t) \le r_\psi$.

graph is undirected, we get [4]

$$\psi_1(s_p^{n+1})\dot{W}_n(s_p^{n+1}) \leq M_0+\sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}\setminus\mathcal{J}_n}\alpha_{ij}\left[\mu_j\left(\tfrac{\nu_j(s_p^{n+1})}{\psi_j(s_p^{n+1})}\right)-\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi_i(s_p^{n+1})}\right)\right]. \quad (9)$$

To arrive at the sought contradiction $\psi_1(s_p^{n+1})\dot{W}_n(s_p^{n+1}) < -N\theta_\psi(1+r_\psi)$ we will exploit that due to (6) the terms $\mu_i\left(\frac{\nu_i(s_p^{n+1})}{\psi_i(s_p^{n+1})}\right)$ will be very large for each $i\in\mathcal{J}_n$ while $\mu_j\left(\frac{\nu_j(s_p^{n+1})}{\psi_j(s_p^{n+1})}\right)$ for $j\notin\mathcal{J}_n$ will not be very large.

By rewriting the definition of W_n we get $\nu_i=\psi_1(t)W_n-\sum_{j\neq i,j\in\mathcal{J}_n}\nu_j$ for each $i\in\mathcal{J}_n$ and hence [5]

$$\begin{aligned}\nu_i(s_p^{n+1}) &> \psi_1(s_p^{n+1})W_n(s_p^{n+1})-\sum_{j\neq i,j\in\mathcal{J}_n}\psi_j(s_p^{n+1})\\ &=\sum_{j\in\mathcal{J}_n}\psi_j(s_p^{n+1})-\frac{\psi_1(s_p^{n+1})}{r_\psi}\varepsilon_p^n-\sum_{j\neq i,j\in\mathcal{J}_n}\psi_j(s_p^{n+1})\\ &\geq \psi_i(s_p^{n+1})(1-\varepsilon_p^n). \quad (10)\end{aligned}$$

Therefore, by monotonicity of the μ-functions and the choice of ε_p^n and δ_0^n we have for each $j\in\mathcal{N}\setminus\mathcal{I}_+$ that

$$\begin{aligned}\mu_j\left(\tfrac{\nu_j(s_p^{n+1})}{\psi_j(s_p^{n+1})}\right) &\leq \mu_j(1-\delta_0^n)\leq\mu_j(1-\delta_p^n)\\ &\overset{(8)}{\leq}(1-\delta_p^n)\mu_i(1-\varepsilon_p^n)\overset{(10)}{\leq}(1-\delta_p^n)\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi_i(s_p^{n+1})}\right).\end{aligned}$$

Since we assumed that (7) does *not* hold, the same outer inequality also holds for all $j\in\mathcal{I}_+\setminus\mathcal{J}_n$.

This allows us to bound, in (9), each μ_j term by $(1-\delta_p^n)\mu_i$ term, i.e.,

$$\begin{aligned}\psi_1(s_p^{n+1})\dot{W}_n(s_p^{n+1}) &\leq M_0-\sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}\setminus\mathcal{J}_n}\alpha_{ij}\delta_p^n\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi_i(s_p^{n+1})}\right)\\ &\leq M_0-\underline{\alpha}\delta_p^n\min_{i\in\mathcal{J}_n}\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi_i(s_p^{n+1})}\right)\\ &\overset{(10)}{<}M_0-\underline{\alpha}\delta_p^n\min_{i\in\mathcal{J}_n}\mu_i(1-\varepsilon_p^n)\overset{(8)}{\leq}-N\theta_\psi(1+r_\psi),\end{aligned}$$

which is the sought contradiction, hence (7) must hold and the proof is complete. □

PROOF OF THEOREM 2. Let

$$\Omega_\psi := \{(t,x)\in\mathbb{R}_{\geq t_0}\times\mathbb{R}^N : |\mathcal{L}_{\psi(t)}x|_\infty<1\}$$

where $\mathcal{L}_{\psi(t)} := \operatorname{diag}(1/\psi_1(t),\ldots,1/\psi_N(t))\mathcal{L}$ and $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^N$. Since the right-hand side of (1) with (3) is locally Lipschitz on $\Omega_\psi\subseteq\mathbb{R}^{N+1}$, the standard theory of ODEs yields existence and uniqueness of a maximally extended solution $x:[t_0,\omega)\to\mathbb{R}^N$ such that $(t,x(t))\in\Omega_\psi$ for all $t\in[t_0,\omega)$ (Hartman, 1964). If $\omega=\infty$, then nothing is to show anymore.

Thus, suppose that $\omega<\infty$ for a given maximal solution. Then, again from (Hartman, 1964), there exists a time sequence $\{\tau_k\}\to\omega$ such that $\lim_{k\to\infty}|\mathcal{L}_{\psi(\tau_k)}x(\tau_k)|_\infty=1$. In fact, there exist a subsequence $\{k_p\}\to\infty$ and an index i, such that

$$\lim_{p\to\infty}\frac{\nu_i(\tau_{k_p})}{\psi_i(\tau_{k_p})}=1 \quad\text{or}\quad \lim_{p\to\infty}\frac{\nu_i(\tau_{k_p})}{\psi_i(\tau_{k_p})}=-1.$$

This is because, otherwise we can continue the solution after ω, which violates the fact that x is maximal.

However, since we have $\omega<\infty$, there exists $M>0$ which can possibly depend on $(t_0,\omega,x_1(t_0),\ldots,x_N(t_0))$, such that $|x_i(t)|<M$ for all $i\in\mathcal{N}$ and $t\in[t_0,\omega)$, by Lemma 1. Therefore, by Assumption 1 considered for the compact set $[-M,M]$, there exists $\overline{M}_f$ which can possibly depend on $(t_0,\omega,x_1(t_0),\ldots,x_N(t_0))$, such that (5) is satisfied for all $t\in[t_0,\omega)$. Lemma 3 now shows that $\omega<\infty$ is impossible. This completes the proof. □

Remark 4 *The assumption that the initial values of the agents are such that $|\nu_i(t_0)|<\psi_i(t_0)$ may seem restrictive. However, each agent can pick a suitable ψ_i so that this assumption is satisfied at the initial time. Furthermore, it is possible to adapt the proof of Theorem 2 to also allow for a performance funnel ψ_i with $\psi_i(t_0)=\infty$, e.g., $\psi_i(t)=\frac{1}{t-t_0}$, so that $|\nu_i(t_0)|<\psi_i(t_0)$ is trivially satisfied; however, we omit the technical details as we believe that in most applications some (physical) bounds on the states are known and can be used to choose a suitable finite ψ_i for each agent.* ⋄

Remark 5 *The proof technique used in Theorem 2 and Lemma 3, can easily be extended to* non-scalar *fully actuated agents with dynamics*

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= F_i(t,\boldsymbol{x}_i(t))+\boldsymbol{u}_i(t,\boldsymbol{\nu}_i(t))\in\mathbb{R}^n,\\ \boldsymbol{\nu}_i(t) &= \sum_{j\in\mathcal{N}_i}\alpha_{ij}(\boldsymbol{x}_j(t)-\boldsymbol{x}_i(t)) =: \operatorname{col}(\nu_i^1(t),\ldots,\nu_i^n(t)).\end{aligned}$$

[4] For undirected graphs, we have for any $\chi\in\mathbb{R}^N$ and index set $\mathcal{J}\subsetneq\mathcal{N}$ the following identity:

$$\sum_{i\in\mathcal{J}}\sum_{j\in\mathcal{N}}\alpha_{ij}(\chi_j-\chi_i)=\sum_{i\in\mathcal{J}}\sum_{j\in\mathcal{N}\setminus\mathcal{J}}\alpha_{ij}(\chi_j-\chi_i)$$

where the fact that $\sum_{i\in\mathcal{J}}\sum_{j\in\mathcal{J}}\alpha_{ij}(\chi_j-\chi_i)=0$ is used.

[5] Note that $-\psi_1(t)/r_\psi\geq-\min_j\psi_j(t)\geq-\psi_i(t)$.

In this case, the multi-dimensional funnel coupling can be chosen as an element-wise *funnel coupling*

$$\boldsymbol{u}_i(t,\boldsymbol{\nu}_i) := \mathrm{col}\left(\mu_i\left(\frac{\nu_i^1}{\psi_i(t)}\right),\ldots,\mu_i\left(\frac{\nu_i^n}{\psi_i(t)}\right)\right), \quad (11)$$

or a maximum gain *funnel coupling:*

$$\boldsymbol{u}_i(t,\boldsymbol{\nu}_i) := \boldsymbol{\mu}_i\left(\frac{\boldsymbol{\nu}_i}{\psi_i(t)}\right) := \gamma_i\left(\frac{|\boldsymbol{\nu}_i|_\infty}{\psi_i(t)}\right)\frac{\boldsymbol{\nu}_i}{\psi_i(t)}. \quad (12)$$

The more interesting (and probably more relevant) case is when there are fewer inputs (and outputs) than states which seems to be more complicated and is currently ongoing research. ⋄

Acknowledging the advantages of the funnel coupling law (3) guaranteed by Theorem 2, we have to be however careful about its nonlinear structure. Since the funnel coupling law does not have a memory and the size of it can increase arbitrarily large, the input used for the synchronization might grow unboundedly as time flows. Therefore, in addition to proving that the diffusive term resides inside the funnel, we have to ensure separately that the input used in our system is uniformly bounded on the time interval $[t_0,\infty)$. Luckily, Lemma 3 provides us with some sufficient conditions, which guarantee boundedness of the input as follows.

Corollary 6 *In addition to the assumptions of Theorem 2, assume that one of the following conditions hold.*

(a) $f_i(t,x) \equiv F(t,x) + g_i(t,x)$ *where* $F(t,x)$ *is globally Lipschitz with respect to* x *uniformly in* t *and there exists* $\overline{M}_g$ *such that* $|g_i(t,x)| \le \overline{M}_g$ *for all* $i \in \mathcal{N}$, $t \ge t_0$, *and* $x \in \mathbb{R}$.
(b) There exists $\overline{M}_x$ *such that* $|x_i(t)| \le \overline{M}_x$ *for all* $i \in \mathcal{N}$ *and* $t \ge t_0$.

Then the input $u_i(t,\nu_i(t)) \equiv \mu_i(\nu_i(t)/\psi_i(t))$ *is uniformly bounded on* $[t_0,\infty)$, *i.e., there exists* $\overline{M}_u > 0$ *such that for all* $t \in [t_0,\infty)$ *and* $i \in \mathcal{N}$, *we have* $|u_i(t,\nu_i(t))| \le \overline{M}_u$. ⋄

PROOF. Note first that, if the condition (5) of Lemma 3 is satisfied with $\omega = \infty$, then this implies that the input $u_i(t,\nu_i(t))$ is uniformly bounded on $[t_0,\infty)$. This is because, then by Lemma 3 the index sets $\mathcal{I}_+$ and $\mathcal{I}_-$ of Lemma 3 with $\omega = \infty$ are empty, i.e., there exists $\delta > 0$ such that $|\nu_i(t)/\psi_i(t)| < 1-\delta$ for all $t \ge t_0$ and $i \in \mathcal{N}$, which implies $|u_i(t,\nu_i(t))| \le \max_i \mu_i(1-\delta) =: \overline{M}_u$ for all $t \ge t_0$ and $i \in \mathcal{N}$.

Now, condition (a) ensures (5) with $\omega = \infty$, because

$$\begin{aligned}&|f_j(t,x_j(t)) - f_i(t,x_i(t))|\\ &\le |F(t,x_j(t)) - F(t,x_i(t))| + |g_j(t,x_j)| + |g_i(t,x_i)|\\ &\le \overline{L}|x_j(t)-x_i(t)| + 2\overline{M}_g \le 2\overline{L}\sqrt{N}\overline{\psi}/\lambda_2 + 2\overline{M}_g\end{aligned}$$

where $\overline{L}$ is the Lipschitz constant of F and the third inequality follows from (A.6). On the other hand, condition (b) combined with the properties of the vector fields guarantees that the condition (5) of Lemma 3 is again satisfied with $\omega = \infty$. This completes the proof. □

Remark 7 *It is interesting to note that uniform boundedness of the vector fields is actually not required for Theorem 2 and Corollary 6 (a). In fact, we only need boundedness of* f_i *on each compact subset of* $[t_0,\infty)\times\mathbb{R}$. *It is also quite remarkable that even the assumption on the performance functions* ψ_i *regarding the constant* r_ψ, *which is somewhat natural, is not needed in Theorem 2.* ⋄

Remark 8 *A key assumption in funnel control so far was that the funnel boundary* ψ *is bounded away from zero because otherwise, the gain would grow unbounded which seems undesirable. Therefore, asymptotic synchronization (or asymptotic tracking of an arbitrary reference signal) with prescribed performance seems not possible. On the contrary, we note that this is a misconception and we can show that the inputs can be still bounded as shown above in Corollary 6. The emphasis is that as the performance function* $\psi_i(t)$ *converges to zero as* $t \to \infty$, *the diffusive term* $\nu_i(t)$ *also converges to zero, and in particular, the fraction* $\nu_i(t)/\psi_i(t)$ *is contained in some compact interval* $[-1+\delta, 1-\delta]$ *for all* $t \ge t_0$, *with some positive constant* δ. *This ensures boundedness of the input and also the boundedness of the new funnel gain* $\gamma_i(\nu_i/\psi_i(t))$ *that we are proposing in this paper. We also refer to (Lee & Trenn, 2019) for more on asymptotic convergence by funnel control with bounded inputs.* ⋄

Remark 9 *It is interesting to note that the stability of the agents might not be necessary to guarantee boundedness of the input. In particular, in the condition (a) of Corollary 6, there is no stability restriction on the homogeneous part* $F(t,x)$, *hence the dynamics* $\dot{x} = F(t,x)$ *might even be unstable, e.g., the state might grow unboundedly. The utility of this remark can be found, for instance, when synchronizing heterogeneous oscillators, where the homogeneous part is of the form* $\dot{x}(t) = \Omega(t)$ *which is unstable in many cases (since it is just an integrator), where* $\Omega(\cdot)$ *is a time-varying angular velocity. Furthermore, recall that the time-varying vector fields* $f_i(t,x_i)$ *may depend on an external input; this external input could be used to ensure boundedness of the agents' states, so that condition (b) is ensured. Finally, note that a perturbation in the state such as* $x_i(t) + \tilde{x}_i(t)$, *which yields bounded continuous perturbation* $g_i(t) = \dot{\tilde{x}}_i(t)$, *as in (a) of Corollary 6, which can very likely to happen for practical systems, does not alter the fact that the inputs are bounded.* ⋄

Remark 10 *We want to note that, in theory, our approach can also achieve finite-time synchronization, i.e., for a given* $T > 0$ *we have* $\lim_{t\to t_0+T}|x_i(t) - x_j(t)| = 0$ *for* $i, j \in \mathcal{N}$, *with the same funnel coupling law. In* *particular, for any $T > 0$, if we set our performance functions to satisfy $\psi_i(t_0 + T) = 0$ and $\psi_i(t) > 0$ for $t \in [t_0, t_0 + T)$, the proof of Theorem 2 can be modified to prove that the solution exists such that $(t, x(t)) \in \Omega_\psi$ for all $t \in [t_0, t_0 + T)$ (we can just replace ∞ by $t_0 + T$ to obtain this). Moreover, since the solution exists only for the finite time interval, hence the solution is uniformly bounded, we naturally obtain as in Corollary 6 that the input is also uniformly bounded. However, in such cases, the solution stops to exists after time $t_0 + T$ and an alternative approach is needed to continue synchronization, e.g., sliding mode controller.* $\diamond$

Before concluding Section 2, we want to mention that there are some cases, where it can be explicitly shown that the solution trajectory is uniformly bounded, for instance when the dynamics introduced in Assumption 3 generates a uniformly bounded solution for the infinite time interval. This happens if, for instance, the dynamics $\dot{x} = f_i(t, x)$ are contractive for all $i \in \mathcal{N}$, i.e., for each $i \in \mathcal{N}$ there exists $c_i > 0$ such that $(\partial f_i/\partial x)(t, x) \leq -c_i$ for all $t \geq t_0$ and $x \in \mathbb{R}$ (the utility of this case can be seen in Section 4.5). In this case, the arguments in the proof of Lemma 1 ensure that the condition (b) of Corollary 6 holds.

3 Emergent behavior under funnel coupling

In Section 2 the system (1) is proven to achieve synchronization with respect to the performance functions ψ_i by the funnel coupling law (3) under only mild assumptions on the individual vector field f_i and under only the connectivity of the undirected network. In this section, we will answer the question: What is the emergent collective behavior that arises from funnel coupling?

We will show that the achieved approximately synchronized behavior follows that of the single scalar 'emergent dynamics' given as

$$\dot{\xi} = h_\mu^\psi(t, f_1(t, \xi), \ldots, f_N(t, \xi)) =: f_{\mathrm{em}}(t, \xi) \qquad (13)$$

with suitably chosen initial value and where the function h_μ^ψ that maps $\mathrm{col}(t, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$ to $h = h_\mu^\psi(t, f_1, \ldots, f_N) \in \mathbb{R}$ is defined as the unique solution of the following algebraic equation:

$$H(h, t, f_1, \ldots, f_N) := \sum_{i=1}^{N} \psi_i(t) \mu_i^{-1}(h - f_i) = 0. \qquad (14)$$

The existence and uniqueness of the solution of (14) for each $\mathrm{col}(t, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$ and also the continuity of the function h_μ^ψ is given in the following lemma.

Lemma 11 *Consider a given collection of $\mu_i(\cdot)$ and $\psi_i(\cdot)$ as in* (3) *where in addition to Assumption 2 the functions μ_i are assumed to be continuously differentiable and $\gamma_i(0) > 0$. Then we have for each $\mathrm{col}(t, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$ a unique value $h_\mu^\psi(t, f_1, \ldots, f_N) \in \mathbb{R}$ which satisfies the algebraic equation* (14)*. Furthermore, the map $(t, f_1, \ldots, f_N) \mapsto h_\mu^\psi(t, f_1, \ldots, f_N)$ is differentiable.* $\diamond$

PROOF. Note from its definition that μ_i^{-1} is a continuous function defined over $\mathbb{R}$ which is strictly increasing. This implies that for each fixed $(t, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$ the map $h \mapsto H(h, t, f_1, \ldots, f_N)$ is strictly increasing. By noting also that the value $H(h, t, f_1, \ldots, f_N)$ is positive (negative) when h is bigger than $\max_i f_i$ (smaller than $\min_i f_i$), we have, by the continuity of μ_i^{-1} and ψ_i for each $\mathrm{col}(t, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$ a unique value $h = h_\mu^\psi(t, f_1, \ldots, f_N)$ which satisfies (14).

Differentiability of h_μ^ψ follows from the classical Implicit Function Theorem because $\partial H(h, t, f_1, \ldots, f_N)/\partial h > 0$ as utilized already above. $\square$

Remark 12 *If all agents use a scaled version of the same underlying funnel* shape *ψ, i.e., $\psi_i = r_i \psi$ for some $r_i > 0$, then $\psi(t)$ can be divided out from* (14)*, hence the emergent behavior becomes* time-invariant *and does not depend on the shape of the funnel ψ but only on the scaling factors r_i.* $\diamond$

One of the reasons that emergent dynamics (13) approximates the synchronized behavior of the network (1) coupled via (3) is the point-wise convergence shown in the following lemma. For this and the results that come afterward, we need the following technical assumption.

Assumption 5 *Consider vector fields f_i, coupling gain functions γ_i, and performance functions ψ_i satisfying Assumptions 1 and 2. Assume additionally for all $i \in \mathcal{N}$: a) $|(\partial f_i/\partial t)(t, x)| \leq \theta_f$ for all $t \geq t_0$ and $x \in \mathbb{R}$ for some $\theta_f > 0$, b) $\gamma_i(0) > 0$, c) γ_i is continuously differentiable, and d) $|\dot{\psi}_i(t)| \leq \lambda_\psi \psi_i(t)$ for all $t \geq t_0$ for $\lambda_\psi > 0$.* $\diamond$

Note that the above assumption d) only becomes relevant when (some of) the performance functions asymptotically converge to zero. It is quite remarkable that even these somewhat natural assumptions are not needed in Theorem 2.

The next lemma considers the point-wise limit behavior when the funnels get tighter; indeed the resulting behavior then approaches that of the emergent dynamics.

Lemma 13 *Consider vector fields f_i, coupling gain functions γ_i, performance functions ψ_i, and adjacency elements α_{ij} satisfying Assumptions 1–5. Furthermore, consider a family of solutions $x_{i,\varepsilon}(\cdot)$, $\varepsilon > 0$, of the coupled system* (1) *and* (3) *with funnels $\psi_{i,\varepsilon}(t) = \varepsilon \psi_i(t)$ and with initial conditions*

$x_{i,\varepsilon}(t_0) = x^*_{i,\varepsilon} \in \mathbb{R}$, $i \in \mathcal{N}$; *in particular, we assume that* $|\sum_{j\in\mathcal{N}_i} \alpha_{ij}(x^*_{j,\varepsilon} - x^*_{i,\varepsilon})| =: |\nu^*_{i,\varepsilon}| = |\nu_{i,\varepsilon}(t_0)| < \psi_{i,\varepsilon}(t_0)$ *for all* $i \in \mathcal{N}$ *and* $\varepsilon > 0$. *In addition, we assume that*

$$\lim_{\varepsilon\to 0} x^*_{i,\varepsilon} = x^*_{s,0}, \quad \lim_{\varepsilon\to 0} \frac{\nu^*_{i,\varepsilon}}{\psi_{i,\varepsilon}(t_0)} = \sigma^*_{i,0}, \quad \forall i \in \mathcal{N},$$

with some $x^*_{s,0} \in \mathbb{R}$ *and* $\sigma^*_{i,0} \in (-1, 1)$, $i \in \mathcal{N}$. *Then, for arbitrary* $\eta > 0$ *and* $T > \tau > 0$, *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$,

$$\begin{aligned} |x_{i,\varepsilon}(t) - \xi(t)| &\leq \eta, \\ \left| \frac{\nu_{i,\varepsilon}(t)}{\psi_{i,\varepsilon}(t)} - \mu_i^{-1}(f_{em}(t, x_{s,\varepsilon}(t)) - f_i(t, x_{s,\varepsilon}(t))) \right| &\leq \eta, \end{aligned} \tag{15}$$

for all $t \in [t_0 + \tau, t_0 + T]$ *and* $i \in \mathcal{N}$, *where* $x_{s,\varepsilon}(t) := (1/N)\sum_{i=1}^N x_{i,\varepsilon}(t)$ *and* $\xi(\cdot)$ *is the solution of the emergent dynamics* (13) *with the initial condition* $\xi(t_0) = x^*_{s,0}$.[6] *In particular, we have*

$$\lim_{\varepsilon\to 0} x_{i,\varepsilon}(t) = \xi(t)$$

for each $t > t_0$ *and* $i \in \mathcal{N}$.

If we assume additionally that the emergent dynamics (13) *is contractive, i.e., there exists* $c > 0$ *such that*

$$\frac{\partial f_{em}(t, \xi)}{\partial \xi} \leq -c, \quad \forall t \geq t_0, \quad \xi \in \mathbb{R}.$$

Then, for any $\eta > 0$ *and* $\tau > 0$, *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$, (15) *is satisfied for all* $t \in [t_0 + \tau, \infty)$ *and* $i \in \mathcal{N}$. ◇

The proof is given in Appendix A.1.

Remark 14 *We like to emphasize that in the second part of Lemma 13 only "stability" of the emergent dynamics is assumed but that individual agents are still allowed to be unstable.* ◇

Note that the inequality (15) illustrates that as synchronization with arbitrary precision is achieved to ξ at time $t > t_0$, the coupling term $\mu_i(\nu_i/\psi_i(t))$ generates a compensation $f_{\text{em}}(t, \xi) - f_i(t, \xi)$ that can resolve the heterogeneity among the agents, and thus, the vector field of the network are aligned to $f_{\text{em}}(t, \xi)$. Now, an intuition for the emergent dynamics (13) (or the definition of the function h^ψ_μ) can be given by the above illustrated relation $\mu_i(\nu_i/\psi_i(t)) = f_{\text{em}}(t, \xi) - f_i(t, \xi)$, $i \in \mathcal{N}$, in accordance with the algebraic constraint $\sum_{i=1}^N \nu_i \equiv 0$, as

$$0 \equiv \sum_{i=1}^N \nu_i = \sum_{i=1}^N \psi_i(t)\mu_i^{-1}(f_{\text{em}}(t, \xi) - f_i(t, \xi))$$

which gives $f_{\text{em}}(t, \xi) = h^\psi_\mu(t, f_1(t, \xi), \ldots, f_N(t, \xi))$ by (14). This also allows us to consider the given ε-parameterized network (1) coupled via (3) as a singularly perturbed system, where the quasi-steady-state subsystem is the emergent dynamics (13) and Lemma 13 corresponds to Tikhonov's theorem.

Remark 15 *We want to emphasize that this methodology of finding emergent behavior that arises as we enforce synchronization to a heterogeneous network is universal. In particular, when high-gain linear coupling* $u_i(t, \nu_i) = k\nu_i$ *is used to achieve arbitrary precision synchronization as illustrated in the Introduction, if the states are synchronized to* ξ *at time* $t > t_0$ *and if the vector fields are also synchronized to* f_s, *i.e.,* $f_i(t, \xi) + k\nu_i = f_s$ *for all* $i \in \mathcal{N}$, *then* f_s *should satisfy* $\sum_{i=1}^N (f_s - f_i(t, \xi)) \equiv 0$, *by the algebraic constraint* $\sum_{i=1}^N \nu_i \equiv 0$. *The solution of this algebraic equation is the average among the vector fields, and thus, we can guess that the synchronized behavior can be illustrated by the solution trajectory of the blended dynamics* (4) *introduced in the Introduction.* ◇

Remark 16 *Recall from Remark 10 that finite-time synchronization is achieved with bounded inputs by performance functions* ψ_i *satisfying* $\lim_{t\to t_0+T} \psi_i(t) = 0$. *This gives us the intuition that the initial condition assumption of Lemma 13 can be satisfied at some time* $t'_0 > t_0$ *with any family of performance functions* $\psi_{i,\varepsilon}$ *satisfying the initial condition assumption in Theorem 2* ($|\nu_i(t_0)| < \psi_{i,\varepsilon}(t_0)$, $i \in \mathcal{N}$) *and* $\psi_{i,\varepsilon}(t'_0) = \varepsilon\psi^0_i < \varepsilon\overline{\psi}$ *with some* $\psi^0_i > 0$. *We note that it is hard to exactly identify the value* $x^*_{s,0}$, *in this case, which depends on the network topology and the performance functions* $\psi_{i,\varepsilon}$ *at* $[t_0, t'_0)$.[7] *However, we can still ensure a reasonable estimate, because we can show that for any* $\eta > 0$ *there exists* $t'_0 > t_0$ *which is sufficiently close to* t_0 *such that* $\min_i x_{i,\varepsilon}(t_0) - \eta \leq (1/N)\sum_{i=1}^N x_{i,\varepsilon}(t'_0) \leq \max_i x_{i,\varepsilon}(t_0) + \eta$ *according to the arguments in the proof of Lemma 1. With this estimate, we can, for instance, make a transient error arbitrary small after an arbitrarily short time by making the stability of the emergent dynamics sufficiently strong.*

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[6] Note that the emergent dynamics do not depend on the scaling factor ε in $\psi_{i,\varepsilon}$.

[7] We want to emphasize that characterization of $x^*_{s,0}$ is important only when we are interested in the approximation of the transient behavior because (under the assumption that the emergent dynamics is contractive) the long term behavior of the emergent dynamics is independent of the initial values.

Finally, we conjecture that the limit

$$\lim_{t_0' \to t_0} \lim_{\varepsilon \to 0} \frac{1}{N} \sum_{i=1}^{N} x_{i,\varepsilon}(t_0') = \lim_{t_0' \to t_0} x_{s,0}^*$$

equals a weighted median of a collection χ^* *of the initial values* $x_i(t_0)$ *with the weights* ψ_i^0*, defined as a real number that belongs to the set* $\mathcal{M}_\psi(\chi^*)$ *defined in Section 4.5.* $\diamond$

Remark 17 *Note that Lemma 13 not only guarantees uniform convergence of* $x_{i,\varepsilon}(t)$ *to* $\xi(t)$ *as* $\varepsilon \to 0$ *but also ensures that we can find* $\overline{M}_u > 0$ *such that each input* $u_{i,\varepsilon}(t) := \mu_i(\nu_{i,\varepsilon}(t)/\psi_{i,\varepsilon}(t))$ *satisfies*

$$|u_{i,\varepsilon}(t)| < \overline{M}_u, \quad \forall \varepsilon \in (0, \varepsilon^*), \quad t \geq t_0, \quad i \in \mathcal{N}.$$

This is because, by the uniformly bounded trajectory $\xi(t)$ *of the emergent dynamics* (13) *(which is contractive) we have uniformly bounded averaged trajectory* $x_s(t)$*, which further implies that* $f_{em}(t, x_s(t)) - f_i(t, x_s(t))$ *is uniformly bounded on* $[t_0, \infty)$ *for all* $i \in \mathcal{N}$*. For sufficiently small* η*, this ensures that the fraction* $\nu_{i,\varepsilon}(t)/\psi_{i,\varepsilon}(t)$ *resides inside some compact set contained in the interval* $(-1, 1)$*, hence we can find such* $\overline{M}_u$*.* $\diamond$

We are now ready to state our main result about the limit behavior of the synchronized multi-agent system which needs as additional assumptions that all trajectories $x_i(\cdot)$ remain bounded and that the emergent dynamics (13) are contractive.

Theorem 18 *In addition to Assumptions 1–5, assume that* $|\nu_i(t_0)| < \psi_i(t_0)$ *for all* $i \in \mathcal{N}$*, the emergent dynamics* (13) *is contractive, that the performance functions* ψ_i *asymptotically converges to zero, i.e.,* $\lim_{t\to\infty} \psi_i(t) = 0$*, and that the solution* $x_i(t)$ *of* (1) *under the funnel coupling law* (3) *with the performance functions* ψ_i *and initial condition* $x_i(t_0)$*,* $i \in \mathcal{N}$ *is uniformly bounded, i.e., there exists* $\overline{M}_x > 0$ *such that* $|x_i(t)| \leq \overline{M}_x$ *for all* $t \in [t_0, \infty)$ *and* $i \in \mathcal{N}$*. Then, the steady-state behavior of the network follows that of the emergent dynamics, i.e.,*

$$\lim_{t\to\infty} |x_i(t) - \xi(t)| = 0, \quad i \in \mathcal{N},$$

where $\xi(\cdot)$ *is the solution of the emergent dynamics* (13) *with some initial condition* $\xi(t_0) \in \mathbb{R}$*.* [8] $\diamond$

The proof of this theorem is given in Appendix A.2.

Now, given the characterization of the emergent dynamics (13), and given the analysis which shows that heterogeneous agents (1) under node-wise funnel coupling (3) behaves accordingly with the emergent dynamics when the performance function is sufficiently narrow, we can, for instance, construct a heterogeneous network achieving a specific purpose as noted in the Introduction, if the emergent dynamics is contractive. Note that under the assumption that all agents use the same funnel $\psi_i = \psi$ then the emergent dynamics (13) only depend on the individual vector field f_i and the coupling function μ_i for all $i \in \mathcal{N}$, and thus, can be designed prior without knowing the performance function and the network topology. This scheme of constructing a network with the desired collective behavior is first introduced in (Lee & Shim, 2020) and has many interesting applications. Since the blended dynamics (4) introduced in (Lee & Shim, 2020) (which corresponds to the emergent dynamics in this paper) takes clearly different form to the emergent dynamics (13), a new application might occur. In fact, for any collection of coupling functions μ_i, the function h_μ^ψ can never be linear, i.e., for each $t \geq t_0$ and $\mathrm{col}(a_1, \ldots, a_N) \in \mathbb{R}^N$ there exists $\mathrm{col}(f_1, \ldots, f_N) \in \mathbb{R}^N$ such that $h_\mu^\psi(t, f_1, \ldots, f_N) \neq \sum_{i=1}^N a_i f_i$. In this regard, we further inspect the properties of the emergent dynamics (13), especially the properties of the function h_μ^ψ in the following section.

4 Discussions on the emergent dynamics

4.1 Comments on calculating h_μ^ψ for some special μ_i

In this subsection, we discuss two special coupling rules μ_i and how $h_\mu^\psi(t, f_1, \ldots, f_N)$ could be calculated for given $(t, f_1, \ldots, f_N) \in \mathbb{R}^{N+1}$. Consider first the "classical" funnel coupling law $\mu_i(\eta) = \eta/(1-|\eta|)$, $i \in \mathcal{N}$, then its inverse can be calculated as $\mu_i^{-1}(s) = s/(1+|s|)$, and the algebraic equation (14) becomes

$$H(h, t, f_1, \ldots, f_N) = \sum_{i=1}^{N} \psi_i(t) \frac{h - f_i}{1 + |h - f_i|} \equiv 0.$$

Now, the solution of this equation can be found by solving a piecewise N-th order polynomial, i.e., by evaluating the steps of the following algorithm.

1. Find an index set $\{i_1, \ldots, i_N\}$ such that $f_{i_j} \leq f_{i_{j+1}}$ for all $j = 1, \ldots, N-1$.
2. Set $j = 1$.
3. Solve $\sum_{k=1}^{j} \psi_{i_k}(t) \frac{h - f_{i_k}}{1+h-f_{i_k}} + \sum_{k=j+1}^{N} \psi_{i_k}(t) \frac{h-f_{i_k}}{1-h+f_{i_k}} = 0$, whose roots coincide with those of a polynomial (in h) of order at most N.
4. If there is a root h such that $f_{i_j} \leq h \leq f_{i_{j+1}}$ then return $h_\mu^\psi(t, f_1, \ldots, f_N) = h$.
5. If not, increase j by 1 and go back to Step 3.

[8] Note that the initial condition $\xi(t_0)$ is irrelevant in the statement of Theorem 18, as any two trajectories of the emergent dynamics (13) asymptotically converge to each other by the assumption that (13) is contractive.

Similarly, if we use the funnel coupling law given as

$$\mu_i(\eta) = \begin{cases} \ln(1/(1-\eta)) & \text{if } \eta \geq 0, \\ \ln(1+\eta) & \text{if } \eta < 0, \end{cases}$$

then the inverse can be calculated as

$$\mu_i^{-1}(s) = \begin{cases} 1 - e^{-s} & \text{if } s \geq 0, \\ -1 + e^{s} & \text{if } s < 0. \end{cases}$$

For this special case, we only have to solve a piecewise 2nd order polynomial, and the solution of (14) can be found by the same algorithm as above but with Step 3 replaced by

3. Solve $\sum_{k=1}^{j} \psi_{i_k}(t)(1-e^{-h+f_{i_k}}) + \sum_{k=j+1}^{N} \psi_{i_k}(t)(-1+e^{h-f_{i_k}}) = 0$ which is equivalent to solving a 2nd order polynomial in the variable $\overline{h} = e^h$.

4.2 Simulating emergent dynamics without h_μ^ψ

If the vector fields f_i are differentiable, as in Section 3 (by Assumption 5), then one can perform a simulation in a more efficient way, by observing that the partial derivative of $h_\mu^\psi(t,\, f_1, \ldots, f_N)$ with respect to their arguments $t,\, f_i$ can be described with only the performance functions ψ_i, coupling functions μ_i, arguments f_i, and h_μ^ψ. In particular, we have (invoking the Implicit Function Theorem)

$$\frac{\partial h_\mu^\psi}{\partial f_i}(t, f_1, \ldots, f_N) = \frac{\psi_i(t)(\mu_i^{-1})'(h_\mu^\psi - f_i)}{\sum_{j=1}^{N} \psi_j(t)(\mu_j^{-1})'(h_\mu^\psi - f_j)}$$

and

$$\frac{\partial h_\mu^\psi}{\partial t}(t, f_1, \ldots, f_N) = -\frac{\sum_{j=1}^{N} \dot{\psi}_j(t)\mu_j^{-1}(h_\mu^\psi - f_j)}{\sum_{j=1}^{N} \psi_j(t)(\mu_j^{-1})'(h_\mu^\psi - f_j)}.$$

Thus, the solution of the emergent dynamics can be generated by introducing the new variable $\chi = h_\mu^\psi(t, f_1(t,\xi), \ldots, f_N(t,\xi))$ and the two-dimensional dynamical system given by

$$\begin{aligned} \dot{\xi} &= \chi \\ \dot{\chi} &= \frac{\sum_{j=1}^{N} \psi_j(t)(\mu_j^{-1})'(\chi - f_j(t,\xi)) \left[\frac{\partial f_j}{\partial t}(t,\xi) + \frac{\partial f_j}{\partial \xi}(t,\xi)\chi\right]}{\sum_{j=1}^{N} \psi_j(t)(\mu_j^{-1})'(\chi - f_j(t,\xi))} \\ &\quad - \frac{\sum_{j=1}^{N} \dot{\psi}_j(t)\mu_j^{-1}(\chi - f_j(t,\xi))}{\sum_{j=1}^{N} \psi_j(t)(\mu_j^{-1})'(\chi - f_j(t,\xi))} \end{aligned}$$

with initial value $\xi(t_0)$ and

$$\chi(t_0) = h_\mu^\psi(t_0, f_1(t_0, \xi(t_0)), \ldots, f_N(t_0, \xi(t_0))).$$

Note that when $\psi_i = r_i\psi$ for some common ψ and some individual $r_i > 0$ the partial derivative of h_μ^ψ with respect to time is zero and the dynamics simplify accordingly.

As an example consider the network from (Shim & Trenn, 2015), where the i-th agent is given by

$$\begin{aligned} \dot{x}_i &= (-1+\delta_i)x_i + c_i(t) + \frac{1}{1 - |\nu_i|/\psi(t)} \frac{\nu_i}{\psi(t)} \\ c_i(t) &= 10\sin t + 10m_i^1 \sin(0.1t + \theta_i^1) + 10m_i^2 \sin(10t + \theta_i^2) \end{aligned}$$

where $t_0 = 0$, $\psi_i(t) = 2 + 38e^{-t}$, $i \in \mathcal{N}$, and $N = 5$. Then, the synchronized behavior of the given network can be approximated by the solution trajectory of a two-dimensional emergent dynamics given by

$$\begin{aligned} \dot{\xi} &= \chi \\ \dot{\chi} &= \frac{\sum_{i=1}^{N} [(-1+\delta_i)\chi + \dot{c}_i]/(1 + |\chi + (1-\delta_i)\xi - c_i|)^2}{\sum_{i=1}^{N} 1/(1 + |\chi + (1-\delta_i)\xi - c_i|)^2}. \end{aligned} \tag{16}$$

It is now also clear that this emergent behavior is invariant under the change of a graph topology if it is still undirected and also under the change of performance function $\psi(\cdot)$, as numerically shown in Figures 4, 6, and 7 of (Shim & Trenn, 2015), see also Figure 3.

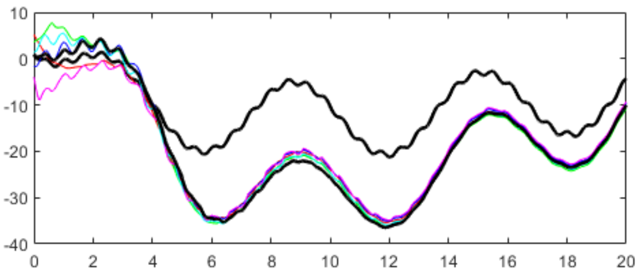

Fig. 3. Simulation result in (Shim & Trenn, 2015) has been revisited. Now, the unsolved question of synchronized behavior is answered as a lower black line, which illustrates the solution trajectory of the emergent dynamics (16).

The remaining three subsections will discuss how to utilize the flexibility in the design towards achieving a desired emergent behavior.

4.3 Temporary leader-follower

In this subsection, we want to highlight that for an arbitrary index $i^* \in \mathcal{N}$, the vector field of the emergent dynamics can be made arbitrarily close to the specific vector field f_{i^*} at time t by making $\psi_{i^*}(t)$ very large compared to $\psi_j(t)$ for $j \neq i^*$. In particular, it can be shown that for $r_{i^*}(t) := \psi_{i^*}(t)/(\sum_{j \neq i^*} \psi_j(t))$ we have

$$\lim_{r_{i^*}(t) \to \infty} h_\mu^\psi(t, f_1, \ldots, f_N) = f_{i^*}.$$

Note that this observation can be utilized, in the situation when the agent i^* temporarily wants to become the leader, and hence, the other agents its followers; this can easily be achieved by a simple action of the agent i^*, which is to make ψ_{i^*} sufficiently large.

4.4 Locally linear coupling functions

In this subsection, we emphasize that by making the coupling functions μ_i linear around the origin, we can design the emergent dynamics (13) semi-globally as a linear combination of the individual vector fields f_i, like the blended dynamics (4) introduced in (Lee & Shim, 2020). This observation can be useful, for instance, when our goal is to preserve the utility of the blended dynamics (4) while making it eligible for fully decentralized design and also making it global.

In particular, when the blended dynamics (4) is utilized for the synthesis of a network with some specific purposes, in many cases, the blended dynamics is stable and there exists a region of interest, which is compact say $[-\overline{M}_x, \overline{M}_x] \subset \mathbb{R}$. Then, we can recover the desired collective behavior, by the emergent dynamics (13), in the region of interest, for instance, by making μ_i^{-1} linear on $[-2\overline{M}_f, 2\overline{M}_f]$ where $\overline{M}_f := \sup_{i,t,|x|\le\overline{M}_x} |f_i(t,x)|$, e.g.,

$$\mu_i(\eta) = \begin{cases} 4\overline{M}_f\eta, & \text{if } |\eta| < 0.5, \\ \frac{\overline{M}_f}{1-\eta}, & \text{if } |\eta| \in [0.5, 1), \end{cases}$$

and using identical performance function $\psi_i = \psi$. Note that if $\xi \in [-\overline{M}_x, \overline{M}_x]$, then $f_i(t,\xi) \in [-\overline{M}_f, \overline{M}_f]$ for all $i \in \mathcal{N}$ and $t \ge t_0$, and thus, $h_\mu^\psi(t, f_1(t,\xi), \dots, f_N(t,\xi)) \in [\min_i f_i(t,\xi), \max_i f_i(t,\xi)] \subset [-\overline{M}_f, \overline{M}_f]$ satisfies

$$0 = \sum_{i=1}^N \mu_i^{-1}(h_\mu^\psi - f_i(t,\xi)) = \sum_{i=1}^N \frac{h_\mu^\psi - f_i(t,\xi)}{4\overline{M}_f},$$

in other words, we have $h_\mu^\psi(t, f_1(t,\xi), \dots, f_N(t,\xi)) = (1/N)\sum_{i=1}^N f_i(t,\xi)$. Now, by designing the emergent dynamics to be globally stable, we then achieve our goal.

Remark 19 *This also explains now why the behavior of the blended dynamics has been recovered by increasing κ in the numerical study in (Shim & Trenn, 2015), because the coupling function $\mu(\eta) = \kappa\eta/(1-|\eta|)$ becomes more and more linear in the sense that the second derivative of its inverse $\mu^{-1}(s) = \frac{s}{\kappa+|s|}$ (which is actually the one determining the emergent dynamics) converges to zero faster than the derivative of $\mu^{-1}(s)$ when $\kappa \to \infty$.* ◇

4.5 Distributed median solver

In this subsection, we introduce an interesting observation that the vector field $f_{\text{em}} = h_\mu^\psi(t, f_1, \dots, f_N)$ of the emergent dynamics (13) can be designed arbitrarily close to a weighted median of the individual vector fields f_i, by making the inverse of the coupling functions μ_i arbitrarily close to the signum function defined as

$$\text{sgn}(s) := \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0. \end{cases}$$

Here, a weighted median of a collection $\mathcal{F}$ of real numbers f_i with the weights $\psi_i > 0$ is defined as a real number that belongs to the set

$$\mathcal{M}_\psi(\mathcal{F}) = \begin{cases} \{f_{s_j}\}, & \text{if } \exists j \in \mathcal{N},\ \sum_{k=1}^j \psi_{s_k} > \psi_{\text{thr}} \\ & \qquad \text{and } \sum_{k=1}^{j-1} \psi_{s_k} < \psi_{\text{thr}}, \\ [f_{s_j}\ f_{s_{(j+1)}}], & \text{if } \exists j \in \mathcal{N},\ \sum_{k=1}^j \psi_{s_k} = \psi_{\text{thr}}, \end{cases}$$

where $\psi_{\text{thr}} := (1/2)\sum_{i=1}^N \psi_i$ and $\{s_k\}$ is the rearrangement of the sequence $\{1, \dots, N\}$ such that $f_{s_1} \le f_{s_2} \le \dots \le f_{s_N}$.

In particular, if the function μ_i approximates the signum function in the sense of

$$|\mu_i^{-1}(s)| \ge 1 - \varepsilon, \quad \forall s \quad \text{s.t.} \quad |s| \ge \eta, \tag{17}$$

with sufficiently small $\varepsilon > 0$ and $\eta > 0$, then the vector field $f_{\text{em}} = h_\mu^\psi(t, f_1, \dots, f_N)$ becomes η-close to a weighted median $\mathcal{M}_\psi(\mathcal{F})$ of the vector fields f_i as shown in the following lemma.

Lemma 20 *Consider a given collection of $\mu_i(\cdot)$ and $\psi_i(\cdot)$ as in (3), let us assume that (17) is satisfied for some $\eta > 0$ and $\varepsilon < 4\delta/(2\delta+1)$, where $\delta > 0$ is such that $\sum_{i\in\mathcal{K}} \psi_i(t) \ge (\frac{1}{2}+\delta)\sum_{i\in\mathcal{N}} \psi_i(t)$ for any $\mathcal{K} \subseteq \mathcal{N}$ satisfying $\sum_{i\in\mathcal{K}} \psi_i(t) > \frac{1}{2}\sum_{i\in\mathcal{N}} \psi_i(t)$. Then, we have $|h_\mu^\psi(t, f_1, \dots, f_N)|_{\mathcal{M}_{\psi(t)}(\mathcal{F})} \le \eta$.* ◇

The proof is carried out in Appendix A.3. Note that in the case when all ψ_i's are identical, then the median $\mathcal{M}(\mathcal{F})$ becomes unweighted and independent from ψ; furthermore, in this case, we can choose $\delta \ge 1/(2N)$, and thus $\varepsilon < 2/(N+1)$, to satisfy the assumptions of Lemma 20.

We illustrate the utility of the above observation by constructing a multi-agent system where each agent has a fixed number $f_i^* \in \mathbb{R}$ and all agents find asymptotically the median of all the collection $\mathcal{F}^* = \{f_1^*, \dots, f_N^*\}$. A key feature of our proposed approach is that the median can be found without communicating the values f_i^* to the neighbors, hence preserving privacy and increasing security.

An interesting application of finding the median is the ability to extract malicious attacks in a cyber-physical system as in (Lee, Kim, & Shim, 2020b).

The proposed multi-agent system to solve the median problem has the form of

$$\dot{x}_i(t) = f_i^* - x_i(t) + \mu_i\left(\frac{\nu_i(t)}{\psi(t)}\right), \quad x_i(t_0) = x_i^0, \quad (18)$$

where $\lim_{t\to\infty}\psi(t) = 0$ and there exists $\lambda_\psi > 0$ such that $|d\psi(t)/dt| \le \lambda_\psi \psi(t)$ for all $t \ge t_0$. Then, by Theorem 2 it will achieve asymptotic synchronization.

With a similar argument as in Lemma 1, we can show that the solutions $x_i(\cdot)$ are uniformly bounded hence Theorem 18 yields that the steady-state behavior of the network follows that of the contractive emergent dynamics given as

$$\begin{aligned}\dot{\xi}(t) &= h_\mu(f_1^* - \xi(t), \ldots, f_N^* - \xi(t)) \\ &= h_\mu(f_1^*, \ldots, f_N^*) - \xi(t),\end{aligned} \quad (19)$$

where $h = h_\mu^\psi(t, f_1, \ldots, f_N) = h_\mu(f_1, \ldots, f_N)$ solves $0 = \sum_{i=1}^N \mu_i^{-1}(h - f_i)$. Since any solution of (19) exponentially converges to the constant $h_\mu(f_1^*, \ldots, f_N^*)$, which can be made arbitrarily close to a median $\mathcal{M}_\psi(\mathcal{F}^*)$ as shown in Lemma 20, the proposed scalar network finds a median with arbitrary precision. The emphasis is that the error η in Lemma 20 only depends on the characteristics (17) of the coupling functions μ_i, hence it is independent of the collection $\mathcal{F}^*$. Note that the design can be done in a fully decentralized manner, with the only prior agreement on ε and η in Lemma 20, and the agreement

5 Conclusion

This paper introduces funnel coupling law which guarantees synchronization for a heterogeneous multi-agent system under only mild assumptions. Some sufficient conditions which guarantee boundedness of the inputs are also provided, and the analysis on the emergent collective behavior that appears as we enforce synchronization by the proposed funnel coupling law has been conducted. In fact, the paper introduced emergent dynamics that can illustrate the synchronized behavior of the whole network, and from its nonlinear structure, some new applications have been discovered, e.g., distributed median solver. Our future work is to extend our result to its vector counterpart, hence utilizing its interesting features, and to further derive useful applications.

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A Proofs

In order to prove all the claims in Section 3 in a compact way, let us introduce some new variables and some inequalities associated with them that are utilized in every proof given in the following subsections.

For this, note first that we can show in every proof either at the beginning or eventually that the trajectories $x_i(t)$ are uniformly bounded by $\overline{M}_x$ on the time interval $\mathbb{R}_{\overline{M}} \subset [t_0, \infty)$ of interest, where $\mathbb{R}_{\overline{M}} = [t_0, t_0 + T]$ (or $[t_0, \infty)$) for Lemma 13 and $\mathbb{R}_{\overline{M}} = [t_0, \infty)$ for Theorem 18, i.e., we have $|x_i(t)| \le \overline{M}_x$ for all $t \in \mathbb{R}_{\overline{M}}$ and $i \in \mathcal{N}$. Given this $\overline{M}_x$, let us denote $L_f(\overline{M}_x)$ and $M_f(\overline{M}_x)$ as a maximum among the Lipschitz constants and a norm bounds of f_i on the compact set $[-\overline{M}_x, \overline{M}_x]$. Note that by its definition, this implies the same for $f_{\rm em}$. Now, this further ensures that there exists $\delta(\overline{M}_x) > 0$ such that

$$\left|\mu_i^{-1}\left(f_{\rm em}(t,a) - f_i(t,a)\right)\right| \le 1 - 2\delta(\overline{M}_x), \tag{A.1}$$

for all $a \in [-\overline{M}_x, \overline{M}_x]$, $t \ge t_0$, and $i \in \mathcal{N}$.[9] Let us finally denote $L_\mu(\overline{M}_x)$ as a maximum among the Lipschitz constants of the coupling function μ_i on the compact set $[-1 + \delta(\overline{M}_x), 1 - \delta(\overline{M}_x)] \subset (-1, 1)$ for $i \in \mathcal{N}$.

Now, recall that $x_s := (1/N)\sum_{i=1}^N x_i$, and let $\underline{\psi}(t) := \min_i \psi_i(t)$, $\widehat{\psi}(t) := \max_i \psi_i(t)$, and[10]

$$y := -\frac{1}{\underline{\psi}(t)}\Lambda R^T \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} - \frac{1}{\underline{\psi}(t)} R^T \begin{bmatrix} \psi_1(t)\mu_1^{-1}(f_{\rm em}^s - f_1^s) \\ \vdots \\ \psi_N(t)\mu_N^{-1}(f_{\rm em}^s - f_N^s) \end{bmatrix}$$

where the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_2, \dots, \lambda_N)$ (with $0 < \lambda_2 < \cdots < \lambda_N$) and the matrix $R \in \mathbb{R}^{N\times(N-1)}$ are such that $\left[(1/\sqrt{N})1_N \;\; R\right]$ is an orthogonal matrix and $\mathcal{L} = R\Lambda R^T$. Then, we have

$$\frac{\nu_i(t)}{\psi_i(t)} - \mu_i^{-1}(f_{\rm em}^s - f_i^s) = \frac{\underline{\psi}(t)}{\psi_i(t)}\mathfrak{r}_i y(t) \tag{A.2}$$

where $\mathfrak{r}_i \in \mathbb{R}^{1\times(N-1)}$ is the i-th row of R, and thus,

$$\begin{aligned} \dot{x}_s &= f_{\rm em}(t, x_s) + \frac{1}{N}\sum_{i=1}^N [f_i(t,x_i) - f_i(t,x_s)] \\ &\quad + \frac{1}{N}\sum_{i=1}^N \left[\mu_i\left(\frac{\underline{\psi}(t)}{\psi_i(t)}\mathfrak{r}_i y + \mu_i^{-1}(f_{\rm em}^s - f_i^s)\right)\right. \\ &\qquad \left. - \mu_i\left(\mu_i^{-1}(f_{\rm em}^s - f_i^s)\right)\right] \end{aligned} \tag{A.3}$$

[9] An example is $\delta(\overline{M}_x) := (1 - \max_i \mu_i^{-1}(2\overline{M}_f(\overline{M}_x)))/2 > 0$.
[10] Here and after, $f_{\rm em}(t, x_s)$ and $f_i(t, x_s)$ are simply written as $f_{\rm em}^s$ and f_i^s respectively.

and

$$\dot{y} = \frac{\dot{\underline{\psi}}}{\underline{\psi}^2}\Lambda R^T \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} - \frac{1}{\underline{\psi}}\Lambda R^T \begin{bmatrix} f_1(t,x_1) - f_1(t,x_s) \\ \vdots \\ f_N(t,x_N) - f_N(t,x_s) \end{bmatrix}$$
$$- \frac{1}{\underline{\psi}}\Lambda R^T \begin{bmatrix} \mu_1\Big(\frac{\underline{\psi}}{\psi_1}\mathfrak{r}_1 y + \mu_1^{-1}(f_{\rm em}^s - f_1^s)\Big) - \mu_1\big(\mu_1^{-1}(f_{\rm em}^s - f_1^s)\big) \\ \vdots \\ \mu_N\Big(\frac{\underline{\psi}}{\psi_N}\mathfrak{r}_N y + \mu_N^{-1}(f_{\rm em}^s - f_N^s)\Big) - \mu_N\big(\mu_N^{-1}(f_{\rm em}^s - f_N^s)\big) \end{bmatrix}$$
$$-R^T \begin{bmatrix} \frac{\underline{\psi}\dot{\psi}_1 - \dot{\underline{\psi}}\psi_1}{\underline{\psi}^2}\mu_1^{-1}(f_{\rm em}^s - f_1^s) \\ \vdots \\ \frac{\underline{\psi}\dot{\psi}_N - \dot{\underline{\psi}}\psi_N}{\underline{\psi}^2}\mu_N^{-1}(f_{\rm em}^s - f_N^s) \end{bmatrix} - R^T \begin{bmatrix} \frac{\psi_1}{\underline{\psi}}\frac{d}{dt}\mu_1^{-1}(f_{\rm em}^s - f_1^s) \\ \vdots \\ \frac{\psi_N}{\underline{\psi}}\frac{d}{dt}\mu_N^{-1}(f_{\rm em}^s - f_N^s) \end{bmatrix}. \tag{A.4}$$

From these, we now obtain the following inequalities when $|x_i(t)| \le \overline{M}_x$, $|\nu_i(t)| < \psi_i(t)$, and $|\nu_i(t)/\psi_i(t)| \le 1 - \delta(\overline{M}_x)$ (or $|y(t)| \le \delta(\overline{M}_x)$), which gives for all $i \in \mathcal{N}$,

$$\left|\mu_i\left(\frac{\underline{\psi}}{\psi_i}\mathfrak{r}_i y + \mu_i^{-1}(f_{\rm em}^s - f_i^s)\right) - \mu_i\left(\mu_i^{-1}(f_{\rm em}^s - f_i^s)\right)\right|$$
$$\le L_\mu(\overline{M}_x)\left|\frac{\underline{\psi}}{\psi_i}y\right| \le L_\mu(\overline{M}_x)|y|.$$

In particular, we have by (A.3),

$$|\dot{x}_s| \le M_f(\overline{M}_x) + \frac{L_f(\overline{M}_x)}{N}\sum_{i=1}^N |x_i - x_s| + L_\mu(\overline{M}_x)|y|$$
$$\le M_f(\overline{M}_x) + \frac{L_f(\overline{M}_x)\sqrt{N}}{\lambda_2}\overline{\psi} + 2L_\mu(\overline{M}_x) =: M_s(\overline{M}_x) \tag{A.5}$$

where the second inequality follows from the relation:

$$\max_{i\in\mathcal{N}} |x_i - x_s| = |RR^T \mathrm{col}(x_1,\ldots,x_N)|_\infty =: |RR^T \boldsymbol{x}|_\infty$$
$$= |R\Lambda^{-1}R^T \mathrm{diag}(\psi_1(t),\ldots,\psi_N(t))\mathcal{L}_\psi(t)\boldsymbol{x}|_\infty$$
$$\le \sqrt{N}|R\Lambda^{-1}R^T|_2 |\mathrm{diag}(\psi_1(t),\ldots,\psi_N(t))|_\infty |\mathcal{L}_\psi(t)\boldsymbol{x}|_\infty$$
$$\le \sqrt{N}\widehat{\psi}(t)/\lambda_2 \le \sqrt{N}\overline{\psi}/\lambda_2. \tag{A.6}$$

Similarly, we have for $V := |x_s(t) - \xi(t)|$ that

$$\dot{V} = \frac{x_s - \xi}{|x_s - \xi|}(\dot{x}_s - \dot{\xi}) \le |\dot{x}_s - \dot{\xi}|$$
$$\le L_f(\overline{M}_x)V + M_V(\overline{M}_x)\widehat{\psi}(t) + L_\mu(\overline{M}_x)|y|, \tag{A.7}$$

where $M_V(\overline{M}_x) := L_f(\overline{M}_x)\sqrt{N}/\lambda_2$. On the other hand, if the emergent dynamics (13) is contractive as in Lemma 13 and Theorem 18 so that $(x_s - \xi)(f_{\rm em}(t,x_s) - f_{\rm em}(t,\xi)) \le -c|x_s - \xi|^2$, then we similarly obtain

$$\dot{V} \le -cV + M_V(\overline{M}_x)\widehat{\psi}(t) + L_\mu(\overline{M}_x)|y|. \tag{A.8}$$

Finally, $W := y^T\Lambda^{-1}y$ satisfies

$$\dot{W} \le 2\left|\frac{\dot{\underline{\psi}}}{\underline{\psi}}\right|\left[W + \frac{1}{\underline{\psi}}\left|y^T\Lambda^{-1}R^T\begin{bmatrix}\psi_1\mu_1^{-1}(f_{\rm em}^s - f_1^s) \\ \vdots \\ \psi_N\mu_N^{-1}(f_{\rm em}^s - f_N^s)\end{bmatrix}\right|\right]$$
$$+\frac{2}{\underline{\psi}}|y|\sqrt{\sum_{i=1}^N |f_i(t,x_i) - f_i(t,x_s)|^2}$$
$$-\frac{2}{\underline{\psi}}\sum_{i=1}^N(\mathfrak{r}_i y)\Big[\mu_i\Big(\frac{\underline{\psi}}{\psi_i}\mathfrak{r}_i y + \mu_i^{-1}(f_{\rm em}^s - f_i^s)\Big) - \mu_i\big(\mu_i^{-1}(f_{\rm em}^s - f_i^s)\big)\Big]$$
$$+2|\Lambda^{-1}y|\sqrt{N}\max_i\left|\frac{\psi_i}{\underline{\psi}}\frac{d}{dt}\mu_i^{-1}(f_{\rm em}(t,x_s(t)) - f_i(t,x_s(t)))\right|$$
$$+\frac{2}{\underline{\psi}^2}|\Lambda^{-1}y|\sqrt{N}\max_i\left|2\lambda_\psi\widehat{\psi}(t)^2\right|$$
$$\le \frac{2N}{\lambda_2}\left[\left(\lambda_\psi + L_f(\overline{M}_x)\right)r_\psi\right]|y| + 2\lambda_\psi W + \frac{4\sqrt{N}}{\lambda_2}\lambda_\psi r_\psi^2|y|$$
$$-\frac{2}{\underline{\psi}}\sum_{i=1}^N(\mathfrak{r}_i y)\Big[\mu_i\Big(\frac{\underline{\psi}}{\psi_i}\mathfrak{r}_i y + \mu_i^{-1}(f_{\rm em}^s - f_i^s)\Big) - \mu_i\big(\mu_i^{-1}(f_{\rm em}^s - f_i^s)\big)\Big]$$
$$+2|y|\frac{\sqrt{N}}{\lambda_2}r_\psi\max_i\left|\frac{d}{dt}\mu_i^{-1}(f_{\rm em}(t,x_s) - f_i(t,x_s))\right| \tag{A.9}$$

where the first inequality follows from the identity:

$$-y^TR^T\boldsymbol{x} = y^T\Lambda^{-1}\left(-\Lambda R^T\boldsymbol{x}\right) = \underline{\psi}(t)y^T\Lambda^{-1}y$$
$$+ y^T\Lambda^{-1}R^T\mathrm{col}(\psi_1\mu_1^{-1}(f_{\rm em}^s - f_1^s),\ldots,\psi_N\mu_N^{-1}(f_{\rm em}^s - f_N^s)).$$

Now, if we note that (see Section 4.2)

$$\left|\frac{d}{dt}\mu_i^{-1}(f_{\rm em}(t,x_s(t)) - f_i(t,x_s(t)))\right|$$
$$= (\mu_i^{-1})'(f_{\rm em}^s - f_i^s)\left|\frac{d}{dt}\left[f_{\rm em}(t,x_s(t)) - f_i(t,x_s(t))\right]\right|$$
$$\le \frac{1}{\underline{\gamma}}\left[2L_f(\overline{M}_x)|\dot{x}_s(t)| + 2\theta_f + \frac{\lambda_\psi}{\overline{\gamma}}\right]$$

where $\overline{\gamma} := \min_{|s|\le 1-2\delta(\overline{M}_x)}\min_j(\mu_j^{-1})'(s) > 0$, $\underline{\gamma} := \min_i\gamma_i(0) > 0$, and the inequality follows from

$$(\mu_i^{-1})'(f) = \frac{1}{\mu_i'(\mu_i^{-1}(f))}$$
$$= \frac{1}{\gamma_i'(|\mu_i^{-1}(f)|)|\mu_i^{-1}(f)| + \gamma_i(|\mu_i^{-1}(f)|)} \le \frac{1}{\gamma_i(0)},$$

for all $f \neq 0$ and $i \in \mathcal{N}$ (where the assumption that γ_i is strictly increasing is utilized for its derivation), and by also noting that we have for all $-\infty < a \leq b < \infty$,[11]

$$(b-a)(\mu_i(b) - \mu_i(a)) \geq \underline{\gamma}(b-a)^2, \tag{A.10}$$

we finally obtain the following:[12]

$$\begin{aligned}\dot{W} \leq{}& \frac{2N}{\lambda_2}\left[\left(\lambda_\psi + L_f(\overline{M}_x)\right) r_\psi\right]|y| + 2\lambda_\psi W - \frac{2}{\widehat{\psi}}\sum_{i=1}^N \underline{\gamma}(\mathfrak{r}_i y)^2 \\ &+ 2|y|\frac{\sqrt{N}}{\lambda_2}\frac{r_\psi}{\underline{\gamma}}\left[2L_f(\overline{M}_x)M_s(\overline{M}_x) + 2\theta_f + \frac{\lambda_\psi}{\overline{\gamma}} + 2\lambda_\psi r_\psi \underline{\gamma}\right] \\ =:{}& M_W(\overline{M}_x)|y| + 2\lambda_\psi W - \frac{2}{\widehat{\psi}}\underline{\gamma}|y|^2. \end{aligned} \tag{A.11}$$

A.1 Proof of Lemma 13

Now, in the case of Lemma 13, note that we deal with ε parameterized sets of solutions $x_{i,\varepsilon}$, however, there exists $\overline{M}_x > 0$ such that

$$|x_{i,\varepsilon}(t)| \leq \overline{M}_x, \quad \forall t \in [t_0, t_0+T], \quad i \in \mathcal{N}, \quad \varepsilon > 0,$$

by Lemma 1. This ensures that the argument at the beginning of Appendix A holds for ε parameterized sets of solutions $x_{i,\varepsilon}$, uniformly in ε.

Let $\overline{M}_\mu := \max_i |\mu_i(\sigma^*_{i,0})|$, $\overline{M}_{\mathrm{vec}} := 3(\overline{M}_\mu + 1 + M_f(\overline{M}_x))$, $L_{\mathrm{vec}} := \max\{L_f(\overline{M}_x), 3(\lambda_\psi \max_i \mu_i'(\mu_i^{-1}(\overline{M}_{\mathrm{vec}} + M_f(\overline{M}_x))) + \theta_f)/\overline{M}_{\mathrm{vec}}\} < \infty$, and $\tau' := \ln(1.5)/L_{\mathrm{vec}}$. Then, there exists $\bar{\varepsilon} > 0$ such that

$$\left|\mu_i\left(\frac{\nu_{i,\varepsilon}(t_0)}{\psi_{i,\varepsilon}(t_0)}\right)\right| \leq \overline{M}_\mu + 1, \quad \forall \varepsilon \in (0, \bar{\varepsilon}), \quad i \in \mathcal{N},$$

hence for $t = t_0$, for all $i \in \mathcal{N}$,

$$|\omega_{i,\varepsilon}(t)| := \left|f_i(t, x_{i,\varepsilon}(t)) + \mu_i\left(\frac{\nu_{i,\varepsilon}(t)}{\psi_{i,\varepsilon}(t)}\right)\right| \leq \frac{\overline{M}_{\mathrm{vec}}}{3}.$$

Now, choosing a time-varying index $J(t) \in \mathcal{N}$ such that $\omega_{J(t),\varepsilon}(t) = \max_i \omega_{i,\varepsilon}(t)$ and $\dot{\omega}_{J(t),\varepsilon}(t) \geq \dot{\omega}_{k,\varepsilon}(t)$ for all those $k \in \mathcal{N}$ with $\omega_{k,\varepsilon}(t) = \max_i \omega_{i,\varepsilon}(t)$, the upper right Dini derivative of $\omega_{J(t),\varepsilon}(t)$ satisfies

$$\begin{aligned} D^+\omega_{J(t),\varepsilon} ={}& \frac{\partial f_{J(t)}}{\partial t}(t, x_{J(t),\varepsilon}) + \frac{\partial f_{J(t)}}{\partial x}(t, x_{J(t),\varepsilon})\omega_{J(t),\varepsilon} \\ &- \mu'_{J(t)}\left(\frac{\nu_{J(t),\varepsilon}}{\psi_{J(t),\varepsilon}}\right)\frac{\dot{\psi}_{J(t),\varepsilon}\nu_{J(t),\varepsilon}}{\psi^2_{J(t),\varepsilon}} \\ &+ \frac{1}{\psi_{J(t),\varepsilon}}\mu'_{J(t)}\left(\frac{\nu_{J(t),\varepsilon}}{\psi_{J(t),\varepsilon}}\right)\sum_{j \in \mathcal{N}_{J(t)}} \alpha_{J(t)j}[\omega_{j,\varepsilon} - \omega_{J(t),\varepsilon}] \\ \leq{}& \theta_f + L_f\omega_{J(t),\varepsilon} + \lambda_\psi \mu'_{J(t)}\left(\frac{\nu_{J(t),\varepsilon}}{\psi_{J(t),\varepsilon}}\right), \end{aligned}$$

where the inequality follows from the fact that $\mu'_{J(t)}$ and $\psi_{J(t),\varepsilon}$ are positive and $\omega_{J(t),\varepsilon}$ is a maximum. Therefore,

$$D^+\omega_{J(t),\varepsilon}(t) \leq L_{\mathrm{vec}}\omega_{J(t),\varepsilon}(t) + L_{\mathrm{vec}}\frac{\overline{M}_{\mathrm{vec}}}{3}$$

whenever $\max_i \mu_i(\nu_{i,\varepsilon}(t)/\psi_{i,\varepsilon}(t)) \leq \overline{M}_{\mathrm{vec}} + M_f(\overline{M}_x)$ (or $\omega_{J(t),\varepsilon}(t) \leq \overline{M}_{\mathrm{vec}}$). Since $\omega_{J(t_0),\varepsilon}(t_0) \leq \overline{M}_{\mathrm{vec}}/3$, we have

$$\begin{aligned} &\omega_{J(t),\varepsilon}(t) \\ &\quad \leq e^{L_{\mathrm{vec}}(t-t_0)}\omega_{J(t_0),\varepsilon}(t_0) + \int_{t_0}^t e^{L_{\mathrm{vec}}(t-s)}L_{\mathrm{vec}}\frac{\overline{M}_{\mathrm{vec}}}{3}ds \\ &\quad \leq e^{L_{\mathrm{vec}}\tau'}2\frac{\overline{M}_{\mathrm{vec}}}{3} = \overline{M}_{\mathrm{vec}} \end{aligned}$$

for all $t \in [t_0, t_0+\tau']$, and thus, we get

$$\left|\frac{\nu_{i,\varepsilon}(t)}{\psi_{i,\varepsilon}(t)}\right| \leq \max_i \mu_i^{-1}\left(\overline{M}_{\mathrm{vec}} + \overline{M}_f(\overline{M}_x)\right),$$

for all $t \in [t_0, t_0+\tau']$, $\varepsilon \in (0, \bar{\varepsilon})$, and $i \in \mathcal{N}$.

Now, without loss of generality assume that

$$\max_i \mu_i^{-1}(\overline{M}_{\mathrm{vec}} + \overline{M}_f(\overline{M}_x)) \leq 1 - \delta(\overline{M}_x).$$

Then, (A.3)–(A.11) are all valid for $t \in [t_0, t_0+\tau']$, and (A.11) implies that we have

$$\dot{W}_\varepsilon \leq M_W(\overline{M}_x)\sqrt{\lambda_N W_\varepsilon} + 2\lambda_\psi W_\varepsilon - \frac{2}{\widehat{\psi}_\varepsilon}\underline{\gamma}\lambda_2 W_\varepsilon,$$

hence for all $t \in [t_0, t_0+\tau']$ and $\varepsilon \in (0, \bar{\varepsilon})$,

$$\dot{\sqrt{W_\varepsilon}} \leq \frac{M_W(\overline{M}_x)\sqrt{\lambda_N}}{2} + \left[\lambda_\psi - \frac{1}{\widehat{\psi}_\varepsilon}\underline{\gamma}\lambda_2\right]\sqrt{W_\varepsilon}.$$

Therefore, since $\lim_{\varepsilon\to 0}\widehat{\psi}_\varepsilon = 0$ and by the fact that

$$W_\varepsilon(t_0) \leq \frac{1}{\lambda_2}|y_\varepsilon|^2 \leq \frac{N}{\lambda_2}\max_i |\mathfrak{r}_i y_\varepsilon|^2 \leq \frac{4Nr_\psi^2}{\lambda_2},$$

[11] Note that we have $\mu_i(b) - \mu_i(a) = \mu_i'(c)(b-a)$ with some $c \in (a,b)$ by the mean value theorem and this implies (A.10) with $\mu_i'(c) = \gamma_i'(|c|)|c| + \gamma_i(|c|) \geq \gamma_i(0) \geq \underline{\gamma}$.
[12] Note that $\sum_{i=1}^N (\mathfrak{r}_i y)^2 = y^T R^T R y = y^T y$.

we further have $\lim_{\varepsilon\to 0} W_\varepsilon(t_0+\tau'')=0$, where

$$\tau'' := \min\left\{\tau,\ \tau',\ \frac{\eta}{12e^{L_f(\overline{M}_x)T}}\frac{1}{\overline{M}_{\rm vec}+M_f(\overline{M}_x)}\right\} > 0.$$

Now, there exists $\widehat{\varepsilon}\in(0,\bar{\varepsilon})$ such that for each $\varepsilon\in(0,\widehat{\varepsilon})$, we have $W_\varepsilon(t)<\min\{\delta(\overline{M}_x)^2,\eta^2\}/\lambda_N$ for all $t\in[t_0+\tau,t_0+T]\subset[t_0+\tau'',t_0+T]$. In particular, if we let $\widehat{\varepsilon}\in(0,\bar{\varepsilon})$ such that

$$\widehat{\psi}_\varepsilon(t) < \min\left\{\frac{\underline{\gamma}\lambda_2\delta_\eta^2}{M_W(\overline{M}_x)\delta(\overline{M}_x)\lambda_N},\ \frac{\underline{\gamma}\lambda_2}{2\lambda_\psi}\right\},\quad \forall t\in[t_0,\infty)$$

and $W_\varepsilon(t_0+\tau'')<\delta_\eta^2/\lambda_N$ for all $\varepsilon\in(0,\widehat{\varepsilon})$, where

$$\delta_\eta := \min\left\{\delta(\overline{M}_x),\ \eta,\ \frac{\min\{L_f(\overline{M}_x),c\}\eta}{6L_\mu(\overline{M}_x)e^{L_f(\overline{M}_x)T}}\right\},$$

then it is sufficient because when $W_\varepsilon=\delta_\eta^2/\lambda_N$, we have

$$\begin{aligned}\dot{W}_\varepsilon &\le M_W(\overline{M}_x)\sqrt{\lambda_N W_\varepsilon}+2\lambda_\psi W_\varepsilon-\frac{2}{\widehat{\psi}_\varepsilon}\underline{\gamma}\lambda_2 W_\varepsilon\\ &\le M_W(\overline{M}_x)\delta(\overline{M}_x)-\frac{1}{\widehat{\psi}_\varepsilon}\underline{\gamma}\lambda_2\frac{\delta_\eta^2}{\lambda_N}<0.\end{aligned}$$

This further ensures, from (A.2):

$$\begin{aligned}&\left|\frac{\nu_{i,\varepsilon}(t)}{\psi_{i,\varepsilon}(t)}-\mu_i^{-1}(f_{\rm em}(t,x_{s,\varepsilon}(t))-f_i(t,x_{s,\varepsilon}(t)))\right|\\ &=\frac{\underline{\psi}_\varepsilon}{\psi_{i,\varepsilon}}|\mathfrak{r}_i y_\varepsilon(t)|\le|y_\varepsilon(t)|\le\sqrt{\lambda_N W_\varepsilon(t)}<\delta_\eta\le\eta\end{aligned}\tag{A.12}$$

for all $i\in\mathcal{N}$ and for all $t\in[t_0+\tau,t_0+T]\subset[t_0+\tau'',t_0+T]$, which proves the second inequality of (15).

Now, proven that $|y_\varepsilon(t)|<\delta_\eta\le\delta(\overline{M}_x)$ for all $t\in[t_0+\tau'',t_0+T]$, we have (A.7) for all $t\in[t_0+\tau'',t_0+T]$ and $\varepsilon\in(0,\widehat{\varepsilon})$. This further ensures

$$\dot{V}_\varepsilon\le L_f(\overline{M}_x)V_\varepsilon+M_V(\overline{M}_x)\widehat{\psi}_\varepsilon+L_\mu(\overline{M}_x)\delta_\eta,$$

and thus, by the calculus of variations, that

$$\begin{aligned}V_\varepsilon(t)&\le e^{L_f(\overline{M}_x)(t-t_0-\tau'')}V_\varepsilon(t_0+\tau'')\\ &\quad+\int_{t_0+\tau''}^{t}e^{L_f(\overline{M}_x)(t-s)}[M_V(\overline{M}_x)\widehat{\psi}_\varepsilon+L_\mu(\overline{M}_x)\delta_\eta]ds\\ &\le e^{L_f(\overline{M}_x)(t-t_0)}\left[V_\varepsilon(t_0+\tau'')+\frac{M_V(\overline{M}_x)\widehat{\psi}_\varepsilon+L_\mu(\overline{M}_x)\delta_\eta}{L_f(\overline{M}_x)}\right]\\ &\le e^{L_f(\overline{M}_x)T}V_\varepsilon(t_0+\tau'')+e^{L_f(\overline{M}_x)T}\frac{M_V(\overline{M}_x)}{L_f(\overline{M}_x)}\widehat{\psi}_\varepsilon+\frac{\eta}{6},\end{aligned}$$

for all $t\in[t_0+\tau'',t_0+T]$ and $\varepsilon\in(0,\widehat{\varepsilon})$. Therefore, if we let $\varepsilon^*<\widehat{\varepsilon}$ such that

$$\widehat{\psi}_\varepsilon(t)<\min\left\{\frac{\min\{L_f(\overline{M}_x),c\}\eta}{6M_V(\overline{M}_x)e^{L_f(\overline{M}_x)T}},\ \frac{\lambda_2\eta}{2\sqrt{N}}\right\},\quad\forall t\in[t_0,\infty)$$

and $V_\varepsilon(t_0)<\eta/(12e^{L_f(\overline{M}_x)T})$ for all $\varepsilon\in(0,\varepsilon^*)$, then by

$$\dot{V}_\varepsilon\le\overline{M}_{\rm vec}+M_f(\overline{M}_x),\quad\forall t\in[t_0,t_0+\tau''],$$

we get $V_\varepsilon(t_0+\tau'')\le\eta/(6e^{L_f(\overline{M}_x)T})$, hence $V_\varepsilon(t)\le\eta/2$ for all $t\in[t_0+\tau'',t_0+T]$, and thus,

$$\begin{aligned}|x_{i,\varepsilon}-\xi|\le V_\varepsilon+|x_{s,\varepsilon}-x_{i,\varepsilon}|&\le V_\varepsilon+|RR^T\boldsymbol{x}_\varepsilon|_\infty\\ &<\frac{\eta}{2}+\sqrt{N}\frac{\widehat{\psi}_\varepsilon}{\lambda_2}\le\eta,\end{aligned}\tag{A.13}$$

for all $i\in\mathcal{N}$, $t\in[t_0+\tau,t_0+T]\subset[t_0+\tau'',t_0+T]$, and $\varepsilon\in(0,\varepsilon^*)$, which proves the first inequality of (15).

For the case when the emergent dynamics (13) is contractive, without loss of generality assume that $\overline{M}_x\ge M_\xi+\eta$, where M_ξ is such that $|\xi(t)|\le M_\xi$ for all $t\ge t_0$, which exists because the emergent dynamics is contractive. Then, we have

$$|x_i(t)|\le\overline{M}_x,\quad\forall t\in[t_0,\infty),\quad i\in\mathcal{N},$$

whenever $V_\varepsilon=|x_s(t)-\xi(t)|\le\eta/2$ and $\varepsilon\in(0,\varepsilon^*)$ (which ensures $\psi_{i,\varepsilon}<\lambda_2\eta/(2\sqrt{N})$, hence $|x_i(t)-x_s(t)|\le\eta/2$ by (A.6)). This further implies that $W_\varepsilon(t)<\delta_\eta^2/\lambda_N$ until $V_\varepsilon\le\eta/2$ holds. On the other hand, we have by (A.8),

$$\dot{V}_\varepsilon\le -cV_\varepsilon+M_V(\overline{M}_x)\widehat{\psi}_\varepsilon+L_\mu(\overline{M}_x)\delta_\eta,$$

until $V_\varepsilon<\eta/2$ holds. Now, noting that by the calculus of variations, we have

$$\begin{aligned}&V_\varepsilon(t)\\ &\le e^{-c(t-t_0-\tau'')}V_\varepsilon(t_0+\tau'')+\frac{M_V(\overline{M}_x)\widehat{\psi}_\varepsilon}{c}+\frac{L_\mu(\overline{M}_x)\delta_\eta}{c}\\ &<\frac{\eta}{6}+\frac{\eta}{6}+\frac{\eta}{6}=\frac{\eta}{2},\end{aligned}$$

for all $t\in[t_0+\tau,\infty)\subset[t_0+\tau'',\infty)$, the proof concludes.

A.2 Proof of Theorem 18

Note that in Theorem 18 we have an assumption that $|x_i(t)|\le\overline{M}_x$ for all $t\in[t_0,\infty)$. We further have by Corollary 6, that the input is uniformly bounded, and therefore, we still have (A.8) and (A.11), even without the condition $|y(t)|\le\delta(\overline{M}_x)$, but possibly with a different Lipschitz constant, say L'_μ.

Then, since the performance functions ψ_i asymptotically converge to zero, i.e., $\lim_{t\to\infty}\psi_i(t)=0$, we can conclude by (A.11) that W is also asymptotically converging to zero, i.e., $\lim_{t\to\infty}W(t)=0$. This is because, otherwise, there exists $\underline{W}>0$ such that $W(t)\geq\underline{W}$ for all $t\geq t_0$, which is a contradiction since, we have

$$\begin{aligned}\dot{W}&\leq M_W(\overline{M}_x)\sqrt{\lambda_N W}+2\lambda_\psi W-\frac{2}{\widehat{\psi}}\underline{\gamma}\lambda_2 W\\&<-\frac{2}{3\widehat{\psi}}\underline{\gamma}\lambda_2 W\leq-\frac{2}{3\psi^*}\underline{\gamma}\lambda_2\underline{W}<0\end{aligned}$$

whenever $W\geq\underline{W}$ and

$$\widehat{\psi}(t)<\min\left\{\frac{\underline{\gamma}\lambda_2}{3\lambda_\psi},\ \frac{2\underline{\gamma}\lambda_2\sqrt{\underline{W}}}{3M_W(\overline{M}_x)\sqrt{\lambda_N}}\right\}=:\psi^*,$$

hence W becomes smaller than $\underline{W}$ in a finite time. Note that $\lim_{t\to\infty}\psi_i(t)=0$ ensures that $\psi_i(t)<\psi^*$ for all $t\geq t_0+T$ and $i\in\mathcal{N}$ with some $T>0$.

Now, similarly, we can conclude by (A.8) that V is also asymptotically converging to zero, i.e., $\lim_{t\to\infty}V(t)=0$, which then completes the proof. For this, if we assume that there exists $\underline{V}>0$ such that $V(t)\geq\underline{V}$ for all $t\geq t_0$, then we arrive at a contradiction since we have

$$\dot{V}\leq-cV+M_V(\overline{M}_x)\widehat{\psi}+L_\mu(\overline{M}_x)\sqrt{\lambda_N W}\leq-\frac{c}{3}\underline{V}<0$$

whenever $V\geq\underline{V}$ and

$$W(t)<\frac{c^2\underline{V}^2}{9L_\mu(\overline{M}_x)^2\lambda_N}=:W^*,\quad \widehat{\psi}(t)<\frac{c\underline{V}}{3M_V(\overline{M}_x)}=:\psi^{**}.$$

Note that $\lim_{t\to\infty}W(t)=\lim_{t\to\infty}\psi_i(t)=0$ ensures $W(t)<W^*$, $\widehat{\psi}(t)<\psi^{**}$ for $t\geq t_0+T$ with some $T>0$.

A.3 Proof of Lemma 20

By the constraint (17), we have either

$$\left|\mu_i^{-1}(h_\mu^\psi(t,f_1,\ldots,f_N)-f_i)\right|\geq 1-\varepsilon, \tag{A.14}$$

or

$$\left|h_\mu^\psi(t,f_1,\ldots,f_N)-f_i\right|<\eta. \tag{A.15}$$

Now, let us consider two separate situations, (i) when there exists $j\in\mathcal{N}$ such that $\mathcal{M}_{\psi}(t)(\mathcal{F})=\{f_{s_j}\}$ and (ii) when there exists $j\in\mathcal{N}$ such that $\mathcal{M}_{\psi}(t)(\mathcal{F})=[f_{s_j},f_{s_{(j+1)}}]$, where $\{s_k\}$ is the rearrangement of the sequence $\{1,\ldots,N\}$ such that $f_{s_1}\leq f_{s_2}\leq\cdots\leq f_{s_N}$.

For the case (ii), if (A.15) is satisfied for either s_j or $s_{(j+1)}$, then we are done. However, if this is not the case, we have

$$\begin{aligned}\mu_{s_j}^{-1}(h_\mu^\psi(t,f_1,\ldots,f_N)-f_{s_j})&\geq 1-\varepsilon\\ \mu_{s_{(j+1)}}^{-1}(h_\mu^\psi(t,f_1,\ldots,f_N)-f_{s_{(j+1)}})&\leq-(1-\varepsilon)\end{aligned}\tag{A.16}$$

or there exists an index set $\mathcal{K}\subseteq\mathcal{N}$ such that $\sum_{i\in\mathcal{K}}\psi_i(t)>(1/2)\sum_{i\in\mathcal{N}}\psi_i(t)$, and that

$$\mu_i^{-1}(h_\mu^\psi(t,f_1,\ldots,f_N)-f_i)\geq 1-\varepsilon,\quad\forall i\in\mathcal{K}, \tag{A.17}$$

or

$$\mu_i^{-1}(h_\mu^\psi(t,f_1,\ldots,f_N)-f_i)\leq-(1-\varepsilon),\quad\forall i\in\mathcal{K}.$$

Also for the case (i), if (A.15) is satisfied for s_j, then we are done, and if this is not the case, we have $\mathcal{K}\subseteq\mathcal{N}$ as above. Since (A.16) implies that $h_\mu^\psi(t,f_1,\ldots,f_N)\in\mathcal{M}_\psi(t)(\mathcal{F})$, we now only have to consider the case when there exists an index set $\mathcal{K}$ with the aforementioned properties. However, this yields a contradiction, because, if without loss of generality assume that (A.17) hold, then we have

$$\begin{aligned}0&=\sum_{i=1}^N\psi_i(t)\mu_i^{-1}(h_\mu^\psi(t,f_1,\ldots,f_N)-f_i)\\&\geq\sum_{i\in\mathcal{K}}\psi_i(t)(1-\varepsilon)-\sum_{i\in\mathcal{N}\setminus\mathcal{K}}\psi_i(t)\\&=(2-\varepsilon)\sum_{i\in\mathcal{K}}\psi_i(t)-\sum_{i\in\mathcal{N}}\psi_i(t)\\&\geq\left[(2-\varepsilon)\left(\frac{1}{2}+\delta\right)-1\right]\sum_{i\in\mathcal{N}}\psi_i(t)>0\end{aligned}$$

where the last term is positive by the definition of ε.