# A time-varying Gramian based model reduction approach for linear switched systems

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# Problem description

## Switched linear systems:

$$u \longrightarrow \boxed{S_\sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) \end{cases}} \longrightarrow y$$

- piecewise constant switching signal $\sigma : \mathbb{R} \to M = \{0, 1, \dots, f\}$
- $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{p \times n}$, $i \in M$, and $x(0) = \mathbf{0}$
- $n$ is the number of state variables, called the order of the subsystems, $n \gg m, p$

## Question

Is there a reduced order model which approximates the input-output sufficiently well?

# Existing approaches

## Arbitrary switching signals, differential inclusion

- Monshizadeh, Trentelman & Camlibel, IEEE TAC 2012
- Shaker & Wisniewski, IJICIC 2012
- Batug, Petreczky, Wisniewski & Leth, IEEE TAC 2016
- Papadopoulus & Prandini, Automatica 2016
- Gosea, Petreczky, Antoulas & Fiter, ACM 2018

## Viewing switching signal as input

- Schulze & Unger, SIAM JCO 2018
- Pontes Duff, Grundel & Benner, IEEE TAC 2020

## Fixed switching signal

No results available for fixed switching signal

# View switched system as time-varying system

## New approach

Assume $\sigma$ is known and fixed $\hookrightarrow$ switched system becomes

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t)$$

## Existing result for time-varying model reduction

- Shokoohi, Silverman & Dooren, IEEE TAC 1983
- Sandberg & Rantzer, IEEE TAC 2004

## Existing approaches not applicable

Coefficient matrices not continuous

# Naive approach: Example

## Switched system

$$(A_0, B_0, C_0) := \left( \begin{bmatrix} -0.5 & 0.01 \\ 0.01 & -0.5 \end{bmatrix}, \begin{bmatrix} 0.001 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.001 & 1 \end{bmatrix} \right)$$

$$(A_1, B_1, C_1) := \left( \begin{bmatrix} -0.5 & 0.01 \\ 0.01 & -0.5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.001 \end{bmatrix}, \begin{bmatrix} 1 & 0.001 \end{bmatrix} \right)$$

with switching signal: $\sigma(t) = 0$ on $t \in [0, 1)$ and $\sigma(t) = 1$ on $[1, 2)$.

## Modewise reduced system

$$\dot{\widehat{x}}(t) = -0.5 \cdot \widehat{x}(t) + 1 \cdot u(t),$$
$$\widehat{y}(t) = 1 \cdot \widehat{x}(t).$$

# Simulation for $u(t) = (\sin(5t) + 0.05)e^{-.5t}$

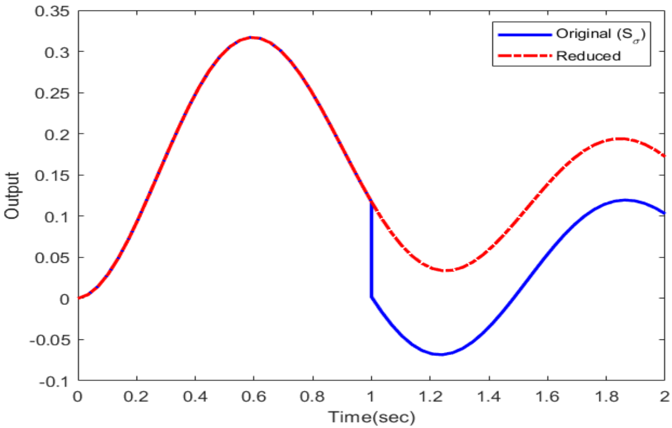



Figure: Comparison between the output of full and 1st order reduced system.

## Observations:

A naive mode-wise model reduction method will not result in good approximations of a switched system in general

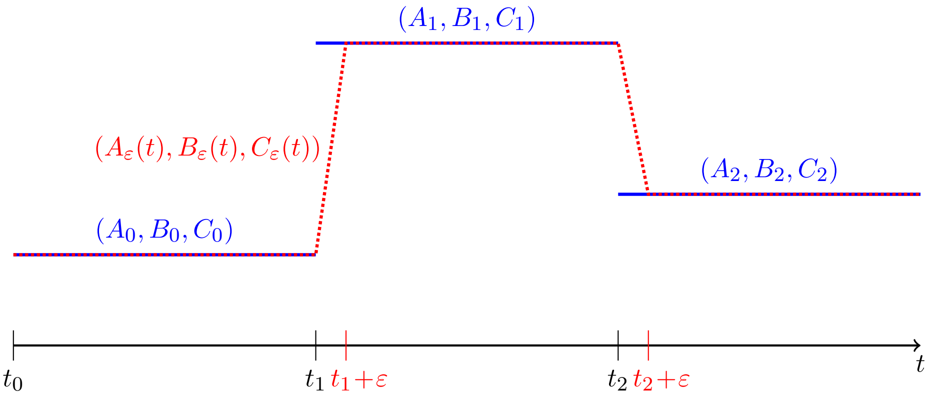
## Our proposed approach

**Step 1:** Approximate switched system with continuous time-varying system
**Step 2:** Use existing model-reduction methods for linear time-varying systems

$$\begin{aligned}\dot{x} &= A_\sigma x + B_\sigma u\\ y &= C_\sigma x\end{aligned} \longrightarrow \begin{aligned}\dot{x}_\varepsilon &= A_\varepsilon(t)x_\varepsilon + B_\varepsilon(t)u\\ y_\varepsilon &= C_\varepsilon(t)x_\varepsilon\end{aligned} \longrightarrow \begin{aligned}\dot{\widehat{x}}_\varepsilon &= \widehat{A}_\varepsilon(t)\widehat{x}_\varepsilon + \widehat{B}_\varepsilon(t)u\\ \widehat{y}_\varepsilon &= \widehat{C}_\varepsilon(t)\widehat{x}_\varepsilon\end{aligned}$$

# Contents

# Illustration of piecewise linear approximation



# Formal definition of approximation

## Approximate time-varying system

For $\varepsilon > 0$ let

$$S_\varepsilon : \begin{cases} \dot{x}_\varepsilon(t) = A_\varepsilon(t)x_\varepsilon(t) + B_\varepsilon(t)u(t), \quad x_\varepsilon(0) = \mathbf{0} \\ y_\varepsilon(t) = C_\varepsilon(t)x_\varepsilon(t), \end{cases}$$

where $(A_\varepsilon(t), B_\varepsilon(t), C_\varepsilon(t)) := (A_0, B_0, C_0)$, for $t \in [0, t_1)$ and for $i = 1, \cdots, f$

$$A_\varepsilon(t) = \begin{cases} A_{i-1} + \frac{t-t_i}{\varepsilon}(A_i - A_{i-1}) & : t \in [t_i, t_i + \varepsilon), \\ A_i & : t \in [t_i + \varepsilon, t_{i+1}) \end{cases}$$

$B_\varepsilon(.)$ and $C_\varepsilon(.)$ are analogous to $A_\varepsilon(.)$

# Properties of approximation

## Remark

$A_\varepsilon(\cdot), B_\varepsilon(\cdot), C_\varepsilon(\cdot)$ absolutely continuous, however, derivatives grow with $1/\varepsilon$.

## Approximation accuracy

Let $x_\sigma$ and $x_\varepsilon$ solutions of original and its approximation for given input. Then for every finite interval $[0,T]$ there exist $\bar{\varepsilon} > 0$ and a constant $c > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$

$$\|x_\varepsilon(t) - x_\sigma(t)\| < c\,\varepsilon, \quad \forall t \in [0,T].$$

## Attention

A similar error bound for the output does not hold!
Reason: $y_\varepsilon(\cdot)$ is continuous while $y_\sigma(\cdot)$ is discontinuous at switching times.

# Validations

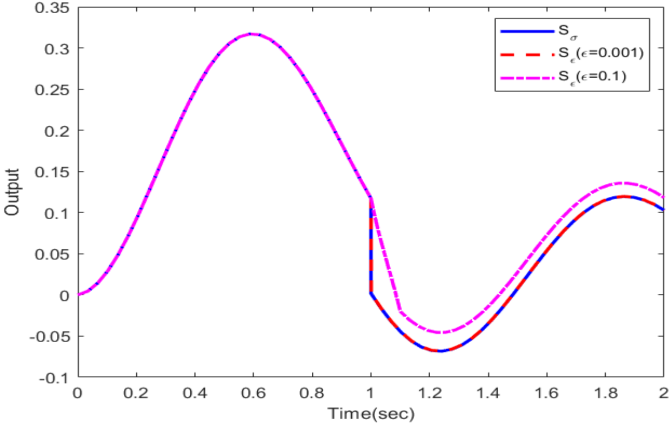



Figure: Comparison between the output of original and proposed approximation systems.

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# Transition matrix for time-varying linear systems

## Time-varying system

$$S_\varepsilon : \quad \begin{cases} \dot{x}(t) = A_\varepsilon(t)x(t) + B_\varepsilon(t)u(t) \\ y(t) = C_\varepsilon(t)x(t) \end{cases}$$

- $A_\varepsilon(.)$, $B_\varepsilon(.)$ and $C_\varepsilon(.)$ are continuous and bounded
- Let $\Phi_\varepsilon(\cdot, t_0)$ be transition matrix of $S_\varepsilon$, i.e.

$$\dot{\Phi}_\varepsilon(\cdot, t_0) = A_\varepsilon \Phi_\varepsilon(\cdot, t_0) \text{ and } \Phi_\varepsilon(t_0, t_0) = I$$

- Solution formula: $x(t) = \Phi_\varepsilon(t, t_0)x_0 + \int_{t_0}^{t} \Phi_\varepsilon(t, \tau)B_\varepsilon(\tau)u(\tau)d\tau$

# Gramians for linear time-varying system

## Controllability and observability Gramians on finite interval

For $t \in [0, T]$ and some symmetric positive definite $P_0$, $Q_T$ let

- **Controllability Gramian:** $P_\varepsilon(t) := P_0 + \int_0^t \Phi_\varepsilon(t,\tau) B_\varepsilon(\tau) B_\varepsilon^\top(\tau) \Phi_\varepsilon^\top(t,\tau) d\tau,$
- **Observability Gramian:** $Q_\varepsilon(t) := Q_T + \int_t^T \Phi_\varepsilon^\top(\tau,t) C_\varepsilon^\top(\tau) C_\varepsilon(\tau) \Phi_\varepsilon(\tau,t) d\tau$

## Lemma (cf. Behr, Benner & Heiland 2019)

*The controllability and observability Gramiens are the unique solutions of*

$$\dot{P}_\varepsilon(t) = A_\varepsilon(t) P_\varepsilon(t) + P_\varepsilon(t) A_\varepsilon^\top(t) + B_\varepsilon(t) B_\varepsilon^\top(t), \qquad P_\varepsilon(0) = P_0$$

$$\dot{Q}_\varepsilon(t) = -\left( A_\varepsilon^\top(t) Q_\varepsilon(t) + Q_\varepsilon(t) A_\varepsilon(t) + C_\varepsilon^\top(t) C_\varepsilon(t) \right), \qquad Q_\varepsilon(T) = Q_T.$$

# Balanced truncation for linear time-varying system

**Step 1:** Calculate Gramians $P_\varepsilon(t)$ and $Q_\varepsilon(t)$
**Step 2:** Compute Cholesky decomp.: $P_\varepsilon(t) = R(t)R(t)^\top$ and $Q_\varepsilon(t) = L(t)L(t)^\top$.
**Step 3:** Compute singular value decomposition

$$R(t)^\top L(t) = U(t)\Sigma(t)V(t)^\top = \begin{bmatrix} U_1(t) & U_2(t) \end{bmatrix} \begin{bmatrix} \Sigma_1(t) & \\ & \Sigma_2(t) \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}^\top$$

with $\Sigma_1(t) = \text{diag}\{\sigma_1(t), \cdots, \sigma_r(t)\}$ and $\Sigma_2(t) = \text{diag}\{\sigma_{r+1}(t), \cdots, \sigma_n(t)\}$
**Step 4:** Compute: $V(t) = L(t)V_1(t)\Sigma_1(t)^{-\frac{1}{2}}$ and $W(t) = R(t)U_1(t)\Sigma_1(t)^{-\frac{1}{2}}$
**Step 5:** Reduced model:

$$\begin{aligned} \dot{\widehat{x}}_\varepsilon(t) &= \widehat{A}_\varepsilon(t)\widehat{x}_\varepsilon(t) + \widehat{B}_\varepsilon(t)u(t) \\ \widehat{y}_\varepsilon(t) &= \widehat{C}_\varepsilon(t)\widehat{x}_\varepsilon(t) \end{aligned}$$

with $\widehat{A}_\varepsilon(t) = V(t)^\top(A_\varepsilon(t)W(t) - \dot{W}(t))$, $\widehat{B}_\varepsilon(t) = V(t)^\top B_\varepsilon(t)$, $\widehat{C}_\varepsilon(t) = C_\varepsilon(t)W(t)$

# Error bounds

## Lemma (Sandberg and Rantzer (2004))

Under some technical assumptions on the interval $[0,T]$:

$$\|y_\varepsilon - \widehat{y}_\varepsilon\|_{L_2} \leq 2 \sum_{i=r+1}^{n} \sup_{t\in[0,T]} \sigma_i(t)\ \|u\|_{L_2}$$

## Currently unclear

When are technical assumptions for approximation of switched system satisfied?
Dependence on $\varepsilon$?

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# Reduced system for motivating example

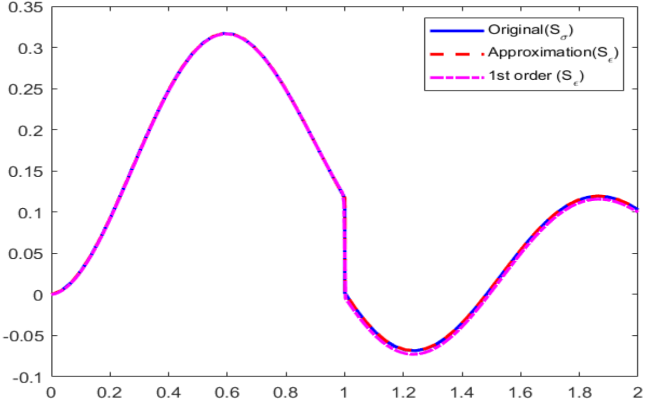


Figure: Comparison between the output of original, approximation and 1st order reduced system.

# Results for slightly bigger system

## Original switched system

$$A_0 = \begin{bmatrix} -0.74 & 0.3 & 0.2 & -0.01 & -0.06 \\ 0.965 & -1.43 & -0.5 & 0.8 & -0.26 \\ 0.922 & -0.0487 & -0.44 & 0.03 & 0.054 \\ -0.98 & 0.28 & 0.31 & -0.764 & 0.07 \\ -0.634 & -1.26 & 0.534 & 0.662 & -0.48 \end{bmatrix}, B_0 = \begin{bmatrix} 2 \\ 1.4 \\ 1.1 \\ -0.06 \\ 0.08 \end{bmatrix}, C_0^T = \begin{bmatrix} 2.5 \\ 2 \\ 1.6 \\ 0.02 \\ -0.03 \end{bmatrix}$$

$$A_1 = A_0 - 0.5 * I_5, B_1 = \begin{bmatrix} 2.5 \\ 1.8 \\ 0.3 \\ 0.6 \\ -1 \end{bmatrix}, C_1^T = \begin{bmatrix} 1.5 \\ 1.4 \\ 0.7 \\ 0.1 \\ 0.2 \end{bmatrix}.$$

- $x_0 := \mathbf{0}$, $u(t) = (\sin(5t) + 0.05)e^{-0.5t}, \varepsilon = 10^{-3}$.
- Switching signal:

$$\sigma(t) = \begin{cases} 0 & : t \in [0,1) \cup [2,4), \\ 1 & : t \in [1,2) \cup [4,6] \end{cases}$$

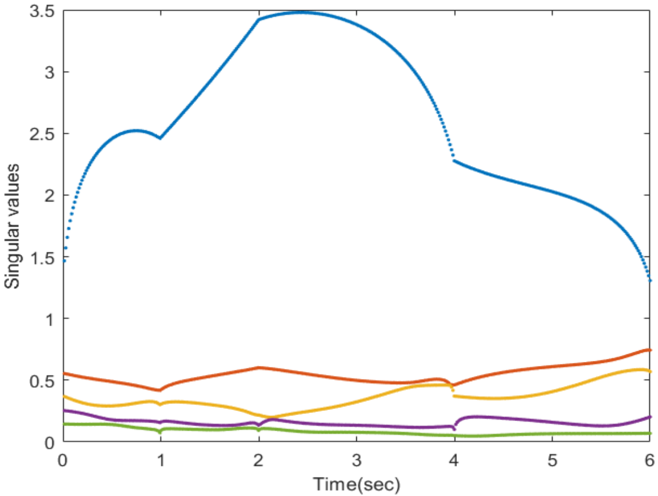


Figure: Hankel singular values of pointwise Gramians.

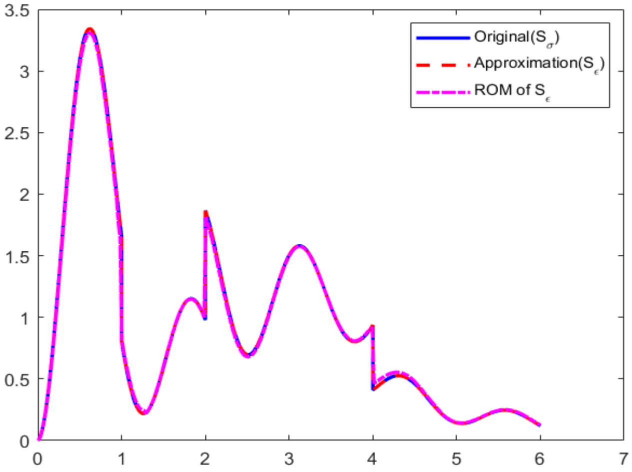


Figure: Comparison between the output of original, proposed approximation and 1st order reduced systems.

## Overall Summary

- Novel viewpoint: Switched system as given time-varying system
- Approximate discontinuous system by continuous one
- Use available balanced truncation methods for time-varying linear systems
- Small scale numerical experiments look promising

## Remaining challenges

- Feasibility for large scale systems?
- Reduced system is fully time-varying (and not piecewise constant)
- Limiting behavior for $\varepsilon \to 0$?
- Extension to switched differential-algebraic equations (DAEs)?