Impulse controllability of switched differential-algebraic equations

Paul Wijnbergen and Stephan Trenn

***Abstract*— This paper addresses impulse controllability of switched DAEs on a finite interval. First we present a forward approach where we define certain subspaces forward in time. These subpsaces are then used to provide a sufficient condition for impulse controllability. In order to obtain a full characterization we present afterwards a backward approach, where a sequence of subspaces is defined backwards in time. With the help of the last element of this backward sequence, we are able to fully characterize impulse controllability. All results are geometric results and thus independent of a coordinate system.**

I. INTRODUCTION

We consider *switched differential algebraic equations (switched DAEs)* of the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p, n, m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in [1]. We study impulse controllability of (1) where impulse controllability means that for every initial value there exist an input such that the resulting trajectory is impulse free (see Definition 9 for details).

Differential algebraic equations (DAEs) arise naturally when modeling physical systems with certain algebraic constraints on the state variables; examples of applications of DAEs in electrical circuits (with distributional solutions) can be found, e.g., in [2]. These constraints are often eliminated such that the system is described by ordinary differential equations (ODEs). However, in the case of switched systems, the elimination process of the constraints is in general different for each individual mode and therefore there does not exist a description as a switched ODE with a common state variable for every mode in general. This problem can be overcome by studying switched DAEs directly.

Ever since control systems have been considered, the question whether the control objective can be achieved with minimal (quadratic) cost has been of great interest. In the non-switched case, optimal control of DAEs has been studies in e.g. [3]–[5]. It is proven in both [3] and [4] that impulse controllability is a necessary condition for the existence of finite (quadratic) cost regardless of the initial condition. This follows from the fact that the integral over the square of a Dirac impulse is not well defined and therefore such an integral is assigned an infinite value. Trajectories resulting in finite cost must thus be free of impulsive behavior. This argument is independent of the underlying system model and hence trajectories of switched DAEs need to be impulse free as well in order to achieve finite quadratic cost. Therefore, there is a need for a characterization of all switched DAEs that are impulse controllable.

Several other structural properties of (switched) DAEs have been studied recently. Among those are controllability [6], stability [7] and observability [8]. However, impulse controllability has thus far only been studied in the non-switched case [9]–[12] and, to the best of the authors knowledge, there are no results yet for the switched case. An obvious sufficient condition is to demand each mode of the switched system to be impulse controllable. This is however not a necessary condition. If the first mode is impulse controllable and any initial condition can be steered to a point at the switching time where it doesn't produce Dirac impulses the system would be impulse controllable as well. A more detailed example of such a system will be given in Section III.

In this paper we give a characterization of impulse controllability on a given interval. We will consider two points of view with respect to impulse controllability: a forward and a backward approach. In the forward approach we consider the first mode and consider the space where the state can be steered to at the switching times. It follows that in order to ensure an impulse free solution, all future modes need to be taken into account. Therefore it is necessary for finding a characterization of impulse controllability to consider a time interval with a finite number of switches. The second method is concerned with the backward approach. In this approach the space from which the last mode can be reached in an impulse free way is considered. This space needs to contain all allowed initial conditions in order for the system to be impulse controllable. Our results are geometric characterizations, i.e. they do not depend on a specific choice of a coordinate system. Furthermore, we do not make an a-priori assumption on the index of the individual DAEs.

The outline of the paper is as follows: notations and results for non-switched DAEs are presented in Section II. The main results on impulse controllability are presented in Section III, followed by a brief discussion on the interpretation of the results. Conclusions and discussions on future work are given in Section IV.

The authors are with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, Nijenborgh 9, 9747 AG, Groningen, The Netherlands.
Email: p.wijnbergen@rug.nl; s.trenn@rug.nl
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II. Mathematical Preliminaries

In this section we recall some notation and properties related to the non-switched DAE

$$E\dot{x} = Ax + Bu. \tag{2}$$

A. Properties and definitions for regular matrix pairs

In the following, we call a matrix pair (E, A) and the associated DAE (2) *regular* iff the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form* [13].

Proposition 1: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{3}$$

where $J \in \mathbb{R}^{n_1 \times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_1 := n - n_1$, is a nilpotent matrix.
The matrices S and T can be calculated by using the so-called *Wong sequences* [13], [14]:

$$\begin{array}{lll} \mathcal{V}_0 := \mathbb{R}^n, & \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), & i = 0, 1, ... \\ \mathcal{W}_0 := \{0\}, & \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), & i = 0, 1, ... \end{array} \tag{4}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup \mathcal{W}_i. \tag{5}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (3) holds.

Based on the Wong sequences we define the following projector and selectors.

Definition 2: Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (3). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential selector* is given by

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

and the *impulse selector* is given by

$$\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$\begin{array}{ll} A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, & E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E, \\ B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B, & B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B. \end{array}$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). An important feature for DAEs is the so called consistency space, defined as follows:

Definition 3: Consider the DAE (2), then the *consistency space* is defined as

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } x \text{ of} \\ E\dot{x} = Ax, \text{ with } x(0) = x_0 \end{array} \right\},$$

and the *augmented consistency space* is defined as

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solutions } (x, u) \text{ of} \\ E\dot{x} = Ax + Bu \text{ and } x(0) = x_0 \end{array} \right\}.$$

In order to express (augmented) consistency spaces in terms of the Wong limits we need the following notation for matrices A,B of suitable sizes:

$$\langle A \mid B \rangle := \operatorname{im}[B, AB, \ldots, A^{n-1}B].$$

Proposition 4 ([15]): Consider the regular DAE (2), then $\mathcal{V}_{(E,A)} = \mathcal{V}^* = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\text{diff}}_{(E,A)}$ and $\mathcal{V}_{(E,A,B)} = \mathcal{V}^* \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

For studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [1] as the solution space, that is, we seek a solution $(x, u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+m}$ to the following initial-trajectory problem (ITP):

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{6a}$$

$$(E\dot{x})_{[0,\infty)} = (Ax + Bu)_{[0,\infty)}, \tag{6b}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In [1] it is shown that the ITP (6) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. As a direct consequence, the switched DAE (1) with regular matrix pairs is also uniquely solvable (with piecewise-smooth distributional solutions) for any switching signal with locally finitely many switches.

B. Properties of DAE's

Recall the following definitions and characterization of (impulse) controllability [15].

Proposition 5: The reachable space of the regular DAE (2) defined as

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \; \exists \text{ smooth solution } (x, u) \text{ of (2)} \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\}$$

satisfies $\mathcal{R} = \langle A^{\text{diff}} \mid B^{\text{diff}} \rangle \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

It is easily seen that the reachable space for (2) coincides with the controllable space, i.e.

$$\mathcal{R} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \; \exists \text{ smooth solution } (x, u) \text{ of (2)} \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Corollary 6: The augmented consistency space of (2) satisfies $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}}, B^{\text{imp}} \rangle$.

Definition 7: The DAE (2) is impulse controllable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x, u) of the ITP (6) such that $x(0^-) = x_0$ and $(x, u)[0] = 0$, i.e. the state and the input are impulse free at $t = 0$. The space of impulse controllable states of the DAE (2) is given by

$$\mathcal{C}^{\text{imp}}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } (x, u) \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \text{ of (6)} \\ \text{s.t. } x(0^-) = x_0 \text{ and } (x, u)[0] = 0. \end{array} \right\}.$$

In particular, the DAE (2) is impulse controllable if and only if $\mathcal{C}^{\text{imp}}_{(E,A,B)} = \mathbb{R}^n$.

The impulse controllable space can be characterized as follows [16].

Proposition 8: Consider the DAE (2) then

$$\begin{aligned}\mathcal{C}^{\text{imp}}_{(E,A,B)} &= \mathcal{V}_{(E,A,B)} + \ker E \\ &= \mathcal{V}_{(E,A)} + \mathcal{R} + \ker E \\ &= \mathcal{V}_{(E,A)} + \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle + \ker E.\end{aligned}$$

According to [17] if the input $u(\cdot)$ is sufficiently smooth, trajectories of (2) are continuous and given by

$$\begin{aligned}x(t) &= x_u(t,t_0;x_0) = e^{A^{\text{diff}}(t-t_0)}\Pi_{(E,A)}x_0 \\ &+ \int_{t_0}^{t} e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s)\, ds - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \quad (7)\end{aligned}$$

In particular, all trajectories can be written as the sum of an autonomous part $x_{\text{aut}}(t,t_0;x_0) = e^{A^{\text{diff}}t}\Pi_{(E,A)}x_0$ and a controllable part $x_u(t,t_0)$ as follows

$$x_u(t,t_0;x_0) = x_{\text{aut}}(t,t_0;x_0) + x_u(t,t_0).$$

In fact, this solution formula remains valid also for the switched case (by evaluating the initial value at t_0^-).

III. Impluse controllability of switched DAE's

The concepts introduced in the previous section are now utilized to obtain necessary and sufficient conditions for impulse controllability of switched DAEs. We will consider impulse-controllability on some *finite* interval (t_0,t_f) and assume that the switching signal only has finitely many switches in that interval; without restricting generality we can then assume that the switching signal has the following form:

$$\sigma(t) = p \quad \text{ if } t \in [t_p, t_{p+1}), \quad (8)$$

where $t_1 < t_2 < \dots t_{\mathtt{n}}$ are the $\mathtt{n} \in \mathbb{N}$ switches in (t_0,t_f) and for notational convenience let $t_{\mathtt{n}+1} := t_f$. We will now first introduce the definition of impulse controllability for switched DAEs and then the necessity of considering a finite interval.

A. Impulse controllability: definition

Definition 9: The switched DAE (1) with some fixed switching signal σ is called *impulse controllable on the interval* (t_0,t_f), if for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists a solution $(x,u) \in \mathbb{D}^{n+m}_{\text{pw}\mathcal{C}^\infty}$ of (1) with $x(t_0^+) = x_0$ which is impulse free.

If the interval (t_0,t_f) does not contain a switch, then the corresponding switched DAE is *always* impulse controllable on that interval due the definition of the augmented consistency space in terms of smooth (in particular, impulse free) solutions. This seems counter intuitive, because the active mode on that interval is not necessarily impulse controllable; however, recall that impulse controllability for a single mode (see Definition 7) is formulated in terms of the ITP (6), which can be interpreted as a switched system with one switch at $t_1 = 0$. In fact, letting $t_0 = -\varepsilon$, $t_f = \varepsilon$, $(E_0,A_0,B_0) = (I,0,0)$ and $(E_1,A_1,B_1) = (E,A,B)$, the DAE (2) is impulse controllable if, and only if, the corresponding ITP (reinterpreted as a switched DAE) is impulse controllable on $(-\varepsilon,\varepsilon)$.

Clearly, impulse controllability of *each* mode is a sufficient condition for impulse controllability of the overall switched DAE, however, the following example shows that this is in fact not necessary.

Example 10: Consider the switched DAE

$$\Sigma_\sigma : \begin{cases} \begin{bmatrix} 1&0&0\\0&0&1\\0&0&0 \end{bmatrix}\dot{x}(t) = x(t) + \begin{bmatrix}1\\0\\1\end{bmatrix}u(t), & 0 \leqslant t < t_1, \\ \begin{bmatrix} 0&1&0\\0&0&0\\0&0&1 \end{bmatrix}\dot{x}(t) = x(t) + \begin{bmatrix}0\\0\\1\end{bmatrix}u(t), & t_1 \leqslant t. \end{cases}$$

The first mode in the example is impulse controllable, but the second mode is not. However, since the first mode is completely controllable, any initial condition can be steered to the impulse controllable space of the second mode in an impulse free manner. Hence for any initial condition there exists an input such that the resulting trajectory is impulse free and thus the system is impulse controllable on any interval containing the switch.

B. Impulse controllability: forward approach

We consider the following sequence of subspaces.

$$\begin{aligned}\mathcal{K}_0^f &= \mathcal{V}_{(E_0,A_0)} + \mathcal{R}_0, \\ \mathcal{K}_i^f &= e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i(\mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}) + \mathcal{R}_i, \ i > 0,\end{aligned}$$

where $\mathcal{C}_i^{\text{imp}} := \mathcal{C}^{\text{imp}}_{(E_i,A_i,B_i)}$, $\Pi_i := \Pi_{(E_i,A_i)}$ and A_i^{diff} is the A^{diff}-matrix corresponding to (E_i,A_i). The intuition behind the definition is as follows: $\mathcal{K}_0^f$ are all values for $x(t_1^-)$ which can be reached before the first switch in an impulse free (in fact, smooth) way. Now, inductively, we calculate the set $\mathcal{K}_i^f$ of points which can be reached just before the switching time t_{i+1} by first consider the points $\mathcal{K}_{i-1}^f$ which can be reached in an impulse free way just before t_i, then pick those which can be continued in mode i impulse-freely by intersecting them with $\mathcal{C}_i^{\text{imp}}$, propagate this set forward according to the evolution operator and finally add the reachable space. This intuition is verified by the following two lemmas.

Lemma 11: Consider the switched system (1) on some interval (t_0,t_f) with the switching signal given by (8). If $(x,u) \in \mathbb{D}^{n+m}_{\text{pw}\mathcal{C}^\infty}$ is a solution of (1) which is impulse free on (t_0,t_f) then for all $i \in \{1,...,\mathtt{n}\}$ it holds that

$$x(t_i^-) \in \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}.$$

Proof: Let $(x,u) \in \mathbb{D}^{n+m}_{\text{pw}\mathcal{C}^\infty}$ be an impulse-free solution of (1). Then by definition of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ there exists $\varepsilon > 0$ such that (x,u) is a smooth solution of $E_0\dot{x} = A_0x + B_0u$ on $(t_1-\varepsilon,t_1)$. Hence $x(t_1^-) \in \mathcal{V}_{(E_0,A_0,B_0)}$. Furthermore, (x,u) is an impulse-free solution of $E_1\dot{x} = A_1x + B_1u$ on $[t_1,t_2)$, hence $x(t_1^-) \in \mathcal{C}_1^{\text{imp}}$. This shows the claim for $i=1$ and we conclude the proof inductively by assuming that the statement holds for i and proving that it holds for $i+1$. Since u is impulse-free on $[t_i,t_{i+1})$ we have

$$x(t_{i+1}^-) = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i x(t_i^-) + x_u(t_{i+1},t_i),$$

which by assumption does not exhibit impulses at the switch, therefore we have $x(t_{i+1}^-) \in \mathcal{C}_{i+1}^{\text{imp}}$. By assumption we have $x(t_i^-) \in \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}$ and $x_u(t_{i+1}, t_i) \in \mathcal{R}_i$, thus $x(t_{i+1}) \in \mathcal{K}_i$, which completes the proof. ∎

Lemma 1 showed that the sets $\mathcal{K}_i^f \cap \mathcal{C}_{i+1}^{\text{imp}}$ are large enough to contain all impulse-free solutions, the next lemma shows that they are in fact minimal in the sense that each point in $\mathcal{K}_i^f \cap \mathcal{C}_{i+1}^{\text{imp}}$ can be reached impulse freely on (t_0, t_i).

Lemma 12: Consider the switched system (1) on the interval (t_0, t_f) with switching signal (8). Then for any $i \in \{1, \ldots, \mathtt{n}\}$ and all $\xi \in \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}$ there exists a solution (x, u) of (1) with $x(t_i^-) = \xi$ which is impulse-free on (t_0, t_i).

Proof: The proof is again by induction. For $i = 1$ we have for all $\xi \in \mathcal{C}_1^{\text{imp}} \cap \mathcal{K}_0^f$ that $\xi \in \mathcal{K}_0^f = \mathcal{V}_{(E_0, A_0)} + \mathcal{R}_0 = \mathcal{V}_{(E_0, A_0, B_0)}$. By definition of the augmented consistency space (and taking into account the time-invariance of the definition), we have that there exists a smooth solution (x, u) of $E_0 \dot{x} = A_0 x + B_0 u$ on (t_0, t_1) with $x(t_1^-) = \xi$.

Assuming now that the statement holds for i, we now consider the case for $i + 1$. Let $\xi \in \mathcal{K}_i^f \cap \mathcal{C}_{i+1}^{\text{imp}}$, then by definition of $\mathcal{K}_i^f$ we have $\xi \in e^{A_i^{\text{diff}}(t_{i+1} - t_i)} \Pi_i (\mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}) + \mathcal{R}_i$, i.e. there exists $\widehat{\xi} \in \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}$ and $\xi_{\mathcal{R}} \in \mathcal{R}_i$ such that

$$\xi = e^{A_i^{\text{diff}}(t_{i+1} - t_i)} \Pi_i \widehat{\xi} + \xi_{\mathcal{R}}.$$

By the induction assumption, there exists a solution $(\widehat{x}, \widehat{u})$ of (1) with $\widehat{x}(t_i^-) = \widehat{\xi}$ which is impulse free on (t_0, t_i). Since $\widehat{x}(t_i^-) \in \mathcal{C}_i^{\text{imp}}$, this solution can be assumed to be impulse free also on $[t_i, t_{i+1})$. We will now alter this solution on $[t_i, t_{i+1})$ such that at t_{i+1}^- the desired value ξ is reached and no additional impulses occur. From the solution formula (7) it follows that $\widehat{\xi}_{\mathcal{R}} := \widehat{x}(t_{i+1}^-) - e^{A_i^{\text{diff}}(t_{i+1} - t_i)} \Pi_i \widehat{\xi} \in \mathcal{R}_i$ and hence $\xi_{\mathcal{R}} - \widehat{\xi}_{\mathcal{R}} \in \mathcal{R}_i$. By definition of the reachable space of mode i there exists a (smooth) solution $(\widetilde{x}, \widetilde{u})$ of $E_i \dot{x} = A_i x + B_i u$ such that $\widetilde{x}(t_i^-) = 0$ and $\widetilde{x}(t_{i+1}^-) = \xi_{\mathcal{R}} - \widehat{\xi}_{\mathcal{R}}$. In fact, it can be assumed that $(\widetilde{x}, \widetilde{u})$ is identically zero on (t_0, t_i), hence $(\widetilde{x}, \widetilde{u})$ is then also a solution of the switched DAE (1). By linearity, $(x, u) = (\widehat{x} + \widetilde{x}, \widehat{u} + \widetilde{u})$ is a solution of (1) that is impulse free on (t_0, t_{i+1}) with $x(t_{i+1}^-) = \widehat{x}(t_{i+1}^-) + \widetilde{x}(t_{i+1}^-) = (e^{A_i^{\text{diff}}(t_{i+1} - t_i)} \Pi_i \widehat{\xi} + \widehat{\xi}_{\mathcal{R}}) + (\xi_{\mathcal{R}} - \widehat{\xi}_{\mathcal{R}}) = \xi$, which concludes the proof. ∎

It is important to note that in general not from all $\xi \in \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}$ there is a solution (x, u) with $x(t_i^-) = \xi$ which is also impulse free on $[t_i, t_f)$. To illustrate this, we present the following example.

Example 13: Consider the following switched DAE, where (A, B) is controllable.

$$\Sigma_\sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \quad 0 \leqslant t < t_1, \\ \dot{x}(t) &= 0 \quad t_1 \leqslant t < t_2, \\ \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right] \dot{x}(t) &= x(t) \quad t_2 \leqslant t. \end{cases}$$

In order to have impulse free solutions, the state of the system needs to be in $\operatorname{span}\{e_1, e_2\}$ at $t = t_2$, where e_1, e_2 are the standard base vectors in $\mathbb{R}^3$. However, $\mathcal{K}_0^f \cap \mathcal{C}_1^{\text{imp}} = \mathbb{R}^n$ and therefore we can reach e_3 impulse freely on $(0, t_1]$, but this would lead to an impulse at $t = t_2$.

It thus becomes clear from Example 13 that considering the elements $\mathcal{K}_i^f \cap \mathcal{C}_{i+1}^{\text{imp}}$ will not lead to necessary conditions for impulse controllability. Indeed, the example shows that in general, the impulse controllable spaces of future modes need to be taken into account in order to ensure an overall impulse free solution.

Nevertheless does the forward approach lead to some useful results. Exploiting Lemma 11 and Lemma 12 we prove the next theorem, which gives a sufficient condition.

Theorem 14: Consider the switched system (1) with switching signal (8). If for all $i \in \{1, ..., \mathtt{n}\}$ it holds that

$$\mathcal{K}_{i-1}^f \subseteq \mathcal{C}_i^{\text{imp}} + \mathcal{R}_{i-1}, \tag{9}$$

then the system is impulse controllable.

Proof:

We prove the statement inductively by showing that if (9) holds then for any initial value x_0 there exists a solution (x, u) with $x(t_0^+) = x_0$, $x(t_i^-) \in \mathcal{C}_i^{\text{imp}}$ and which is impulse free on (t_0, t_i). For $i = 1$ we find for any $x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}$ by definition a solution $(\widehat{x}, \widehat{u})$ with $\widehat{x}(t_0^+) = x_0$ which is smooth (and in particular impulse free) on (t_0, t_1). Furthermore, $\widehat{x}(t_1^-) \in \mathcal{V}_{(E_0, A_0, B_0)} \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0$, i.e. there exists $\xi \in \mathcal{C}_1^{\text{imp}}$ and $\eta \in \mathcal{R}_0$ such that $\widehat{x}(t_1^-) = \xi + \eta$. Since η is reachable in mode 0 we can find $\widetilde{u}$ such that the corresponding solution of (1) satisfies $\widetilde{x}(t_0^+) = 0$ and $\widetilde{x}(t_1^-) = -\eta$. Now $(x, u) := (\widehat{x} + \widetilde{x}, \widehat{u} + \widetilde{u})$ solves (1), is impulse-free on (t_0, t_1) and satisfies $x(t_0^+) = x_0$ and $x(t_1^-) = \xi + \eta - \eta \in \mathcal{C}_1^{\text{imp}}$.

Now assume that any initial condition can be steered to $\mathcal{C}_i^{\text{imp}}$ impulse-freely on (t_0, t_i). This solution can now be extended to an impulse free solution $(\widehat{x}, \widehat{u})$ onto (t_0, t_{i+1}). Similar as in Lemma 11 we can conclude that $\widehat{x}(t_i^-) \in \mathcal{K}_{i-1} \cap \mathcal{C}_i^{\text{imp}}$, and hence $\widehat{x}(t_{i+1}^-) \in \mathcal{K}_i^f \subseteq \mathcal{C}_{i+1}^{\text{imp}} + \mathcal{R}_i$. Hence $\widehat{x}(t_i^-) = \xi + \eta$ for $\xi \in \mathcal{C}_{i+1}^{\text{imp}}$ and $\eta \in \mathcal{R}_i$. Similar as above we find a solution $(\widetilde{x}, \widetilde{u})$ which is smooth on $[t_0, t_{i+1})$, identically zero on (t_0, t_i) and satisfies $\widetilde{x}(t_{i+1}^-) = -\eta$. Then $(x, u) = (\widehat{x} + \widetilde{x}, \widehat{u} + \widetilde{u})$ is a solution which is impulse free on (t_0, t_f), has the same initial value as $\widehat{x}$ satisfies $x(t_{i+1}^-) \in \mathcal{C}_{i+1}^{\text{imp}}$.

Finally, from the fact that for any initial value there is a solution (x, u) with $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}}$ it can be concluded that this solution can be extended to (t_0, t_f) in an impulse free way, i.e. the switched system is impulse controllable.

∎

Remark 15: Besides giving a sufficient conditions for impulse controllability of a system for a given switching signal, the result of Theorem 14 can also be used to design a specific switching signal so that the switched system becomes impulse-controllable.

To illustrate the result of Theorem 14 we will give an example where the subspaces involved become apparent.

Example 16: Consider the following switched DAE, with a switch at $t = t_1$:

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$
$$(E_1, A_1, B_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right).$$

It follows from the computation of the consistency projector and the reachable space that

$$\mathcal{V}_{(E_0,A_0)} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\},\ \mathcal{R}_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\},$$

such that we obtain $\mathcal{K}_0^f = \mathbb{R}^n$. The impulse controllable space is given by

$$\mathcal{C}_1^{\text{imp}} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This means that $\mathcal{C}_1^{\text{imp}} + \mathcal{R}_0 = \mathbb{R}^n$ and hence the condition that $\mathcal{K}_0^f \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0$ of Theorem 14 is satisfied and we can conclude that the system is impulse controllable. Indeed we see that for all elements of the consistency space there exists a reachable point such that the sum of the two are in the impulse controllable space.

In the case of a single switch, then there are no other modes to consider than the first two modes. In that case, the sufficient condition in Theorem 14 is also necessary.

Lemma 17: Consider the switched DAE (1) with a single switch at $t = t_s$. If the system is impulse controllable, then

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0.$$

Proof: Since we have that for all initial conditions there exists an input $u(t)$ such that the resulting trajectory is impulse free, i.e. for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ we have

$$x_u(t_1^-, x_0) = e^{A_0^{\text{diff}} t_1} \Pi_0 x_0 + x_u(t_1^+, 0) \in \mathcal{C}_1^{\text{imp}},$$

which means that

$$e^{A^{\text{diff}} t_1} \Pi_0 \mathcal{V}_{(E_0,A_0,B_0)} = \mathcal{V}_{(E_0,A_0} \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0.$$

from which it follows that $\mathcal{V}_{(E_0,A_0,B_0)} = \mathcal{V}_{(E_0,A_0)} + \mathcal{R}_0 \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0$. ■

C. Impulse controllability: backward approach

In order to incorporate all modes on the bounded interval, we make use of a backwards method. This method considers the set of points from which the impulse controllable space of the last mode can be reached. To that extend we first consider the largest set of points from which the impulse controllable space can be reached impulse freely from the preceding mode. Therefore we define the following sequence of sets regarding the switched DAE (1) with switching signal (8):

$$\begin{aligned} \mathcal{K}_{\mathtt{n}}^b &= \mathcal{C}_{\mathtt{n}}^{\text{imp}}, \\ \mathcal{K}_{i-1}^b &= \operatorname{im} \Pi_{i-1} \cap \left(e^{-A_{i-1}^{\text{diff}}(t_{i-1}-t_i)} \mathcal{K}_i^b + \mathcal{R}_{i-1} \right) \\ &\quad + \langle E_{i-1}^{\text{imp}} \mid B_{i-1}^{\text{imp}} \rangle + \ker E_{i-1}, \\ &\quad i = \mathtt{n}, \mathtt{n}-1, \ldots, 1. \end{aligned}$$

Remark 18: Recall that $\operatorname{im} \Pi_i = \mathcal{V}_{(E_i,A_i)}$ and that $\mathcal{C}_i^{\text{imp}} = \mathcal{V}_{(E_i,A_i)} + \langle E_i^{\text{imp}} \mid B_i^{\text{imp}} \rangle + \ker E_i$. Therefore we have that $\mathcal{K}_i^b \subseteq \mathcal{C}_i^{\text{imp}}$.

With these sets, we can prove the following lemma.

Lemma 19: Consider the switched DAE (1) restricted to the interval $[t_{i-1}, t_i)$. Then $\mathcal{K}_{i-1}^b$ is the largest set of points at time t_{i-1}^- from which $\mathcal{K}_i^b$ can be reached (at t_i^-) in an impulse free way.

Proof: First we show that for all $x_{i-1} \in \mathcal{K}_{i-1}^b$ there exists an input such that the corresponding solution with initial value x_{i-1} at t_{i-1}^- is impulse free on $[t_{i-1}, t_i)$ and $x_u(t_i^-, t_{i-1}^-; x_{i-1}) \in \mathcal{K}_i$. To do so, consider first the case $x_{i-1} \in \operatorname{im} \Pi_{i-1} \cap \big(e^{-A_{i-1}^{\text{diff}}(t_{i-1}-t_i)} \mathcal{K}_i^b + \mathcal{R}_{i-1}\big)$. Since $x_{i-1} \in \operatorname{im} \Pi_{i-1}$ we know that x_{i-1} is a consistent initial value. Hence it will not produce any Dirac impulses for a zero input and the corresponding solution $(\widehat{x}, 0)$ satisfies

$$\widehat{x}(t_i^-) = e^{A_{i-1}^{\text{diff}}(t_i - t_{i-1})} x_{i-1} \in \mathcal{K}_i^b + \mathcal{R}_{i-1},$$

where we used the fact, that $\mathcal{R}_{i-1}$ is A_{i-1}^{diff}-invariant. Let $\widehat{x}(t_i^-) = \xi + \eta$ with $\xi \in \mathcal{K}_i^b$ and $\eta \in \mathcal{R}_{i-1}$. Now we are able to choose a smooth solution $(\widetilde{x}, u)$ on $[t_{i-1}, t_i)$ such that $\widetilde{x}(t_{i-1}^-) = 0$ and $\widetilde{x}(t_i^+) = -\eta$. Then $(x, u) := (\widehat{x} + \widetilde{x}, u)$ is an impulse free solution on $[t_i, t_{i-1})$ with $x(t_{i-1}^-) = \widehat{x}(t_i^-) + \widetilde{x}(t_{i-1}^-) = x_{i-1} + 0$ and $x(t_i^-) = \widehat{x}(t_i^-) + \widetilde{x}(t_i^-) = (\xi + \eta) + (-\eta) = \xi \in \mathcal{K}_i^b$. Next assume that $x_{i-1} \in \langle E_{i-1}^{\text{imp}}, B_{i-1}^{\text{imp}} \rangle \subseteq \mathcal{R}_{i-1}$. Then there exists an input u such that $x_u(t_i^-, t_{i-1}^-; x_{i-1}) = 0 \in \mathcal{K}_i^b$. Finally, suppose that $x_{i-1} \in \ker E_{i-1}$ then by applying a zero input the state jumps to zero (impulse-freely) after switching to the $i-1^{\text{st}}$ mode. Altogether, linearity implies that for all $x_{i-1} \in \mathcal{K}_{i-1}$ there exist an input u such that $\mathcal{K}_i^b$ is reached impulse freely.

Next we show that this is the largest set of points from which $\mathcal{K}_i^b$ is impulse freely reachable. Let (x, u) be any impulse-free solution of (1) on $[t_{i-1}, t_i)$ with $x(t_i^-) \in \mathcal{K}_i^b$. We need to show that $x_{i-1} := x(t_{i-1}^-) \in \mathcal{K}_{i-1}^b$. Clearly by definition, $x_{i-1} \in \mathcal{C}_{i-1}^{\text{imp}} = \operatorname{im} \Pi_{i-1} + \langle E_{i-1}^{\text{imp}} \mid B_{i-1}^{\text{imp}} \rangle + \ker E_{i-1}$. Choose $\xi \in \operatorname{im} \Pi_{i-1}$, $\eta \in \langle E_{i-1}^{\text{imp}} \mid B_{i-1}^{\text{imp}} \rangle$, $\zeta \in \ker E_{i-1}$ such that $x_{i-1} = \xi + \eta + \zeta$. Then the solution x_i can be decomposed into

$$x_i = x_{\text{aut}}(t_i^-, t_{i-1}^-, \xi) + x_u(t_i^-, t_{i-1}^-, \eta) + x_{\text{aut}}(t_i^-, t_{i-1}^-, \zeta).$$

Note that $\ker E \subseteq \ker \Pi$ which implies that $x_{\text{aut}}(t_i^-, t_{i-1}^-, \zeta) = 0$, from $\Pi_{i-1} \eta = 0$ we conclude that $x_u(t_i^-, t_{i-1}^-, \eta) \in \mathcal{R}_{i-1}$ and finally $\xi \in \operatorname{im} \Pi_{i-1}$ implies $\Pi_{i-1} \xi = \xi$, hence

$$\xi = e^{-A_{i-1}^{\text{diff}}(t_i - t_{i-1})} x_i - \theta$$

where $\theta = e^{-A_{i-1}^{\text{diff}}(t_i - t_{i-1})} x_u(t_i^-, t_{i-1}^-, \eta) \in \mathcal{R}_{i-1}$, because $\mathcal{R}_{i-1}$ is A^{diff}-invariant. Hence $\xi \in \operatorname{im} \Pi_{i-1} \cap \big(e^{-A_{i-1}^{\text{diff}}(t_{i-1}-t_i)} \mathcal{K}_i^b + \mathcal{R}_{i-1}\big)$ and the claim is shown. ■

Corollary 20: Consider the switched DAE (1) with switching signal (8). Then $\mathcal{K}_{i-1}^f \cap \mathcal{K}_i^b$ is the smallest set containing states that can be reached in an impulse free way on (t_0, t_i) and that can be extended in an impulse free way on $[t_i, t_f)$.

Proof: By Lemma 12 we have for all $x_i \in \mathcal{K}_{i-1}^f \cap \mathcal{K}_i^b \subseteq \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}$ that there exists an x_0 such that $x_u(t, x_0)$ is impulse free and $x_u(t_i^-, x_0) = x_i$. And by Lemma 19 there exists an input such that $\mathcal{C}_{\mathtt{n}}^{\text{imp}}$ is reached impulse freely from x_i.

Let (x, u) be an impulse free solution. Then by Lemma 11 we have that at t_i $x_i \in \mathcal{K}_{i-1}^f \cap \mathcal{C}_i^{\text{imp}}$ and therefore $x_i \in \mathcal{K}_{i-1}^f$. Since we can reach $\mathcal{C}_{\mathtt{n}}^{\text{imp}}$ impulse freely from x_i it must hold that $x_i \in \mathcal{K}_i^b$. Therefore $x_i \in \mathcal{K}_{i-1}^f \cap \mathcal{K}_i^b$, which proves the result. ∎

Since $\mathcal{K}_{i-1}^b$ is the largest set from which $\mathcal{K}_i^b$ can be reached impulse freely, it follows intuitively that the system is impulse controllable on the whole interval (t_0, t_f) if the initial augmented consistency space is contained in $\mathcal{K}_0^b$. This would mean that all consistent trajectory of the initial mode can be steered impulse freely to the impulse controllable space of the last mode. This idea is formalized in the next theorem.

Theorem 21: Consider the switched system (1) on (t_0, t_f) with switching signal (8). The system is impulse controllable if and only if

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^b.$$

Proof: ($\Leftarrow$) Assume that $\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^b$. This means that for all initial conditions $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there must exists an input u such that the resulting trajectory is impulse free and reaches $\mathcal{K}_{\mathtt{n}}^b$. Hence the system is impulse controllable.

($\Rightarrow$) Assume that the system is impulse controllable. Then for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists an input u such that the resulting trajectory is impulse free. Since it holds for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ it must hold that

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^b,$$

which proves the desired result. ∎

To illustrate the results from Theorem 21 we show the following example where we verify impulse controllability.

Example 22: Consider the switched DAE with the following modes and switching times $t_1 = \ln(4)$ and $t_2 = t_1 + \frac{1}{2}\pi$.

$$\begin{aligned}
(E_0, A_0, B_0) &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right), \\
(E_1, A_1, B_1) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right), \\
(E_2, A_2, B_2) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right).
\end{aligned}$$

Since the second mode rotates the state, it is easy to calculate that.

$$\mathcal{C}_2^{\text{imp}} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{K}_1^b = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\},$$

Then calculating the subspaces involved yields

$$\begin{aligned}
\Pi_0 &= \operatorname{im} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad e^{-A_0^{\text{diff}} \ln(4)} \mathcal{K}_1^b = \operatorname{im} \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ -3 & -4 \end{bmatrix}, \\
\mathcal{R}_0 &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}, \quad \ker E_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\end{aligned}$$

From this it can be calculated that $\mathcal{K}_0^b = \mathbb{R}^n$ and hence the system is impulse controllable.

IV. CONCLUSIONS

We have studied impulse controllability of switched DAEs on a finite interval. Based on sequence of subspace defined forward in time we were able to provide a sufficient condition for impulse-controllability. In order to fully characterize impulse-controllability we introduced a sequence of subspaces defined backwards in time.

As a future direction of research, a natural extension is to obtain results on impulse free stabilization. In the current literature impulse free trajectories are part of the definition of stability of switched (autonomous) DAEs. Therefore, impulse controllability is a necessary condition for switched DAEs with inputs to be stabilizable. However, necessary and sufficient condition for stabilizability of non autonomous switched DAEs are yet to be formulated.

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