The one-step-map for switched singular systems in discrete-time

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System class and motivation

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

- $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathtt{n}\}$ switching signal
- $E_1, E_2, \ldots, E_{\mathtt{n}}, A_1, A_2, \ldots, A_{\mathtt{n}} \in \mathbb{R}^{n\times n}$ with E-matrix singular
- $x : \mathbb{N} \to \mathbb{R}^n$ state

Motivation

- Leontief economic model (*Luenberger 1977*)
- discretization of continuous-time time-varying DAEs
- sampled feedback loop for descriptor systems

Simple question

What can we say about existence and uniqueness of solutions?

Small excursion to continuous time

$$E_\sigma \dot{x} = A_\sigma x$$

Theorem (Existence and uniqueness in continuous time, *Trenn 2012*)

Assume (E_i, A_i) are regular, i.e. $\det(sE_i - A_i)$ is not the zero polynomial. Then for any past trajectory $x^0(\cdot)$ and any $t_0 \in \mathbb{R}$ there exists unique $x(\cdot)$ such that

$$x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$$
$$(E_\sigma \dot{x})_{[t_0,\infty)} = (A_\sigma x)_{[t_0,\infty)}$$

In particular, solution behavior is causal w.r.t. to the switching signal.

Distributional solution framework necessary

Above solution result only holds when solution space is enlarged to allow for jumps and Dirac impulses.

A simple example

$$E_{\sigma(k)} x(k+1) = A_{\sigma(k)} x(k) \qquad \text{(SSS)}$$

Example

Consider (SSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A_1 = I \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ A_2 = I$$

Nonswitched solution behavior

$$\sigma \equiv 1: \quad \left.\begin{aligned} x_1(k+1) &= x_1(k) \\ 0 &= x_2(k) \end{aligned}\right\} \rightsquigarrow \quad x(k) = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \ \forall k \in \mathbb{N}$$

$$\sigma \equiv 2: \quad \left.\begin{aligned} 0 &= x_1(k) \\ x_2(k+1) &= x_2(k) \end{aligned}\right\} \rightsquigarrow \quad x(k) = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \ \forall k \in \mathbb{N}$$

A simple example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

Example

Consider (SSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = I \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = I$$

Switched solution behavior $\sigma(k) = \begin{cases} 1, & k < k_s \\ 2, & k \geq k_s \end{cases}$

For $k < k_s$ we have $x(k) = \left(\begin{smallmatrix} c_1 \\ 0 \end{smallmatrix}\right)$ and for $k = k_s - 1$ also $x_1(k_s) = x_1(k_s - 1) = c_1$
BUT: For $k = k_s$ also $0 = x_1(k_s)$, hence $c_1 = 0$ necessary!
Furthermore $x_2(k_s)$ not constraint by mode 1 $\rightsquigarrow$ $x_2(k) = c_2$ for all $k \geq k_s$
$\rightsquigarrow$ $x(k) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $k < k_s$ and $x(k) = \begin{pmatrix} 0 \\ c_2 \end{pmatrix}$ for $k \geq k_s$

Observations from example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

No existence and uniqueness of solutions!

- Not all solutions from the past can be extended to a global solution
- Single initial value leads to multiple solutions in the future
- Loss of causality w.r.t. to switching signal

Definition

(SSS) is called causal w.r.t. the switching signal $:\Longleftrightarrow$ $\forall \sigma, \widetilde{\sigma}$ $\forall x(\cdot)$ sol. for σ $\forall \widetilde{k} \in \mathbb{N}$:

$$\sigma(k) = \widetilde{\sigma}(k) \ \forall k \leq \widetilde{k} \quad \Longrightarrow \quad \exists\, \widetilde{x}(\cdot) \text{ sol. for } \widetilde{\sigma}: \ \widetilde{x}(k) = x(k) \ \forall k \leq \widetilde{k}$$

Example not causal w.r.t. the switching signal: Let $\sigma \equiv 1$, $\widetilde{\sigma}(k) = \begin{cases} 1, & k < k_s \\ 2, & k \geq k_s \end{cases}$

$\rightsquigarrow$ no solution $\widetilde{x}$ with $\widetilde{x}(k) = c_1 = x(k) \neq 0$ for $k < k_s$.

Causality and One-Step-Map

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

Question

When is (SSS) causal w.r.t. the switching signal?

More specifically: When is $x(k+1)$ uniquely defined for all $x(k)$, $\sigma(k)$ and $\sigma(k+1)$?

In other words: Is there a one-step-map $\Phi_{i,j} \in \mathbb{R}^{n\times n}$, $i,j \in \{1,2,\ldots,\mathtt{n}\}$ such that

$$\forall \text{ sol. } x(\cdot) \text{ of (SSS)}: \quad x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$$

Regularity and index

Theorem (Quasi-Weierstrass Form)

(E, A) is regular $\iff$ $\exists S, T$ invertible with

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \qquad \text{(QWF)}$$

where N is nilpotent

Definition

(E, A) has index-1 $:\iff$ $N = 0$ in (QWF)

Index-1 (together with regularity) is also called:

- causal
- admissable
- impulse-free

Index-1 characterization

Theorem (see e.g. *Griepentrog & März 1986*)

(E, A) *is regular and index-1*

$\iff \mathcal{S} \oplus \ker E = \mathbb{R}^n$, *where* $\mathcal{S} := A^{-1}(\operatorname{im} E) := \{\xi \in \mathbb{R}^n \mid A\xi \in \operatorname{im} E\}$

$\iff \mathcal{S} \cap \ker E = \{0\}$

Furthermore, $T = [T_1, T_2]$ *and* $S = [ET_1, AT_2]^{-1}$ *with* $\operatorname{im} T_1 = \mathcal{S}$ *and* $\operatorname{im} T_2 = \ker E$*:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \qquad \text{(QWF)}$$

Corollary

$Ex(k+1) = Ax(k)$ *being regular + index-1 has unique solution with* $x(0) = x_0 \in \mathbb{R}$

$\iff x_0 \in \mathcal{S}$

In fact, $x(k+1) = \Phi_{(E,A)} x(k)$ *with* $\Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Is this the sought one-step map already?

Attention

$\Phi_{(E,A)}$ is one-step-map for $Ex(k+1) = Ax(k)$
BUT: Only true when system is active for at least two time-steps:

$$Ex(1) = Ax(0) \implies x(1) \in E^{-1}(Ax(0)) = \{\Phi_{(E,A)}x(0)\} + \ker E$$
$$Ex(2) = Ax(1) \implies x(1) \in A^{-1}(Ex(2)) \subseteq \mathcal{S}$$

Hence, invoking $\mathcal{S} \cap \ker E = \{0\}$,

$$Ex(1) = Ax(0) \quad \wedge \quad Ex(2) = Ax(1) \quad \implies \quad x(1) = \Phi_{(E,A)}x(0)$$

$\rightsquigarrow$ Not suitable for switched systems!

Both modes in Example were regular+index-1, but no one-step-map exists!

Problem seems to be overlooked in the literature so far!

A key definition

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

Definition

(SSS) or $\{(E_1, A_1), (E_2, A_2), \ldots, (E_{\mathtt{n}}, A_{\mathtt{n}})\}$ is called (jointly) index-1 $:\Longleftrightarrow$

$$\mathcal{S}_i \cap \ker E_j = \{0\} \qquad \forall i, j \in \{1, 2, \ldots, \mathtt{n}\},\ \mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$$

- Clearly ($i = j$) each pair (E_i, A_i) must be index-1
- In general, (E_j, A_i) is not index-1 (not even regular)

Example

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = I \quad \rightsquigarrow \quad \ker E_1 = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},\ \mathcal{S}_1 = A_1^{-1}(\operatorname{im} E_1) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

$$E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = I \quad \rightsquigarrow \quad \ker E_2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},\ \mathcal{S}_1 = A_2^{-1}(\operatorname{im} E_2) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

Clearly, $\mathcal{S}_i \cap \ker E_j \neq \{0\}$ for $i \neq j$.

The one-step-map

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \qquad \text{(SSS)}$$

Theorem (Anh, Linh, Thuan, T; CDC 2019)

Assume (SSS) is (jointly) index-1. Then $\forall \sigma\ \forall x_0 \in \mathbb{R}^n$*:*

$$x(\cdot) \text{ solves (SSS)} \quad \Longleftrightarrow \quad x_0 \in \mathcal{S}_{\sigma(0)} \ \wedge\ x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$$

where

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \cdot \Phi_{(E_j,A_j)}$$

and $\Pi_{\mathcal{S}_i}^{\ker E_j}$ *is the projector onto* $\mathcal{S}_i$ *along* $\ker E_j$.

Skip proof

Proof idea

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k) \quad \text{with} \quad \Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \cdot \Phi_{(E_j,A_j)}$$

Lemma

For any subspace $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ it holds that

$$\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n \quad \Longrightarrow \quad \mathcal{V} \cap (\{z\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}} z\}$$

- index-1 $\Longrightarrow$ $\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n \rightsquigarrow \Pi_{\mathcal{S}_i}^{\ker E_j}$ well defined
- $E_{\sigma(0)} x(1) = A_{\sigma(0)} x(0) \Longrightarrow x(0) \in \mathcal{S}_{\sigma(0)}$
- Show by induction that $x(k) \in \mathcal{S}_{\sigma(k)} \Longrightarrow \exists!\, x(k+1) \in \mathcal{S}_{\sigma(k+1)}$
  - $E_{\sigma(k)} x(k+1) = A_{\sigma(k)} x(k) \Longrightarrow x(k+1) \in \{\Phi_{(E_{\sigma(k)},A_{\sigma(k)})} x(k)\} + \ker E_{\sigma(k)}$
  - $E_{\sigma(k+1)} x(k+2) = A_{\sigma(k+1)} x(k+1) \Longrightarrow x(k+1) \in A_{\sigma(k+1)}^{-1}(\operatorname{im} E_{\sigma(k+1)}) = \mathcal{S}_{\sigma(k+1)}$
  - $\overset{\text{Lemma}}{\Longrightarrow} x(k+1) = \Pi_{\mathcal{S}_{\sigma(k+1)}}^{\ker E_{\sigma(k)}} \Phi_{(E_{\sigma(k)},A_{\sigma(k)})} x(k)$

Necessity of index-1 asssumption

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = 0 \qquad \text{(SSS)}$$

Theorem

$\forall\sigma$ $x(1) = 0$ is only solution of (SSS) for $k = 0, 1$
$\implies \mathcal{S}_i \cap \ker E_j = \{0\}$ for $i, j \in \{1, 2, \ldots, \mathtt{n}\}$

Proof sketch:

- $k = 0$: $E_j x(1) = A_j x(0) = 0 \iff x(1) \in \ker E_j$
- $k = 1$: $E_i x(2) = A_i x(1) \iff x(1) \in \mathcal{S}_i$
- $x(1) = 0$ is only solution $\implies \ker E_j \cap \mathcal{S}_i = \{0\}$

Summary

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

- Simple example shows that index-1 of each mode is not sufficient for existence and uniqueness of solutions
- (SSS) is index-1 $:\Longleftrightarrow \quad A_i^{-1}(\operatorname{im} E_i) \cap \ker E_j = \{0\} \quad \forall i,j \in \{1,2,\ldots,\mathtt{n}\}$
- (SSS) index-1 $\implies$ existence of one-step-map $\Phi_{i,j}$ such that

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$$

- Unique solvability $\implies$ index-1 of (SSS)

Extensions

- Explicit calculation of $\Phi_{i,j}$ (without QWF)
- Extension to inhomogeneous case
- Stability analysis via joint spectral radius