The Laplace transform and inconsistent initial values

Stephan Trenn

SCAA @ BI(FSE), University of Groningen, Nijenborgh 9, 9747 AG Groningen, email: `s.trenn@rug.nl`

Abstract: Switches in electrical circuits may lead to Dirac impulses in the solution; a real word example utilizing this effect is the spark plug. Treating these Dirac impulses in a mathematically rigorous way is surprisingly challenging. This is in particular true for arguments made in the frequency domain in connection with the Laplace transform. A survey will be given on how inconsistent initials values have been treated in the past and how these approaches can be justified in view of the now available solution theory based on piecewise-smooth distributions.

Keywords:

1. INTRODUCTION

Modeling electrical circuits containing (ideal) switches naturally leads to a description via switched differential-algebraic equations (DAEs) of the form

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$$

where x is the state (including algebraic variables), u is the input and σ is the switching signal (Trenn, 2012). The reason why it is in general not possible to find a more classical model in terms of ordinary differential equations (ODEs) is the fact, that the changing position of the switches changes the algebraic constraints; without including algebraic constraints in the model it would not be possible to incorporate *changing* algebraic constraints. Furthermore, the effect of *inconsistent* initial values can not be studied in a model already in the form of an ODE (because ODEs do not have inconsistent initial values). For a given switching signal σ the switched DAE can be viewed as a repeated initial value problem for the DAE

$$E\dot{x} = Ax + Bu \tag{1}$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$. As mentioned above, at a switching time the algebraic constraints may change, hence the initial value from the past may not be consistent anymore and the meaning of "$x(0) = x_0$" has to be made precise. In the context of electrical circuit this is a long standing question and can be traced back at least to Verghese et al. (1981). There have been different approaches to deal with inconsistent initial values, e.g. Sincovec et al. (1981); Cobb (1982); Opal and Vlach (1990); Rabier and Rheinboldt (1996); Reißig et al. (2002); Frasca et al. (2010), some of which will be discussed later. All have in common that jumps as well as Dirac impulses may occur in the solutions. The Dirac impulse is a distribution (a generalized function) hence one must enlarge the considered solution space to also include distributions.

This extended abstract is a revisitation of Trenn (2013) and will discuss the mathematical and conceptional difficulties arising from the notion of inconsistent initial values, in particular, when the Laplace transform is applied to the equation and arguments from the frequency domain are used. First some required notation from distribution theory is recalled, then a Laplace transform approach is presented on how to deal with inconsistent initial values and finally an alternative approach in the time domain is presented.

⋆ This work was partially supported by Vidi-grant 639.032.733

2. PRELIMINARIES ON DISTRIBUTION THEORY

The classical distribution theory by Schwartz (1957, 1959) is revised in the following. The space of *test functions* is $\mathcal{C}_0^\infty := \{\ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \in \mathcal{C}^\infty \text{ has compact support }\}$, which is equipped with a certain topology [1]. The space of distributions, denoted by $\mathbb{D}$, is then the dual of the space of test functions, i.e.

$$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous }\}.$$

A large class of ordinary functions, namely locally integrable functions, can be embedded into $\mathbb{D}$ via the following injective [2] homomorphism $f \mapsto f_\mathbb{D}$ with $f_\mathbb{D}(\varphi) := \int_\mathbb{R} f\varphi$.

The main feature of distributions is the ability to take derivatives for any distribution $D \in \mathbb{D}$ via $D'(\varphi) := -D(\varphi')$, which is consistent with the classical derivative, i.e. if f is differentiable, then $(f_\mathbb{D})' = (f')_\mathbb{D}$. In particular, the Heaviside unit step $\mathbb{1}_{[0,\infty)}$ has a distributional derivative which can easily be calculated to be

$$(\mathbb{1}_{[0,\infty)_\mathbb{D}})'(\varphi) = \varphi(0) =: \delta(\varphi),$$

hence it results in the well known Dirac impulse δ (at $t = 0$). In general, the Dirac impulse δ_t at time $t \in \mathbb{R}$ is given by $\delta_t(\varphi) := \varphi(t)$. Furthermore, if g is a piecewise differentiable function with one jump at $t = t_J$, then

$$(g_\mathbb{D})' = (g')_\mathbb{D} + (g(t_J+) - g(t_J-))\delta_{t_j}, \tag{2}$$

where g' is the derivative of g on $\mathbb{R} \setminus \{0\}$.

[1] The topology is such that a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of test functions converges to zero if, and only if, 1) the supports of all φ_k are contained within one common compact set $K \subseteq \mathbb{R}$ and 2) for all $i \in \mathbb{N}$, $\varphi_k^{(i)}$ converges uniformly to zero as $k \to \infty$

[2] Two locally integrable functions which only differ on a set of measure zero are identified with each other.

Now it is no problem to consider the DAE (1) (without the initial condition) in a distributional solution space; instead of x and u being vectors of functions they are now vectors of distributions, i.e. $x \in \mathbb{D}^n$ and $f \in \mathbb{D}^m$ where $n \times n$ and $n \times m$ are the size of the matrices E, A and B. The definition of the matrix vector product remains unchanged [3] so that (1) reads as m equations in $\mathbb{D}$.

Considering distributional solutions, however, does *not* help to treat inconsistent initial value; au contraire, distributions cannot be evaluated at a certain time because they are not functions of time, so writing $x(0) = x_0$ makes no sense. Even when assuming that a pointwise evaluation is well defined for certain distributions, the DAE (1) will still not exhibit (distributional) solution with arbitrary initial values. This is easily seen when considering, e.g., the DAE (1) with $(E, A, B) = (0, I, 0)$, which simply reads as $0 = x$.

So what does it then mean to speak of a solution of (1) with inconsistent initial value? The motivation for inconsistent initial values is the situation that the system descriptions gets active at the initial time $t = 0$ and before that the system was governed by different (maybe unknown) rules. This viewpoint was already expressed by Doetsch (1974) in the context of distributional solutions for ODEs:

> The concept of "initial value" in the physical science can be understood only when the past, that is the interval $t < 0$, has been included in our considerations. This occurs naturally for distributions which, without exception, are defined on the entire t-axis.

So mathematically, there is some given past trajectory x^0 for x up to the initial time and the DAE (1) only holds on the interval $[0, \infty)$. This means that a solution of the following *initial trajectory problem* (ITP) is sought:

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax + Bu)_{[0,\infty)}, \end{aligned} \tag{3}$$

where $x^0 \in \mathbb{D}^n$ is an arbitrary past trajectory and D_I for some interval $I \subseteq \mathbb{R}$ and $D \in \mathbb{D}$ denotes a distributional restriction generalizing the restrictions of functions given by$f_I(t) = f(t)$ for $t \in I$ and $f(t) = 0$ otherwise.

A fundamental problem is the fact (Trenn, 2013, Lem. 5.1) that such a distributional restriction does not exist!

This problem was resolved especially in older publication (Campbell, 1980, 1982; Verghese et al., 1981) by ignoring it and/or by arguing with the Laplace transform (see the next section). Cobb (1984) seems to be the first to be aware of this problem and he resolved it by introducing the space of piecewise-continuous distributions; Geerts (1993b,a) was the first to use the space of impulsive-smooth distributions (introduced by Hautus and Silverman (1983)) as a solution space for DAEs. Seemingly unaware of these two approaches, Tolsa and Salichs (1993) developed a distributional solution framework which can be seen as a mixture between the approaches of Cobb and Geerts. The more comprehensive space of piecewise-smooth distributions was later introduced (Trenn, 2009) to combine the advantages of the piecewise-continuous and impulsive-smooth distributional solution spaces. Further details are discussed in Section 4.

Cobb (1982) also presented another approach by justifying the impulsive response due to inconsistent initial values via his notion of *limiting solutions*. The idea is to replace the singular matrix E in (1) by a "disturbed" version E_ε which is invertible for all $\varepsilon > 0$ and $E_\varepsilon \to E$ as $\varepsilon \to 0$. If the solutions of the corresponding initial value ODE problem $\dot{x} = E_\varepsilon^{-1}Ax$, $x(0) = x_0$ converges to a distribution, then Cobb calls this the limiting solution. He is then able to show that the limiting solution is unique and equal to the one obtained via the Laplace-transform approach. Campbell (1982) extends this result also to the inhomogeneous case.

3. LAPLACE TRANSFORM APPROACHES

Especially in the signal theory community it is common to study systems like (1) in the so called *frequency domain* (in contrast to the *time domain*). The transformation between time and frequency domain is given by the *Laplace transform* defined via the Laplace integral:

$$\hat{g}(s) := \int_0^\infty e^{-st} g(t) \; \mathrm{d}t \tag{4}$$

for some function g and $s \in \mathbb{C}$. Note that in general the Laplace integral is not well defined for all $s \in \mathbb{C}$ and a suitable domain for $\hat{g}$ must be chosen (Doetsch, 1974). If a suitable domain exists, then $\hat{g} = \mathcal{L}\{g\}$ is called the *Laplace transform* of g and, in general, $\mathcal{L}\{\cdot\}$ denotes the Laplace transform operator. Again note that it is not specified at this point which class of functions have a Laplace transform and which class of functions are obtained as the image of $\mathcal{L}\{\cdot\}$. The main feature of the Laplace transform is the following property, where g is a differentiable function for which g and g' have Laplace transforms,

$$\mathcal{L}\{g'\}(s) = s\mathcal{L}\{g\}(s) - g(0), \tag{5}$$

which is a direct consequence of the definition of the Laplace integral invoking partial differentiation. If g is not continuous at $t = 0$ but $g(0^+)$ exists and g' denotes the derivative of g on $\mathbb{R} \setminus \{0\}$, then (5) still holds in a slightly altered form:

$$\mathcal{L}\{g'\}(s) = s\mathcal{L}\{g\}(s) - g(0^+). \tag{6}$$

In particular, the Laplace transform does not take into account at all how g behaved for $t < 0$ which is a trivial consequence of the definition of the Laplace integral. This observation will play an important role when studying inconsistent initial values.

Taking into account the linearity of the Laplace transform the DAE (1) is transformed into

$$sE\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s) + Ex(0^+) \tag{7}$$

If the matrix pair (E, A) is regular [4] and $x(0^+) = 0$, the latter can be solved easily algebraically:

$$\hat{x}(s) = (sE - A)^{-1}B\hat{u}(s) =: G(s)\hat{u}(s), \tag{8}$$

where $G(s)$ is a matrix over the field of rational functions and is usually called transfer function.

[3] Some authors (Rabier and Rheinboldt, 2002; Kunkel and Mehrmann, 2006) use a different definition for the matrix vector product which is due to the different viewpoint of a distributional vector x as a map from $(\mathcal{C}_0^\infty)^n$ to $\mathbb{R}$ instead of a map from $\mathcal{C}_0^\infty$ to $\mathbb{R}^n$. The latter seems the more natural approach in view of applying it to (1), but it seems that both approaches are equivalent at least with respect to the solution theory of DAEs.

[4] (E, A) is called regular, if $\det(sE - A)$ is not identically zero.

A first systematic treatment of descriptor systems in the frequency domain was carried out by Rosenbrock (1970). He, however, only considered zero initial values and the input-output behavior. In particular, he was not concerned with a solution theory for general DAEs (1) with possible inconsistent values. Furthermore, he restricted attention to inputs which are exponentially bounded (guaranteeing existence of the Laplace transform), hence formally his framework could not deal with arbitrary (sufficiently smooth) inputs.

The definition of the Laplace transform can be extended to be well defined for certain distributions as well (Doetsch, 1974), therefore consider the following class of distributions:

$$\mathbb{D}_{\geq 0,k} := \left\{ D=(g_{\mathbb{D}})^{(k)} \;\middle|\; \begin{array}{l}\text{where } g:\mathbb{R}\to\mathbb{R} \text{ is continuous}\\ \text{and } g(t)=0 \text{ on } (-\infty,0)\end{array} \right\}.$$

For $D \in \mathbb{D}_{\geq 0,k}$ with $D = (g_{\mathbb{D}})^{(k)}$ the (distributional) Laplace transform is now given by

$$\mathcal{L}_{\mathbb{D}}\{D\}(s) := s^k \mathcal{L}\{g\}(s)$$

on a suitable domain in $\mathbb{C}$. Note that $\delta \in \mathbb{D}_{\geq 0,2}$ and it is easily seen that

$$\mathcal{L}_{\mathbb{D}}\{\delta\} = 1. \tag{9}$$

Furthermore, for every locally integrable function g for which $\mathcal{L}\{g\}$ is defined on a suitable domain it holds

$$\mathcal{L}_{\mathbb{D}}\{g_{\mathbb{D}}\} = s\mathcal{L}\left\{\int_0^{\cdot} g\right\} = s\frac{1}{s}\mathcal{L}\{g\} = \mathcal{L}\{g\}, \tag{10}$$

i.e. the distributional Laplace transform coincides with the classical Laplace transform defined by (4).

A direct consequence of the definition of $\mathcal{L}_{\mathbb{D}}$ is the following derivative rule for all $D \in \bigcup_k \mathbb{D}_{\geq 0,k}$:

$$\mathcal{L}_{\mathbb{D}}\{D'\}(s) = s\mathcal{L}_{\mathbb{D}}\{D\} \tag{11}$$

which seems to be in contrast to the derivative rule (6), because *no initial value occurs.* The latter can actually not be expected because general distributions do not have a well defined function evaluation at a certain time t. However, the derivative rule (11) is consistent with (6); to see this let g be a function being zero on $(-\infty, 0)$, differentiable on $(0,\infty)$ with well defined value $g(0^+)$. Denote with g' the (classical) derivative of g on $\mathbb{R}\setminus\{0\}$, then (invoking linearity of $\mathcal{L}_{\mathbb{D}}$)

$$\begin{aligned}\mathcal{L}_{\mathbb{D}}\{(g_{\mathbb{D}})'\} &\overset{(2)}{=} \mathcal{L}_{\mathbb{D}}\{(g')_{\mathbb{D}} + g(0^+)\delta\}\\ &= \mathcal{L}_{\mathbb{D}}\{(g')_{\mathbb{D}}\} + g(0^+)\mathcal{L}_{\mathbb{D}}\{\delta\} \overset{(9),(10)}{=} \mathcal{L}\{g'\} + g(0^+),\end{aligned}$$

which shows equivalence of (11) and (6). The key observation is that the distributional derivative takes into account the jump at $t = 0$ whereas the classical derivates ignores it, i.e. in the above context

$$(g_{\mathbb{D}})' \neq (g')_{\mathbb{D}}.$$

As it is common to identify g with $g_{\mathbb{D}}$ (even in Doetsch (1974)), the above distinction is difficult to grasp, in particular for inexperienced readers. As this problem plays an important role when dealing with inconsistent initial values, it is not surprising that researchers from the DAE community who are simply using the Laplace transform as a tool, struggle with the treatment of inconsistent initial values, c.f. Lundberg et al. (2007).

Revisiting the treatment of the DAE (1) in the frequency domain one has now to decide wether to use the usual Laplace transform resulting in (7) or the distributional Laplace transform resulting in

$$sE\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s), \tag{12}$$

where the initial value $x(0^+)$ does not occur anymore. In particular, if $u = 0$ the only solution of (12) is $\hat{x}(s) = 0$, which implies $x = 0$. Altogether, the following dilemma occurs:

Dilemma. Consider the regular DAE (1) with zero input but non-zero initial value, then the following conflicting observations can be made:
• An adhoc analysis calls for *distributional solutions* in response to inconsistent initial values. For consistent initial value there exist classical (nonzero) solutions.
• Using the *distributional* Laplace transform to analyze the (distributional) solutions of (1) reveals that the *only* solution is the trivial one. In particular, no initial values (neither inconsistent nor consistent ones) are taken into account at all.

This problem was already observed by Doetsch (1974) and is based on the definition of the distributional Laplace transform which is only defined for distributions vanishing on $(-\infty, 0)$. The following "solution" to this Dilemma was suggested (Doetsch, 1974, p. 129): Define for $D \in \bigcup_k \mathbb{D}_{\geq 0,k}$ the "past-aware" derivative operator $\frac{\mathrm{d}_-}{\mathrm{d}t}$:

$$\tfrac{\mathrm{d}_-}{\mathrm{d}t} D := D' - d_0^- \delta \tag{13}$$

where $d_0^- \in \mathbb{R}$ is interpreted as a "virtual" initial value for $D(0^-)$. Note however, that, by definition $D(0^-) = 0$ for every $D \in \bigcup_k \mathbb{D}_{\geq 0,k}$ hence at this stage it is not clear why this definition makes sense. This problem was also pointed out by Cobb (1982).

Using now the past-aware derivative in the distributional formulation of (1) one obtains:

$$Ex' = Ax + Bu + Ex_0^- \delta \tag{14}$$

where $x_0^- \in \mathbb{R}^n$ is the virtual (possible inconsistent) initial value for $x(0^-)$ and solutions are sought in the space $\left(\bigcup_k \mathbb{D}_{\geq 0,k}\right)^n$, i.e. x is assumed to be zero on $(-\infty, 0)$. Applying the distributional Laplace transform to (14) yields

$$sE\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s) + Ex_0^- \tag{15}$$

In contrast to (7), x_0^- is not the initial value for $x(0^+)$ but is the virtual initial value for $x(0^-)$. If the matrix pair (E, A) is regular, the solution of (15) can now be obtained via $\hat{x}(s) = (sE-A)^{-1}(B\hat{u}(s)+Ex_0^-)$ and using the inverse Laplace transform; there are however the following major drawbacks:
(i) Within the frequency domain it is not possible to motivate the incorporation of the (inconsistent) initial values as in (14); in fact, Doetsch (1974) who seems to have introduced this notion needs to argue with the help of the distributional derivative and (13) within the time domain!
(ii) The Laplace transform ignores everything what was in the past, i.e. on the interval $(-\infty, 0)$; this is true for the classical Laplace transform (by definition of the Laplace integral) as well as for the distributional Laplace transform (by only considering distributions which vanish for $t < 0$). Hence the natural viewpoint of an initial trajectory problem (3) as also informally advocated by Doetsch is not possible to treat with the Laplace transform

approach.
(iii) Making statements about existence and uniqueness of solution with the help of the frequency domain heavily depends on an isomorphism between the time-domain and the frequency domain; there are, however, only a few special isomorphisms between certain special subspaces of the frequency and time domain, no general isomorphismus is available.

4. PIECEWISE-SMOOTH DISTRIBUTIONS

In order to rigorously analyse switched DAEs it was suggested in Trenn (2009) to use as an underlying solution space the space of piecewise-smooth distributions

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, T \subseteq \mathbb{R} \text{ locally} \\ \text{finite}, \forall t \in T: \\ D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \dots\} \end{array} \right\},$$

where $\mathcal{C}^\infty_{\mathrm{pw}}$ is the space of piecewise-smooth functions (with locally finitely many discontinuities). This space is closed under differentiation and therefore removes one shortcoming of Cobb's space of piecewise-continuous distributions and generalized the space of impulsive-smooth distributions, which only considers Dirac impulses at $t = 0$ [5].

A key result for the ITP (3) is then the following equivalence:

Theorem 1. (cf. Thm. 5.3 in Trenn (2013)). Consider the ITP (3) within the piecewise-smooth distributional solution framework with fixed initial trajectory $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ and inhomogeneity $u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$. Then $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ solves the ITP (3) if, and only if, $z := x - x^0_{(-\infty,0)} = x_{[0,\infty)}$ solves

$$\begin{aligned} z_{(-\infty,0)} &= 0 \\ (E\dot{z})_{[0,\infty)} &= (Az + Bu)_{[0,\infty)} + Ex^0(0-)\delta. \end{aligned} \tag{16}$$

Corollary 2. Consider a (possible inconsistent) initial value $x_0 \in \mathbb{R}^n$ for the regular DAE (1). Then for any trajectory $x^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ with $x^0(0^-) = x_0$ and any input u, the solution of the ITP (3) restricted the interval $[0, \infty)$ equals the solution obtained via the Laplace transform approach (15).

5. CONCLUSION

Inconsistent initial values cannot be treated in a meaningful way when studying DAEs in the frequency domain. However, arguments in the time-domain based on piecewise-smooth distribution justify why adding the term Ex_0 to the right-hand side of the distributionally Laplace transformed DAE has something to do with the solution for an inconsistent initial value.

REFERENCES

Campbell, S.L. (1980). *Singular Systems of Differential Equations I.* Pitman, New York.

Campbell, S.L. (1982). *Singular Systems of Differential Equations II.* Pitman, New York.

Cobb, J.D. (1982). On the solution of linear differential equations with singular coefficients. *J. Diff. Eqns.*, 46, 310–323. doi:10.1016/0022-0396(82)90097-3.

Cobb, J.D. (1984). Controllability, observability and duality in singular systems. *IEEE Trans. Autom. Control*, 29, 1076–1082. doi:10.1109/TAC.1984.1103451.

Doetsch, G. (1974). *Introduction to the Theory and Application of the Laplace Transformation.* Springer-Verlag, Berlin.

Frasca, R., Çamlıbel, M.K., Goknar, I.C., Iannelli, L., and Vasca, F. (2010). Linear passive networks with ideal switches: Consistent initial conditions and state discontinuities. *IEEE Trans. Circuits Syst., I: Fundam. Theory Appl.*, 57(12), 3138–3151.

Geerts, A.H.W.T. (1993a). Invariant subspaces and invertibility properties for singular systems: the general case. *Linear Algebra Appl.*, 183, 61–88. doi:10.1016/0024-3795(93)90424-M.

Geerts, A.H.W.T. (1993b). Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant linear systems: The general case. *Linear Algebra Appl.*, 181, 111–130. doi:10.1016/0024-3795(93)90027-L.

Geerts, A.H.W.T. and Schumacher, J.M.H. (1996). Impulsive-smooth behavior in multimode systems. Part I: State-space and polynomial representations. *Automatica*, 32(5), 747–758.

Hautus, M.L.J. and Silverman, L.M. (1983). System structure and singular control. *Linear Algebra Appl.*, 50, 369–402.

Kunkel, P. and Mehrmann, V. (2006). *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, Switzerland. doi:10.4171/017.

Lundberg, K.H., Miller, H.R., and Trumper, D.L. (2007). Initial conditions, generalized functions, and the Laplace transform. *IEEE Control Systems Magazine*, 27(1), 22–35. doi:10.1109/MCS.2007.284506.

Opal, A. and Vlach, J. (1990). Consistent initial conditions of linear switched networks. *IEEE Trans. Circuits Syst.*, 37(3), 364–372.

Rabier, P.J. and Rheinboldt, W.C. (1996). Time-dependent linear DAEs with discontinuous inputs. *Linear Algebra Appl.*, 247, 1–29.

Rabier, P.J. and Rheinboldt, W.C. (2002). Theoretical and numerical analysis of differential-algebraic equations. In P.G. Ciarlet and J.L. Lions (eds.), *Handbook of Numerical Analysis*, volume VIII, 183–537. Elsevier Science, Amsterdam, The Netherlands.

Reißig, G., Boche, H., and Barton, P.I. (2002). On inconsistent initial conditions for linear time-invariant differential-algebraic equations. *IEEE Trans. Circuits Syst., I: Fundam. Theory Appl.*, 49(11), 1646–1648.

Rosenbrock, H.H. (1970). *State Space and Multivariable Theory.* John Wiley and Sons Inc., New York, NY.

Schwartz, L. (1957, 1959). *Théorie des Distributions.* Hermann, Paris.

Sincovec, R.F., Erisman, A.M., Yip, E.L., and Epton, M.A. (1981). Analysis of descriptor systems using numerical algorithms. *IEEE Trans. Autom. Control*, 26, 139–147.

Tolsa, J. and Salichs, M. (1993). Analysis of linear networks with inconsistent initial conditions. *IEEE Trans. Circuits Syst.*, 40(12), 885 – 894. doi:10.1109/81.269029.

Trenn, S. (2009). *Distributional differential algebraic equations.* Ph.D. thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany. URL `http://www.db-thueringen.de/servlets/DocumentServlet?id=13581`.

Trenn, S. (2012). Switched differential algebraic equations. In F. Vasca and L. Iannelli (eds.), *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, chapter 6, 189–216. Springer-Verlag, London. doi:10.1007/978-1-4471-2885-4_6.

Trenn, S. (2013). Solution concepts for linear DAEs: a survey. In A. Ilchmann and T. Reis (eds.), *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, 137–172. Springer-Verlag, Berlin-Heidelberg. doi:10.1007/978-3-642-34928-7_4.

Verghese, G.C., Levy, B.C., and Kailath, T. (1981). A generalized state-space for singular systems. *IEEE Trans. Autom. Control*, 26(4), 811–831.

[5] Rabier and Rheinboldt (1996) seem to be aware of this restriction and they introduce the space $\mathcal{C}_{\mathrm{imp}}(\mathbb{R} \setminus \mathcal{S})$, where $\mathcal{S} = \{ t_i \in \mathbb{R} \mid i \in \mathbb{Z} \}$ is a strictly ordered set with $t_i \to \pm\infty$ as $i \to \pm\infty$ and $D \in \mathcal{C}_{\mathrm{imp}}(\mathbb{R} \setminus \mathcal{S})$ is such that $D_{(t_i,t_{i+1})}$ is induced by the corresponding restriction of a smooth function. A similar idea is proposed in Geerts and Schumacher (1996), however in both cases the resulting distributional space is not studied in detail.