On geometric and differentiation index of nonlinear differential-algebraic equations

Yahao Chen * **Stephan Trenn** *

* *Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, The Netherlands.*

Abstract: In this paper, we discuss two notions of index (the geometric index and the differentiation index), which appear in the studies of the solvability of nonlinear differential-algebraic equations DAEs. First, we analyze the solutions of nonlinear DAEs via a geometric method, then depending on the analysis of solutions, we show that although both of the two indices serve as a measure of the difficulties of solving DAEs, they are actually related to the existence and uniqueness of solutions in a different manner. We also show that the two DAE indices have close relations with each other when some assumptions of smoothness and constant rankness are satisfied. An example of a pendulum system is used to illustrate our geometric method of solving DAEs and also our results of the relations of the two DAE indices.

Keywords: differential-algebraic equations, geometric method, differentiation index, geometric index, existence and uniqueness of solutions

1. INTRODUCTION

We consider nonlinear differential-algebraic equations (DAEs) of the following form

$$\Xi: \ E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is called the generalized state and X is an open subset of $\mathbb{R}^n$ (or n-dimensional manifold), and where $E: X \to \mathbb{R}^{l\times n}$ and $F: X \to \mathbb{R}^l$ are $\mathcal{C}^\infty$-smooth maps. A DAE of form (1) will be denoted by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. A DAE of form (1) is usually called a quasi-linear DAE (see e.g., Rabier and Rheinboldt (2002), Riaza (2008)), which is a special case of DAEs of the following general form:

$$\Xi^{gen}: G(t, x, x') = 0, \tag{2}$$

where $G: I \times TX \to \mathbb{R}^l$ is $\mathcal{C}^\infty$-smooth, $I \subseteq \mathbb{R}$ is an open interval and TX is the tangent bundle of a differentiable manifold X. Notice that Ξ^{gen} can be transformed into a DAE of form (1) by extending the generalized state to $(t, x, z) = (t, x, x')$, i.e., $\dot{t} = 1$, $\dot{x} = z$, $0 = G(t, x, z)$, which is a DAE of form (1), where

$$E = \begin{bmatrix} I_{n+1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+l+1)\times(2n+1)}, \ \ F(t, x, z) = \begin{bmatrix} 1 \\ z \\ G(t, x, z) \end{bmatrix}.$$

A solution of Ξ is a $\mathcal{C}^1$-curve $x: I \to X$ with an open interval I such that for all $t \in I$, $x(t)$ solves (1). An admissible point of (1) is a point $x_0 \in X$ such that through x_0, there exists at least one solution. Denote by S_{adm} the admissible set, i.e., the set of all admissible points, of Ξ. We also consider linear DAEs of the form

$$\Delta: \ E\dot{x} = Ax, \tag{3}$$

where $E \in \mathbb{R}^{l\times n}$ and $A \in \mathbb{R}^{l\times n}$. A linear DAE of form (3) will be denoted by $\Delta_{l,n} = (E, A)$ or, shortly, Δ.

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Note that removing dependent equations and the trivial constrains $0 = 0$ of a DAE (linear or nonlinear) does not change the solutions of the DAE. For example, let $H(x, \dot{x}) = E(x)\dot{x} - F(x)$ and rewrite (1) as the general form $H(x, \dot{x}) = 0$, we assume that two rows $H_1: TX \to \mathbb{R}$ and $H_2: TX \to \mathbb{R}$ of the function $H(x, \dot{x})$ are dependent, i.e., $\exists$ nonzero $\alpha: TX \to \mathbb{R}$ such that $H_1 = \alpha H_2$. Now we remove H_1 and denote H by $\tilde{H}$. Clearly, $\tilde{H}(x, \dot{x}) = 0$ has the same solutions as $H(x, \dot{x}) = 0$. For simplicity, we assume throughout that the DAEs we study do not contain any dependent equations and the trivial algebraic constrains $0 = 0$.

Two main streams of the researches on solutions of nonlinear DAEs are the geometric methods shown in, e.g., Rheinboldt (1984), Reich (1990), Reich (1991), Rabier and Rheinboldt (1994) and the numerical methods illustrated in, e.g., Gear (1988), Brenan et al. (1996), Kunkel and Mehrmann (2006). Moreover, some crossover results of the two methods can be consulted in Griepentrog (1991), Campbell and Griepentrog (1995), Rabier and Rheinboldt (2002) and the references therein. To characterize the different properties of nonlinear DAEs, various notions of index are proposed, see the survey or survey-like papers on index of DAEs: Griepentrog et al. (1992), Campbell (1995), Campbell and Gear (1995), Mehrmann (2015). The most commonly used notions of index seem to be the geometric index (see Reich (1990) and Rabier and Rheinboldt (2002)) and the differentiation index (see the two somewhat related definitions in Definition 4 of Campbell and Gear (1995) and in Section 3 of Griepentrog (1991), their common grounds and differences will be discussed in Section 3 of the present paper). Our formulations of the definitions of the two indices are given in Definition 5 and Definition 9 below, respectively.

The aim of the present paper is to have a comprehensive understanding of the two notions of DAE index by analyzing their relations and differences. We give our analysis of the solvability of DAEs and show how it is related to the geometric index via a geometric method called the maximal invariant submanifold algorithm in Section 2. Campbell and Gear (1995) have claimed that these two indices coincide when sufficient smoothness and assumptions are satisfied, we elaborate this claim and present our results about the relations of the two indices in Section 3. The proofs of the two main theorems and the conclusions of this paper are given in Section 4 and Section 5, respectively. We use the following notations: x' or $\dot{x}$ denotes the derivate of x with respect to t. $\mathcal{C}^k$ denotes the class of functions which are k-times differentiable. For a map $A : X \to \mathbb{R}^{l\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively; $\mathrm{D}A(x)$ denotes the differential of $A(x)$ and $\mathrm{D}_x A(x)$ denotes the Jacobian matrix of $A(x)$ at x. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$.

2. MAXIMAL INVARIANT SUBMANIFOLD ALGORITHM AND GEOMETRIC INDEX OF DAES

The main idea of geometric analysis of DAEs is to view a DAE as a vector field defined on a submanifold, the later is called the constraint submanifold or the configuration space (called the consistence space for linear DAEs see e.g., Trenn (2013)), we will call the same notion the maximal invariant submanifold and denote it by M^*, see the formal definition of (maximal) invariant submanifold below.

Definition 1. (Invariant submanifold). For a DAE Ξ defined on X, a smooth connected submanifold M of X is called *invariant* if for any point $x_0 \in M$, there exists a solution $x : I \to X$ of Ξ such that $x(0) = x_0$ and $x(t) \in M$ for all $t \in I$.

Given an admissible point x_a, we will say that M is a locally invariant submanifold (around x_a) if there exists an open neighborhood $U \subseteq X$ of x_a such that $M \cap U$ is invariant. A locally invariant submanifold M^* is called maximal, if there exists a neighborhood U of x_a such that for any other locally invariant submanifold M, we have $M \cap U \subseteq M^* \cap U$. The following algorithm is a recursive way to construct the locally maximal invariant submanifold M^* (Chen and Respondek (2019b)):

For a DAE $\Xi_{l,n} = (E, F)$, set $M_0 = X$, assume that a point $x_p \in M_0$ satisfies $F(x_p) \in \operatorname{Im} E(x_p)$. Step 1: set

$$M_1 := \{x \in X : F(x) \in \operatorname{Im} E(x)\}; \tag{4}$$

Step k: assume that $M_{k-1} \subsetneq \cdots \subsetneq M_0$, for a certain $k > 1$, have been constructed and for some open neighborhood $U_{k-1} \subseteq X$ of x_p that the intersection $M_{k-1} \cap U_{k-1}$ is a smooth submanifold and denote by M^c_{k-1} the connected component of $M_{k-1} \cap U_{k-1}$ satisfying $x_p \in M^c_{k-1}$. Set

$$M_k := \left\{x \in M^c_{k-1} \,|\, F(x) \in E(x) T_x M^c_{k-1}\right\}. \tag{5}$$

Remark 2. If we apply the above algorithm to a linear DAE $\Delta_{l,n} = (E, A)$ and denote $V = M$, we get the following sequence of subspaces: $V_0 = \mathbb{R}^n$,

$$V_k = \{x \in V_{k-1} \,|\, Ax \in EV_{k-1}\} = A^{-1}EV_{k-1}.$$

The above sequence V_k is one of the Wong sequences (Wong (1974)), which plays an important role in the geometric theory of linear DAEs (see e.g., Berger and Trenn (2012) and Chen and Respondek (2019a)). Thus the sequence of submanifolds M_k is, clearly, a nonlinear generalization of the Wong sequence V_k. Note that the limits of V_k, i.e., $V^* = V_n$ is the largest subspace such that $AV^* \subseteq EV^*$, which coincides with the consistency space of Δ, i.e., the subspace on which the solutions of Δ exist.

The geometric descriptions of (4) and (5) need a constructive application, which can be realized through Algorithm 1 below, some similar ways of constructing the sequence of submanifolds M_k and the maximal invariant submanifold M^* can be consulted in Reich (1990), Rabier and Rheinboldt (2002) and Riaza (2008), called the geometric reduction method.

Algorithm 1. Maximal invariant submanifold algorithm.
Results: locally maximal invariant submanifold M^*.
Initialization. Let $\Xi_{l,n} = (E, F)$, $x_p \in X$ be given. Set $\bar{z}_0 = x$ $\tilde{E}^1_0(\bar{z}_0) = E(x)$, $\tilde{F}^1_0(\bar{z}_0) = F(x)$, $M_0 = M^c_0 = X$, $M_1 \neq X$, $r_0 = l$, $s_0 = n$, $k = 1$.
While locally $M_k \neq M_{k-1}$
Step k: set $E_k(\bar{z}_{k-1}) = \tilde{E}^1_{k-1}(\bar{z}_{k-1})$, $F_k(\bar{z}_{k-1}) = F^1_{k-1}(\bar{z}_{k-1})$, $\Xi_k = (E_k, F_k)$, where $E_k : M^c_{k-1} \to \mathbb{R}^{r_{k-1}\times s_{k-1}}$, $F_k : M^c_{k-1} \to \mathbb{R}^{r_{k-1}}$.
Assume: there exists a neighborhood $U_k \subseteq X$ of x_p such that $\operatorname{rank} E_k(\bar{z}_{k-1}) = const. = r_k \le s_{k-1}$ on $W_k = U_k \cap M^c_{k-1} \subseteq M^c_{k-1}$. Find $Q_k : W_k \to Gl(r_{k-1}, \mathbb{R})$ such that $\operatorname{rank} E^1_k(\bar{z}_{k-1}) = r_k$ for all $\bar{z}_{k-1} \in W_k$:

$$Q_k E_k(\bar{z}_{k-1}) = \begin{bmatrix} E^1_k(\bar{z}_{k-1}) \\ 0 \end{bmatrix}, \quad Q_k F_k(\bar{z}_{k-1}) = \begin{bmatrix} F^1_k(\bar{z}_{k-1}) \\ F^2_k(\bar{z}_{k-1}) \end{bmatrix},$$

where $E^1_k : W_k \to \mathbb{R}^{r_k \times s_{k-1}}$, $F^1_k : W_k \to \mathbb{R}^{r_k}$. Following (5), define

$$M_k = \left\{\bar{z}_{k-1} \in W_k \,|\, F^2_k(\bar{z}_{k-1}) = 0\right\}.$$

Assume: $x_p \in M_k$ and $\operatorname{rank} \mathrm{D}F^2_k(\bar{z}_{k-1}) = const. = s_{k-1} - s_k \le r_{k-1} - r_k$ for $\bar{z}_{k-1} \in M_k \cap U_k$.
Find the independent rows of $\mathrm{D}F^2_k(\bar{z}_{k-1})$ and denote by $\mathrm{d}\varphi^1_k(\bar{z}_{k-1}), ..., \mathrm{d}\varphi^{s_{k-1}-s_k}_k(\bar{z}_{k-1})$.
Choose new coordinates $(\bar{z}_k, z_k) = \psi_k(\bar{z}_{k-1})$ on W_k, where $z_k = (\varphi^1_k(\bar{z}_{k-1}), ..., \varphi^{s_{k-1}-s_k}_k(\bar{z}_{k-1}))$, and where $\bar{z}_k$ are any complementary coordinates such that ψ_k is a local diffeomorphism. Then $M^c_k = \{\bar{z}_{k-1} \in W_k \,|\, z_k = 0\}$. Set

$$\tilde{E}_k(\bar{z}_{k-1}) = Q_k E_k \left(\frac{\partial \psi}{\partial \bar{z}_{k-1}}\right)^{-1} (\bar{z}_{k-1}) = \begin{bmatrix} \tilde{E}^1_k(\bar{z}_k, z_k) & \tilde{E}^2_k(\cdot) \\ 0 & 0 \end{bmatrix},$$

$$\tilde{F}_k(\bar{z}_{k-1}) = Q_k F_k(\bar{z}_{k-1}) = \begin{bmatrix} F^1_k(\bar{z}_{k-1}) \\ F^2_k(\bar{z}_{k-1}) \end{bmatrix} = \begin{bmatrix} \tilde{F}^1_k(\bar{z}_k, z_k) \\ \tilde{F}^2_k(\bar{z}_k, z_k) \end{bmatrix}.$$

where $\tilde{E}^1_k : W_k \to \mathbb{R}^{r_k \times s_k}$, $\tilde{E}^1_k : W_k \to \mathbb{R}^{r_k}$.
Set $z_k = 0$ to get the reduced DAE:

$$\tilde{\Xi}_k|_{M^c_k} : \tilde{E}^1_k(\bar{z}_k, 0)\dot{\bar{z}}_k = \tilde{F}^1_k(\bar{z}_k, 0).$$

do $k := k + 1$
End While
Output: $k^* = k$, $U^* = U_{k^*}$, $M^* = M^c_{k^*}$ and $\tilde{\Xi}_{k^*}|_{M^c_{k^*}}$.

The following theorem shows that under the assumptions of (A1) and (A2) below, the convergence of Algorithm 1 and the existence of solutions of Ξ can be guaranteed.

Theorem 3. Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a point $x_p \in X$. Assume in Algorithm 1 that there exists a neighborhood $U_k \subseteq X$ around x_p that for each $k > 0$,

(A1) $x_p \in M_k$ and $\operatorname{rank} \mathrm{D}F^2_k(\bar{z}_{k-1})$ is constant for all $\bar{z}_{k-1} \in M_k \cap U_k$.
(A2) $\dim E(x) T_x M^c_{k-1} = \operatorname{rank} E_k(\bar{z}_{k-1})$ is constant for all $\bar{z}_{k-1} \in W_k = M^c_{k-1} \cap U_k$.

Then there exists a smallest k, denoted by $k^* \le n$ such that $M_{k^*+1} = M^c_{k^*}$, we have that x_p is an admissible point and $M^* = M^c_{k^*}$ is a locally maximal invariant submanifold. Moreover, we have locally (on $U^* = U_{k^*}$) that

(i) if the admissible set S_{adm} is a smooth connected submanifold, then $M^* = S_{adm}$;
(ii) there exist a diffeomorphism between a solution of Ξ and a solution of
$$\tilde{E}^1_{k^*}(\bar{z}_{k^*})\dot{\bar{z}}_{k^*} = \tilde{F}^1_{k^*}(\bar{z}_{k^*}), z_{k^*} = 0, \ldots, z_1 = 0, \quad (6)$$
where $\tilde{E}^1_{k^*} : M^* \to \mathbb{R}^{r_{k^*} \times s_{k^*}}$, $\tilde{F}^1_{k^*} : M^* \to \mathbb{R}^{r_{k^*}}$ and $\operatorname{rank} \tilde{E}^1_{k^*} = \dim E(x)T_x M^* = r_{k^*}$;
(iii) for any $x_0 \in M^*$, there passes only one solution if and only if $\dim M^* = \dim E(x)T_x M^*$ for all $x \in M^*$, i.e., $s_{k^*} = r_{k^*}$.

The proof is given in Section 4.

Remark 4. (i) Item (i) of Theorem 3 illustrates that for all x in a neighborhood (U^*) of a admissible point $x_a = x_p$, the solutions of nonlinear DAEs exist on the maximal invariant submanifold M^* only, i.e., M^* is locally where the solutions of DAEs exist. Therefore, for any point $x_0 \in U^*/M^*$ around x_p, there exists no solution.

(ii) From (ii) of Theorem 3, it is seen that a solution of a DAE Ξ is isomorphic to that of a reduced DAE $\tilde{E}^1_{k^*}(\bar{z}_{k^*})\dot{\bar{z}}_{k^*} = \tilde{F}^1_{k^*}(\bar{z}_{k^*})$, the later is an "underdetermined" DAE which can be expressed as an ordinary differential equation ODE (see equation (18) in Section 4). Item (iii) of Theorem 3 shows that the uniqueness of the solutions is determined by the presence of free variables of the ODE getting from the reduced DAE.

(iii) The constancy of $\operatorname{rank} E_k(z_k) = \dim E(x)T_x M^c_k$ and in particular, that of $\operatorname{rank} E_{k^*}(z_{k^*}) = \dim E(x)T_x M^c_{k^*}$ is not a necessary condition for the existence of solutions and that of the maximal invariant submanifold M^*. Take the following DAE for example
$$\Xi : x\dot{x} = x^2, \quad (7)$$
where $x \in X = \mathbb{R}$. Clearly, $M^* = X$ is a maximal invariant submanifold since Ξ has an unique solution $x(t) = e^t x_0$ for any $x_0 \in X$ and $x(t)$ will stay on M^* for $t \in \mathbb{R}$. However, $\dim E(x)T_x M^c_{k^*}$ equals to $\dim x = 1$ for $x \neq 0$ and is 0 for $x = 0$, which is not constant in $M^* = X$. Nevertheless, the assumptions of (A2) exclude singular points (in our example the singular point is $x_0 = 0$), at which the DAE may exhibit completely different behavior. Since the aim of this paper is not to classify singular points of DAEs, we will keep (A2) for Theorem 3.

The definition of geometric index is given as follows. The intuition behind the definition of geometric index was described in Rabier and Rheinboldt (2002) as "the index of the problem, defined as the number of steps needed for the procedure to stabilize is equally independent of the solutions of the DAE, as in the linear constant-coefficient case and even independent of their existence."

Definition 5. (Geometric index). The geometric index of $\nu_g \in \mathbb{N}$ of a DAE Ξ is defined by
$$\nu_d := \min \{k \geq 0 \mid (M_k = M_{k+1}) \wedge (M_k \neq \emptyset)\}.$$

Now combining the results of Theorem 3, we give some comments on the geometric index defined above.

Remark 6. (i) The definition of the geometric index needs only the assumption that for each k, M_k is locally a smooth connected submanifold around an admissible point $x_p = x_a$. The constant rank assumptions of (A2) are not necessarily required, e.g., for the DAE given by (7), we have $M_1 = M_0 = X$ and thus $\nu_g = 0$, but $\dim E(x)T_x M_0$ is not of constant rank in $M_0 = X$ as shown in item (iii) of Remark 4.

(ii) The definition of the geometric index is indeed independent of the existence of solutions. Namely, for a given initial point x_0, we do not need the existence of solution at x_0 to define the geometric index since even if Ξ is not solvable at x_0, it is possible that we can still find an admissible point $x_p = x_a$ near to x_0 such that we can construct M_k and then define the geometric index around x_p. However, the geometric index allows for a conclusion about the existence of solutions: suppose that Ξ has a well-defined geometric index ν_g around an admissible point $x_p = x_a$, if $x_0 \in U_{\nu_g}/M_{\nu_g}$ around x_p, we can conclude that there does not exist a solution $x(t)$ such that $x(0) = x_0$.

(iii) The geometric index does not concern the uniqueness of the solutions. As an example, consider two DAEs
$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_2, \end{cases} \quad \text{and} \quad \tilde{\Xi}_{2,3} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_3, \end{cases}$$
where $f : X_1 \times X_2 \to \mathbb{R}$ is smooth. Observe that although for both Ξ and $\tilde{\Xi}$, $\nu_g = 1$, but for any admissible initial point, Ξ has a unique solution isomorphic to the solution of the ODE $\dot{x}_1 = f(x_1, 0)$, $\tilde{\Xi}$ has infinite solutions since x_2 is a free variable of the ODE $\dot{x}_1 = f(x_1, x_2)$.

(iv) For a linear DAE $\Delta_{l,n} = (E, A)$, if Δ is regular, i.e., $l = n$ and $|sE - A| \not\equiv 0$, then $\exists$ invertible $Q \in \mathbb{R}^{l \times l}$ and $P \in \mathbb{R}^{n \times n}$ transforming Δ into the Weierstraß form (see Gantmacher (1959)):
$$(QEP^{-1}, QAP^{-1}) = \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right),$$
where $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent matrix, $J \in \mathbb{R}^{n_1 \times n_1}$ and $n_1 + n_2 = n$. The DAE index ν_{lin} of Δ is often defined by the nilpotency of N (see e.g. Berger and Reis (2015)),
$$\nu_{lin} := \begin{cases} 0, & \text{if } n_1 = n, \\ \min \{ k \in \mathbb{N} \mid N^k = 0 \}, & \text{if } n_1 < n. \end{cases}$$
Note that ν_{lin} coincides with the smallest k such that $V_k = V_{k+1}$ of the Wong sequences (see Remark 2). As a result, the geometric index ν_g of nonlinear DAEs is a nonlinear generalization of the index ν_{lin} of linear DAEs.

Example 7. Consider the following DAE $\Xi_{5,5} = (E, F)$, given by (8), borrowed from Rabier and Rheinboldt (1994), the DAE describes the mathematical model of a pendulum with a mass attached to its end,
$$\Xi : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ x_4 \\ -x_5 x_3 - g \\ x_1^2 + x_3^2 - l^2 \end{bmatrix}. \quad (8)$$

We consider the point $x_0 = (x_{10}, x_{20}, x_{30}, x_{40}, x_{50})$, where $x_{10} = 0$, $x_{20} = 0$, $x_{30} = -l$, $x_{40} = 0$, $x_{50} = g/l$ and apply Algorithm 1 to $\Xi_1 = (E_1, F_1) = \Xi$.

Step 1: since E_1 is already of the desired form, we set $Q_1 = I_5$. It follows that
$$M_1 = \left\{ x \in X \mid x_1^2 + x_3^2 - l^2 = 0 \right\}.$$

Now locally around x_0, we change x_3 to $\bar{x}_3 = x_1^2 + x_3^2 - l^2$, then the local diffeomorphism $\psi_1(x) = (x_1, x_2, x_4, x_5, \bar{x}_3)$ defines new local coordinates on X. The DAE Ξ_1 in the new coordinates is

$$\tilde{\Xi}_1 : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2x_1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ 2x_3 x_4 \\ -x_5 x_3 - g \\ \bar{x}_3 \end{bmatrix},$$

where $x_3 = -(l^2 - \bar{x}_3^2 - x_1^2)^{1/2}$. Set $\bar{x}_3 = 0$, we get

$$\tilde{\Xi}_1|_{M_1} : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ 2x_3 x_4 \\ -x_5 x_3 - g \end{bmatrix},$$

where $x_3 = -(l^2 - x_1^2)^{1/2}$.

Step 2: consider $\Xi_2 = (E_2, F_2) = \tilde{\Xi}_1|_{M_1}$. It is possible to find Q_2 such that

$$Q_2 E_2(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q_2 F_2(x) = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ -x_5 x_3 - g \\ x_3 x_4 + x_1 x_2 \end{bmatrix}, \tag{9}$$

where $x_3 = -(l^2 - x_1^2)^{1/2}$. It follows that

$$M_2 = \{x \in M_1 \mid x_3 x_4 + x_1 x_2 = 0\}.$$

Then define new coordinates on M_1 via the local diffeomorphism $\psi_2(x) = (x_1, x_2, x_5, \bar{x}_4)$, where $\bar{x}_4 = -(l^2 - x_1^2)^{1/2} x_4 + x_1 x_2$. In the new coordinates, we have

$$\tilde{\Xi}_2 : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a(x) & -x_1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_5 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ -x_5 x_3^2 - g x_3 \\ \bar{x}_4 \end{bmatrix},$$

where $a(x) = x_1^2 x_2 (l^2 - x_1^2)^{-1} + x_2$. Set $\bar{x}_4 = 0$ to have

$$\tilde{\Xi}_2|_{M_2} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a(x) & -x_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \\ -x_5 x_3^2 - g x_3 \end{bmatrix}.$$

Step 3: consider $\Xi_3 = (E_3, F_3) = \tilde{\Xi}_2|_{M_2}$, via a similar procedure as Step 1 and 3, we can get

$$\begin{aligned} M_3 &= \left\{x \in M_2 \mid x_1^2 x_2^2 (l^2 - x_1^2)^{-1} + x_2^2 - x_5 l^2 - g x_3 = 0\right\} \\ &= \left\{x \in M_2 \mid x_4^2 + x_2^2 - x_5 l^2 - g x_3 = 0\right\}. \end{aligned}$$

Set $\bar{x}_5 = x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x^2)^{1/2} - l^2 x_5$ and define the new local coordinates $(x_1, x_2, \bar{x}_5)$ on M_2, then we denote Ξ_3 in the new coordinates by $\tilde{\Xi}_3$. It follows that

$$\tilde{\Xi}_3|_{M_3} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \end{bmatrix}, \tag{10}$$

where $x_5 = \frac{1}{l^2}(x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x_1^2)^{1/2})$.

Step 4: Since $\Xi_4 = \tilde{\Xi}_3|_{M_3}$ is clearly an ODE, it is seen that $k^* = 3$ and $M^* = M_4 = M_3$.

Notice that the assumptions of (A1) and (A2) are satisfied around x_0 and $x_0 \in M^*$, we can conclude by Theorem 3 that the solution of Ξ passing through x_0 exists and is unique (since $\dim M^* = \operatorname{rank} E_{k^*} = 2$), and this unique solution is diffeomorphic to the solution of the ODE (10) with the constraints $\bar{x}_3 = \bar{x}_4 = \bar{x}_5 = 0$. Moreover, the geometric index $\nu_g = k^* = 3$. Note that our sequence of submanifold M_k coincides with that of the projections W_k of the tangent bundles TW_k in Rabier and Rheinboldt (1994). But the two methods of constructing sequences of submanifolds are different in many ways, although it is interesting to compare them, it is not our aim to discuss their differences in details in the present paper.

3. GEOMETRIC INTERPRETATION OF THE DIFFERENTIATION INDEX

The notion of differentiation index is originally proposed for DAEs of the general form (2)(see Campbell and Gear (1995)): define the differential array of (2) by

$$H_k(t, x, x', w) = \begin{bmatrix} H \\ D_t H + D_x H x' + D_{x'} H x'' \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k} H \end{bmatrix} (t, x, x', w) = 0, \tag{11}$$

where $w = \left[x^{(2)}, \ldots, x^{(k+1)}\right]$, the differentiation index ν_d is the least integer k such that equation (11) uniquely determines x' as a function of (x, t), i.e., $x' = v(x, t)$. We now illustrate two deficiencies or ambiguities of the above classical definition of differentiation index.

Remark 8. (i) In contrast to the geometric index of Definition 5, the above classical definition of differentiation index does not allow for a conclusion of the existence of solutions. For example, consider two DAEs

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = F(x_1, x_2) \end{cases} \quad \text{and} \quad \tilde{\Xi}_{3,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = F(x_1, x_2) \\ 0 = x_1 \end{cases},$$

where $F(0,0) \neq 0$. Both of $\Xi_{1,1}$ and $\tilde{\Xi}_{2,1}$ already determine $(x_1', x_2') = (x_2, F(x_1, x_2))$, thus the two DAEs have the same differentiation index $\nu_d = 0$ by the above definition. However, Ξ has an unique solution and $\tilde{\Xi}$ has no solutions for any initial point $(x_{10}, x_{20}) \in X \subseteq \mathbb{R}^2$. The reason is that the above definition of differentiation index only implies the existence and uniqueness of some vector field determined by the differential array but does not indicate where the solutions of the DAE should exist (which is indicated by the sequence of submanifolds M_k as shown in Section 2). In particular, if the defined vector field $v(x,t)$ is not tangent to M_k at $x_0 \in M_k$ (as in our example, $(x_2, F(x_1, x_2))$ is not tangent to $M_2 = \{x \mid x_1 = x_2 = 0\}$ at $(x_{10}, x_{20}) = (0,0)$), then the trajectory of $\dot{x} = v(x,t)$ will leave M_k immediately, indicating that the DAE does not have a solution at x_0.

(ii) The above definition of differentiation index does not distinguish the difference of an ODE and an "over-determined" DAE. Consider the DAEs $\Xi_{1,1}$ and $\tilde{\Xi}_{2,1}$ of item (i) but suppose $F(0,0) = 0$. Then both Ξ and $\tilde{\Xi}$ have the same solution $x_1(t) = 0, x_2(t) = 0$ for $(x_{10}, x_{20}) = (0,0)$, and also the same differentiation index $\nu_d = 0$. Nevertheless, Ξ is clearly an ODE which does not need any differentiation to be solved, $\tilde{\Xi}$ needs two times of differentiations in order to deduce a solution. Thus this example also shows that the differentiation index defined above loses its original intention, i.e., as a measure of difficulties of solving a DAE.

To give a geometric background to the notion of differentiation index, a new definition was proposed in Griepentrog (1991). In the present paper, in order to clear out the deficiencies mentioned in Remark 8, we reform Griepentrog's definition of differentiation index as follows. Consider a nonlinear DAE $\Xi_{l,n} = (E,F)$, let $H(x,\zeta_1) = E(x)\zeta_1 - F(x)$, denote $(\frac{\mathrm{d}^k}{\mathrm{d}t^k}H) = H^{(k)}$ and define

$$H_k(x,\bar{\zeta}_{k+1}) = \begin{bmatrix} H^{(0)}(x,\zeta_1) \\ H^{(1)}(x,\zeta_1,\zeta_2) \\ \vdots \\ H^{(k)}(x,\bar{\zeta}_{k+1}) \end{bmatrix} = 0, \tag{12}$$

where $\bar{\zeta}_{k+1} = (\zeta_1,\ldots,\zeta_{k+1})$. Set $\mathcal{M}_0 = X$, $\mathcal{Z}_1^0 = \mathbb{R}^n$ and for $k > 1$, define

$$\begin{aligned} \mathcal{M}_k &:= \left\{x \in X \mid H_{k-1}(x,\bar{\zeta}_k) = 0\right\}, \\ \mathcal{Z}_1^k &:= \left\{\zeta_1 \in \mathbb{R}^n \mid H_{k-1}(x,\bar{\zeta}_k) = 0, x \in \mathcal{M}_k\right\}, \end{aligned} \tag{13}$$

and assume that for each $k > 0$, $\mathcal{M}_k$ is a smooth connected submanifold.

Definition 9. (Differentiation index). Consider a DAE $\Xi_{l,n} = (E,F)$, the differentiation index ν_d of Ξ is defined by $\nu_d :=$

$$\begin{cases} 0, & \text{if } (l=n) \wedge (E : X \to Gl(l,\mathbb{R})), \\ \min\left\{ k > 0 \,\middle|\, \begin{array}{l} \mathcal{M}_k \neq \emptyset \wedge (\mathcal{Z}_1^k = \mathcal{Z}_1^k(x) \text{ is a singleton}) \\ \wedge (\mathcal{Z}_1^k(x) \in T_x\mathcal{M}_k, \forall x \in M_k) \end{array} \right\}, & \text{otherwise.} \end{cases}$$

Now we state our main theorem of this subsection.

Theorem 10. For a DAE $\Xi_{l,n} = (E,F)$, consider the sequence of submanifolds $\mathcal{M}_k$, given by (13). Fix a point x_p and assume locally around x_p that for each $k > 0$,

(A1)' $\mathcal{M}_k$ is a smooth embedded submanifold and $x_p \in \mathcal{M}_k$;

(A2)' $\dim E(x)T_x\mathcal{M}_{k-1}^c$ is constant, where $\mathcal{M}_{k-1}^c$ is the connected component of $\mathcal{M}_{k-1}$ satisfying $x_p \in \mathcal{M}_{k-1}^c$.

Then we have locally around x_p that for each $k \geq 0$, M_k of the maximal invariant submanifold algorithm in Section 2 is a smooth and connected embedded submanifold and

$$\mathcal{M}_k = M_k.$$

Thus there exists a smallest k, denoted by $k^* \leq n$ such that $\mathcal{M}_{k^*+1} = \mathcal{M}_{k^*}^c$ and the geometric index $\nu_g = k^*$. Moreover, the differentiation index ν_d of Ξ exists and satisfies $\nu_d = \nu_g$ if and only if $\dim \mathcal{M}_{k^*}^c = \dim E(x)T_x\mathcal{M}_{k^*}^c$.

The proof is given in Section 4.

Remark 11. (i) The assumptions of (A1)' and (A2)' in Theorem 10 correspond to that of (A1) and (A2) in Theorem 3, respectively. If, a priory, we assume that for each $k \geq 0$, $M_k = \mathcal{M}_k$, then it is not hard to see that (A1) coincides with (A1)' since $\operatorname{rank} E_k(x) = \dim E(x)T_x\mathcal{M}_{k-1}^c = \dim E(x)T_xM_{k-1}^c$ locally for all $x \in \mathcal{M}_{k-1}^c = M_{k-1}^c$, and (A2) is the same as (A2)' since $\mathcal{M}_k = M_k$ is a smooth embedded submanifold if and only if DF_k^2 is of constant rank locally on M_k (see Lee (2001)).

(ii) The geometric background of Definition 9 enables the differentiation index ν_d to conclude the existence of solutions since the sequence of submanifolds $\mathcal{M}_k$ coincides with M_k and thus on the limits $\mathcal{M}_{k^*}^c = M_{k^*}^c = M^*$ of $\mathcal{M}_k = M_k$, we can conclude where the solutions of the DAE exist.

(iii) The differentiation index and the geometric index differ from each other by their different relations with the uniqueness of solutions. As in Theorem 10, there exists at least one solution for any point $x_0 \in \mathcal{M}_{k^*}$ around x_p and the geometric index always exist, but the differentiation index exists if and only if the solution is unique.

Example 12. (continuation of Example 7) Consider the DAE $\Xi = (E,f)$ of equation (8), since the first 4 rows of E are already linear independent, the differentiations of these equations will not create constrains for x, we only need to differentiate the last equation $x_1^2 + x_3^2 - l^2 = 0$. It is not hard to verify that for $k = 1,2,3$, the assumptions (A1)' and (A2)' are satisfied around x_0, and $\mathcal{M}_1 = \left\{x \in X \mid x_1^2 + x_3^2 - l^2 = 0\right\} = M_1$,

$$\begin{aligned} \mathcal{M}_2 &= \{x \in \mathcal{M}_1 \mid 2x_1\dot{x}_1 + 2x_3\dot{x}_3 = 0\} \\ &= \{x \in \mathcal{M}_1 \mid x_1x_2 + x_3x_4 = 0\} = M_2, \end{aligned}$$

$$\begin{aligned} \mathcal{M}_3 &= \{x \in \mathcal{M}_2 \mid x_1\dot{x}_2 + x_2\dot{x}_3 + x_3\dot{x}_4 + x_4\dot{x}_3 = 0\} \\ &= \left\{x \in \mathcal{M}_2 \mid -x_5x_1^2 + x_2^2 - x_5x_3^2 - x_3g + x_4^2 = 0\right\} \\ &= \left\{x \in \mathcal{M}_2 \mid x_4^2 + x_2^2 - x_5l^2 - gx_3 = 0\right\} = M_3. \end{aligned}$$

Moreover, by differentiating $x_4^2 + x_2^2 - x_5l^2 - gx_3 = 0$, we get that

$$\dot{x}_5 = \frac{1}{l^2}(-2x_4(x_5x_3 + g)2x_2(x_5x_1) - gx_4).$$

Combining the above equation with the first 4-equations of (8), we have

$$\mathcal{Z}_1^3 = \{\zeta_1 \mid \zeta_1 = (x_2, -x_5x_1, x_4, -x_5x_3 - g, \dot{x}_5),\ x \in \mathcal{M}_3\}.$$

We can conclude that $\mathcal{Z}_1^3 = \mathcal{Z}_1^3(x)$ is a singleton and $\mathcal{Z}_1^3(x) \in T_x\mathcal{M}_3, \forall x \in M_3$. Therefore the differentiation index $\nu_d = 3$, which coincides with the geometric index ν_g.

4. PROOFS OF THEOREM 3 AND 10

Proof. (of Theorem 3). The proof of M^* is a locally maximal invariant submanifold can be consulted in the proof of Proposition 3.6 of Chen and Respondek (2019b).

(i) Item (i) can be proved by the geometric constructions of M_k in (4) and (5). First, by Definition 1, for any point $x_0 \in M^*$, there exists at least one solution, thus x_0 is admissible i.e., $x_0 \in S_{adm}$. So we have $M^* \subseteq S_{adm}$. Conversely, we prove that locally $S_{adm} \subseteq M^*$ by induction. Consider any point $x_0 \in S_{adm} \cap U^*$, it follows that there exists a solution $x : I \to X$ of Ξ such that $x(0) = x_0$. Thus by $E(x_0)\dot{x}(0) = F(x_0)$, we have that $F(x_0) \in \operatorname{Im} E(x_0)$. The later implies that $x_0 \in M_1$ by equation (4). Then suppose that for a certain $k > 0$, $x_0 \in M_{k-1} \cap U^*$, we get $\dot{x}(0) \in T_{x_0}M_k^c$. Now the equation $E(x_0)\dot{x}(t_0) = F(x_0)$ indicates that $F(x_0) \in E(x_0)T_{x_0}M_k^c$, which means that $x_0 \in M_k$ by equation 5. So for every k, we have $x_0 \in M_k$. As a consequence, we have $x_0 \in M_{k^*} \cap U^* = M_{k^*}^c = M^*$ implying that $S_{adm} \subseteq M^*$ locally on U^*. Now it is clear to see that we have locally $S_{adm} = M^*$.

(ii) Notice that in every k step of Algorithm 1, since Q_k is invertible, the left-multiplication of Q_k preservers the solutions of $E_k(\bar{z}_{k-1})\dot{\bar{z}}_{k-1} = F_k(\bar{z}_{k-1})$ to that of

$$\begin{bmatrix} E_k^1(\bar{z}_{k-1}) \\ 0 \end{bmatrix} \dot{\bar{z}}_{k-1} = \begin{bmatrix} F_k^1(\bar{z}_{k-1}) \\ F_k^2(\bar{z}_{k-1}) \end{bmatrix} \tag{14}$$

and the local diffeomorphism ψ_k maps the solutions of (14) to that of

$$\begin{bmatrix} \tilde{E}_k^1(\bar{z}_k, z_k) & \tilde{E}_k^2(\cdot) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{z}}_k \\ \dot{z}_k \end{bmatrix} = \begin{bmatrix} \tilde{F}_k^1(\bar{z}_k, z_k) \\ F_k^2(\bar{z}_k, z_k) \end{bmatrix}. \tag{15}$$

The solutions of (15) have to be on the the submanifold $M_k^c = \{\bar z_{k-1} \in W_k \mid z_k = 0\}$, thus $(\bar z_k(t), 0)$ is a solution of (15), where $\bar z_k(t)$ is a solution of $\tilde E_k^1(\bar z_k, 0)\dot{\bar z}_k = \tilde F_k^1(\bar z_k, 0)$. Now we can conclude that $(\bar z_k(t), 0)$ is a solution of

$$\begin{cases} \tilde E_k^1(\bar z_k)\dot{\bar z}_k = \tilde F_k^1(\bar z_k) \\ \quad z_k \quad = 0, \end{cases}$$

if and only if $\bar z_{k-1}(t) = \psi_k^{-1}(\bar z_k(t), 0)$ is a solution of $E_k(\bar z_{k-1})\dot{\bar z}_k = F_k(z_{k-1})$. Then by a recursive argument, we can prove that $(\bar z_{k^*}(t), 0, \ldots, 0)$ is a solution of (6) if and only if $\psi_1^{-1}(\ldots \psi_{k^*-1}^{-1}(\psi_{k^*}^{-1}(\bar z_{k^*}(t), 0), 0) \ldots, 0) = \Phi(\bar z_{k^*}(t), 0, \ldots, 0)$ is a solution of Ξ. It is not hard to see that the Jacobian matrix $\Phi_{\bar z_{k^*}, z_k, \ldots, z_1}$ at $x_a = x_p$ is invertible, which proves that Φ is a local diffeomorphism. Therefore there exist a diffeomorphism between a solution of Ξ and a solution of (6). Next we prove that the row rank of $\tilde E_{k^*}^1$ is locally constant and full. Clearly, the constancy of the rank follows from assumption (A2), we only need to prove that $\tilde E_{k^*}^1$ is of full row rank. Suppose that it is not of full row rank, then we can find Q_{k^*+1} such that $E_{k^*}^1$ of $Q_{k^*+1}\tilde E_{k^*}^1 = \begin{bmatrix} E_{k^*+1}^1 \\ 0 \end{bmatrix}$ is of full row rank, denote $Q_{k^*+1}\tilde F_{k^*}^1 = \begin{bmatrix} F_{k^*+1}^1 \\ F_{k^*+1}^2 \end{bmatrix}$, locally we have

$$M_{k^*+1} = \left\{ \bar z_{k^*} \in M_{k^*} \mid F_{k^*+1}^2(\bar z_{k^*}) = 0 \right\}, \qquad (16)$$

where $\bar z_{k^*}$ are local coordinates on M_{k^*}. By $M_{k^*+1} = M_{k^*}$, equation (16) implies that there are some dependent equations in $\tilde E_{k^*}^1(\bar z_{k^*})\dot{\bar z}_{k^*} = \tilde F_{k^*}^1(\bar z_{k^*})$, this contradicts our assumptions in Section 1. Thus $\tilde E_{k^*}^1$ is locally of full row rank.

(iii) We will consider the solutions of (6) rather than that of Ξ since by item (ii), the two DAEs have diffeomorphic solutions. Rewrite $\tilde E_{k^*}^1(\bar z_{k^*})\dot{\bar z}_{k^*} = \tilde F_{k^*}^1(\bar z_{k^*})$ as

$$\left[\bar E_{k^*}^1(z_{k^*}) \ \bar E_{k^*}^2(z_{k^*})\right] \begin{bmatrix} \dot{\bar z}_{k^*}^1 \\ \dot{\bar z}_{k^*}^2 \end{bmatrix} = \tilde F_{k^*}^1(\bar z_{k^*}), \qquad (17)$$

where $\tilde E_{k^*}^1 = \left[\bar E_{k^*}^1 \ \bar E_{k^*}^2\right]$, $\bar E_{k^*}^1 : M^* \to \mathbb{R}^{r_{k^*} \times r_{k^*}}$ and $\bar z_{k^*} = (\bar z_{k^*}^1, \bar z_{k^*}^2)$. Without loss of generality, we can always assume $\bar E_{k^*}^1$ is locally invertible on M^*, since if not, we can permute the columns of $\tilde E_{k^*}^1$ such that the first r^* columns are independent. We can see that the solutions of (17) coincide with that of the ODE

$$\begin{cases} \dot z_{k^*}^1 = f(z_{k^*}) + g(z_{k^*})\, u \\ \dot z_{k^*}^2 = u, \end{cases} \qquad (18)$$

where $f = (\bar E_{k^*}^1)^{-1}\tilde F_{k^*}^1$ and $g = (\bar E_{k^*}^1)^{-1}\bar E_{k^*}^2$. Equation (18) can be seen as a control system with the inputs u. For any initial point $z_{k^*}^0 \in M^*$, if u is present, then equation (18) has infinite numbers of solutions (since u is a free variable). Thus equation (18) has a unique solution if and only if $\dot z_{k^*}^2$ is absent. The later is equivalent to that $\tilde E_{k^*}^1$ is invertible or $s_{k^*} = r_{k^*}$ (recall that $\tilde E_{k^*}^1$ is of full row rank). Therefore, equation (6) and thus Ξ have a unique solution passing through $x_0 \in M^*$ if and only if $\dim M^* = \dim E(x)T_xM^*$. □

Definition 13. (Restriction) Consider $L : X \times \mathbb{R}^n \to \mathbb{R}^l$ and fix a point $x_p \in X$. Assume that M is a smooth embedded submanifold of X containing x_p. Let (x_1, x_2) be local coordinates on X such that locally

$$M = \{(x_1, x_2) \mid x_2 = 0\}.$$

The local restriction of L to M is

$$L|_M = L(x_1, x_2, \zeta_1)|_{x_2=0} = L(x_1, 0, \zeta_1).$$

Remark 14. For a nonlinear map L, the restriction to a smooth embedded submanifold M and the differentiation with respect to t are two commutative operations, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}(L|_M) = (\frac{\mathrm{d}}{\mathrm{d}t}L)|_M$. Indeed, rewrite L as $L(x_1, x_2, \zeta_1) = L(x_1, 0, \xi_1) + x_2\bar L(x_1, x_2, \zeta_1)$ for some $\bar L$, then we have

$$(\frac{\mathrm{d}}{\mathrm{d}t}L)|_M = (\frac{\mathrm{d}}{\mathrm{d}t}(L|_M) + \dot x_2 \bar L + x_2\frac{\mathrm{d}}{\mathrm{d}t}\bar L)|_{x_2=0} = \frac{\mathrm{d}}{\mathrm{d}t}(L|_M).$$

Proof. (of Theorem 10) First, we construct $\mathcal{M}_k$ explicitly via the following procedure: Set $\mathcal{M}_0 = X$. Assume that for certain $k > 0$, $\mathcal{M}_{k-1} \subsetneq \cdots \subsetneq \mathcal{M}_1 \subsetneq \mathcal{M}_0 = X$ have been constructed. Step k: if $k = 1$, set $E_1 = E$ and $F_1 = F$; if $k > 1$, set

$$E_k = \begin{bmatrix} E_{k-1}^1 \\ D_x\tilde F_{k-1}^2 \end{bmatrix}|_{\mathcal{M}_{k-1}} \quad \text{and} \quad F_k = \begin{bmatrix} F_{k-1}^1 \\ 0 \end{bmatrix}|_{\mathcal{M}_{k-1}},$$

where $E_k : \mathcal{M}_{k-1} \to \mathbb{R}^{l\times n}$ and $F_k : \mathcal{M}_{k-1} \to \mathbb{R}^l$. Assume that $\operatorname{rank} E_k = const. = r_k \le n$ in $\mathcal{M}_{k-1}$. After a possible permutation, we may assume that the first r_k rows of E_k are linearly independent. Then we can split the left equation of below to the right:

$$E_kv + F_k = 0 \Leftrightarrow \begin{bmatrix} E_k^1 \\ E_k^2 \end{bmatrix} v + \begin{bmatrix} F_k^1 \\ F_k^2 \end{bmatrix} = 0,$$

where $E_k^1 : \mathcal{M}_{k-1} \to \mathbb{R}^{r_k\times n}$ is of full row rank in $\mathcal{M}_{k-1}$ and $F_k^1 : \mathcal{M}_{k-1} \to \mathbb{R}^{r_k}$. Then, there exists $a_k : \mathcal{M}_{k-1} \to \mathbb{R}^{(l-r_k)\times r_k}$ such that $a_kE_k^1 = E_k^2$. Denote

$$v_k = E_k^1v + F_k^1,$$

we have $E_k^2v + F_k^2 = a_k(E_k^1v + F_k^1) - a_kF_k^1 + F_k^2 = a_kv_k + \tilde F_k^2$, where $\tilde F_k^2 = F_k^2 - a_1F_k^1$. Set

$$\mathcal{M}_k = \left\{ x \in \mathcal{M}_{k-1} \mid \tilde F_k^2(x) = 0 \right\}. \qquad (19)$$

We now show that under assumption (A1)' and (A2)', locally around x_p, for each $k > 0$, the submanifold $\mathcal{M}_k$, given by (19), is indeed the one defined by (13). Observe that for $x \in \mathcal{M}_{k-1}$,

$$\ker E_k(x) = (\ker E(x)) \cap T_x\mathcal{M}_{k-1}.$$

Thus the assumption of (A1)' implies the (locally) constant rankness of E_k. Then consider the differentiation array H_k of equation (12), denote $\xi_1 = v$ and use the same notations v_k as in the above procedure to have

$$H^{(0)}(x, v) = E(x)v + F(x) = \begin{bmatrix} E_1^1(x) \\ E_1^2(x) \end{bmatrix} v + \begin{bmatrix} F_1^1(x) \\ F_1^2(x) \end{bmatrix} = \begin{bmatrix} v_1 \\ \tilde a_1(x, v_1) + \tilde F_1^2(x) \end{bmatrix},$$

where $\tilde a_1(x, v_1) = a_1(x)v_1$. Now by the assumption of (A2)', $\mathcal{M}_k$ is a locally smooth embedded submanifold for each $k > 0$. It follows locally that for $x \in M_1$,

$$H^{(1)}|_{\mathcal{M}_1} = (H|_{\mathcal{M}_1})^{(1)} = \begin{bmatrix} v_1^{(1)} \\ \tilde a_1^{(1)} + \begin{bmatrix} \bar v_1 \\ \tilde a_2 + \tilde F_2^2 \end{bmatrix} \end{bmatrix},$$

where $\bar a_2(x, v_2) = a_2(x)v_2$ and $v_2 = (v_1, \bar v_1)$. In general, locally for $x \in \mathcal{M}_k$, we have

$$H^k|_{\mathcal{M}_k} = \begin{bmatrix} v_1^{(k)} \\ \tilde{a}_1^{(k)} + (\tilde{F}_1^2)^{(k)} \end{bmatrix},$$

where $(\tilde{F}_1^2)^{(k)}$ is given by the following iterative formula:

$$(\tilde{F}_i^2)^{(k)} = \begin{bmatrix} \bar{v}_i^{(k-1)} \\ \tilde{a}_{i+1}^{(k-1)} + (\tilde{F}_{i+1}^2)^{(k-1)} \end{bmatrix},$$

where $\bar{a}_i = \bar{a}_i(x, v_i) = a_i(x)v_i$ and $v_i = (v_{i-1}, \bar{v}_{i-1})$ for $i > 1$. Notice that for each k, since E_k^1 is of full row rank, the differentiation $v_1^{(k)}$ of v_1 and $\bar{v}_i^{(k-1)}$ of $\bar{v}_i$ for $i \geq 1$ do not create any constraint for x. Now it is clear to see that for $k > 0$,

$$\begin{aligned} \mathcal{M}_k &= \left\{x \in X \mid H_{k-1}(x, \bar{\zeta}_k) = 0\right\} \\ &= \left\{x \in \mathcal{M}_{k-1} \mid H^{(k-1)}(x, \bar{\zeta}_k) = 0,\ \zeta_1 \in \mathcal{Z}_1^k\right\} \\ &= \left\{x \in \mathcal{M}_{k-1} \mid \tilde{F}_k^2(x) = 0\right\}, \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_1^k &= \left\{\zeta_1 \mid H_{k-1}(x, \bar{\zeta}_k) = 0\right\} \\ &= \left\{\zeta_1 \mid E_k(x)\zeta_1 + F_k(x) = 0, x \in \mathcal{M}_k\right\} \\ &= \left\{\zeta_1 \mid v_k(\zeta_1, x) = 0, x \in \mathcal{M}_k\right\}. \end{aligned} \tag{20}$$

Next we show that for each $k > 0$, $\mathcal{M}_k$ locally coincides with M_k of the maximal invariant submanifold algorithm. It is obvious to see that $M_1 = \{x \in X : F(x) \in \operatorname{Im} E(x)\} = \{x \in X : F_1(x) \in \operatorname{Im} E_1(x)\} = \left\{x \in X \mid \tilde{F}_1^2(x) = 0\right\} = \mathcal{M}_1$. For $k > 1$, suppose that $M_{k-1} = \mathcal{M}_{k-1}$, then we have

$$\begin{aligned} \mathcal{M}_k &= \{x \in \mathcal{M}_{k-1} \mid E_k(x)v + F_k(x) = 0\} \\ &= \left\{x \in M_{k-1} \mid \begin{bmatrix} E_{k-1}^1(x) \\ D_x\tilde{F}_{k-1}^2(x) \end{bmatrix} v + \begin{bmatrix} F_{k-1}^1(x) \\ 0 \end{bmatrix} = 0\right\} \\ &= \left\{x \in M_{k-1} \mid E_{k-1}^1(x)v + F_{k-1}^1(x) = 0,\ v \in T_xM_{k-1}\right\} \\ &= \left\{x \in M_{k-1} \mid F_{k-1}^1(x) \in E_{k-1}^1(x)T_xM_{k-1}\right\} \\ &= \{x \in M_{k-1} \mid F(x) \in E(x)T_xM_{k-1}\} = M_k. \end{aligned}$$

Therefore, by Definition 5, the geometric index $\nu_g = k^*$, i.e., the smallest k such that locally $M_{k^*+1} = \mathcal{M}_{k^*+1} = \mathcal{M}_{k^*} = M_{k^*}$. Moreover, $\mathcal{Z}_1^k(x) \in \ker D_xF_{k-1}^1(x) \in T_x\mathcal{M}_{k-1}$ by (20), which implies that $\mathcal{Z}_1^{k^*}(x) = \mathcal{Z}_1^{k^*+1}(x) \in T_x\mathcal{M}_{k^*}$. Also by (20),

$$\begin{aligned} \mathcal{Z}_1^{k^*} &= \left\{\zeta_1 \in \mathbb{R}^n \mid v_{k^*} = E_{k^*}^1(x)\zeta_1 + F_{k^*}^1(x) = 0\right\} \\ &= \left\{\zeta_1 \in \mathbb{R}^n \mid \zeta_1 = -(E_{k^*}^1)^\dagger F_{k^*}^1(x)\right\}. \end{aligned}$$

where $(E_{k^*}^1)^\dagger$ is a right inverse of the full row rank matrix $E_{k^*}^1$. Thus $\mathcal{Z}_1^{k^*} = \mathcal{Z}_1^{k^*}(x)$ is a singleton if and only if $E_{k^*}^1$ is invertible, i.e., $\operatorname{rank} E_{k^*} = r_{k^*} = n$ or equivalently, $\dim\ker E_{k^*}(x) = 0$. Since $x_p \in \mathcal{M}_{k^*}$ already indicates $\mathcal{M}_{k^*} \neq \emptyset$, we can now conclude by Definition 9 that $\nu_d = k^* = \nu_g$ if and only if $\dim\ker E_{k^*}(x) = \dim(\ker E(x) \cap T_x\mathcal{M}_{k^*}) = 0$, i.e., $\dim \mathcal{M}_{k^*} = \dim E(x)T_x\mathcal{M}_{k^*}$. □

5. CONCLUSIONS

In this paper, we first discuss the maximal invariant submanifold algorithm and give the conditions of the existence and uniqueness of the solutions for nonlinear differential-algebraic equations. Then we comment the geometric index based on the results of the solutions. We also modify the classical definition of the differentiation index and show the relations and differences with the two indices. We show that under some constant rankness and smoothness assumptions, the sequence of submanifolds defined by the differentiation array coincides with that of the maximal invariant submanifold algorithm and the two indices coincide with each other if and only if the DAE has an unique solution.

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