# Discontinuous Lyapunov functions for discontinous piecewise-affine systems

**Raffaele Iervolino** [*] **Stephan Trenn** [**] **Francesco Vasca** [***]

[*] *Department of Electrical Engineering and Information Technology, University of Naples Federico II, Via Claudio 21, 80125 Napoli, Italy, email:* `rafierv@unina.it`
[**] *SCAA @ BI(FSE), University of Groningen, Nijenborgh 9, 9747 AG Groningen, email:* `s.trenn@rug.nl`
[***] *Department of Engineering, University of Sannio, Piazza Roma 21, 82100 Benevento, Italy, email:* `francesco.vasca@unisannio.it`

**Abstract:** Asymptotic stability of continuous-time piecewise affine systems defined over a polyhedral partition of the state space, with possible discontinuous vector field on the boundaries, is considered. We first introduce the feasible Filippov solution concept by characterizing single-mode Caratheodory, sliding mode and forward Zeno behaviors. Then, a global asymptotic stability result through a (possibly discontinuous) piecewise Lyapunov function is presented. The sufficient conditions are based on pointwise classifications of the trajectories which allow the identification of crossing, unreachable and Caratheodory boundaries. It is highlighted that the sign and jump conditions of the stability theorem can be expressed in terms of linear matrix inequalities by particularizing to piecewise quadratic Lyapunov functions and using the cone-copositivity approach.

*Keywords:* Lyapunov stability, Asymptotic stability, discontinuoes systems, Fillipov solutions

## 1. INTRODUCTION

Lyapunov theory has been widely used for the asymptotic stability analysis of continuous-time piecewise affine (PWA) systems defined over a polyhedral partition of the state space (Johansson, 2003; Christophersen, 2007). When the vector fields are not continuous on the boundaries, which is the case considered in this paper, the stability problem becomes more challenging due to the possible occurrence of sliding mode and Zeno behaviors (Filippov, 1988; Imura and der Schaft, 2000). For this class of discontinuous systems, to find a globally quadratic Lyapunov function is a nontrivial issue (Rodrigues and Boyd, 2005; Sakurama and Sugie, 2005) and the existence of such a function is not ensured either (Pavlov et al., 2007; Lin and Antsaklis, 2009).

A possible direction for overcoming limitations of global quadratic functions, consists of considering continuous piecewise Lyapunov functions (Leth and Wisniewski, 2012). In particular, piecewise quadratic (PWQ) Lyapunov functions and the $\mathcal{S}$-procedure lead to stability conditions for classical solutions which can be expressed in terms of linear matrix inequalities (LMIs), see Waitman et al. (2016); Iervolino et al. (2017a, 2015). In general, to have a continuous PWQ function which is positive definite and decreasing in time in each region it is not sufficient for concluding the asymptotic stability of Filippov solutions of a discontinuous PWA system. Indeed, further conditions for dealing with sliding modes must be added (Johansson, 2003; Hajiahmadi et al., 2016). Other classes of continuous piecewise Lyapunov functions have been considered (Samadi and Rodrigues, 2011). For instance, a backstepping procedure for the construction of a continuous Lyapunov function in the form of sum of squares for piecewise polynomial systems with discontinuous vector field is proposed in Samadi and Rodrigues (2014). In Dezuo et al. (2014) continuous functions given by convex combinations of quadratic forms allow to conclude the stability of some discontinuous PWA systems with sliding modes. The more general class of piecewise smooth Lyapunov functions is considered in Heemels and Weiland (2008) but the continuity on the boundaries is assumed therein too.

In this extended abstract we consider the more general case of possibly discontinuous piecewise Lyapunov functions for discontinuous PWA systems. Discontinuous PWQ Lyapunov functions have been considered in Eghbal et al. (2013); Cheraghi-Shami et al. (2019) for the asymptotic stability of planar PWA systems, but the analysis was restricted to the case of continuous vector fields. The stability conditions proposed in Pettersson and Lennartson (2002) allows discontinuities but the apriori knowledge of the sequence of modes is required. In Bisoffi et al. (2018) a discontinuous Lyapunov function designed by looking at the system structure has been proposed for a point mass subject to Coulomb friction in feedback with a PID controller.

The analysis proposed here originates from the preliminary arguments presented in Iervolino et al. (2017b) where more restrictive classes of PWA systems and PWQ Lyapunov functions were considered. Herein the possibly discontinuous Lyapunov function does not require to have a

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PWQ form, although can be particularized to that class thus allowing the formulation of the stability conditions in terms of LMIs through the copositive programming approach (Sponsel et al., 2012).

This extended abstract will only contain the main ideas of our approach without proofs and detailed discussion, the interested reader is referred to the upcoming full version presented in Iervolino et al. (2020).

## 2. PWA SYSTEM AND SOLUTION CONCEPT

We consider the PWA system

$$\dot{x} = A_s x + b_s, \quad x \in X_s, \quad s \in \Sigma \tag{1}$$

where $A_s \in \mathbb{R}^{n\times n}$, $b_s \in \mathbb{R}^n$ and $\{X_s\}_{s=1}^S$ is a polyhedral partition of $\mathbb{R}^n$ with $S \in \mathbb{N}$ being the *finite* size of the partition; let $\Sigma := \{1, \ldots, S\}$. In particular, every $X_s$ is a closed convex set with positive measure resulting from the finite intersection of (closed) half-spaces. Furthermore, we assume that the intersection $X_i \cap X_j$ is empty or a common face of the polyhedra $X_i$ and $X_j$ for all $i, j \in \Sigma$. Since each $X_s$ is a closed set, neighbouring polyhedra have a nonempty intersection and there is some ambiguity in the system definition on these intersections. This ambiguity needs to be handled carefully when defining solutions and also is crucial in the forthcoming stability analysis.

The (dynamic-independent) index set of *current modes* at $x \in \mathbb{R}^n$ is defined as $\Sigma^x := \{\ s \in \Sigma \mid x \in X_s\ \}$. Note that for those $x \in \mathbb{R}^n$ which are not on a boundary, $\Sigma^x$ just contains one index. For those $x$ which are on the boundaries, $\Sigma^x$ contains the indices of all polyhedra which share that point. Rewriting (2) now as a differential inclusion in the form

$$\dot{x} \in \{\ A_s x + b_s \mid s \in \Sigma^x\ \}, \tag{2}$$

we introduce the following solution concept for (1).

*Definition 1.* (Caratheodory solution). Consider the PWA system (1). We call $\xi : [t_0, T) \to \mathbb{R}^n$, $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, a *Caratheodory solution* of the system iff

(i) $\xi$ is absolutely continuous and
(ii) for almost all $t \in [t_0, T)$:

$$\dot{\xi}(t) \in \left\{\ A_s \xi(t) + b_s \ \middle|\ s \in \Sigma^{\xi(t)}\ \right\}. \tag{3}$$

The set of maximal Caratheodory solutions with intitial condition $x(0) = x_0 \in \mathbb{R}^n$ is denoted by $\mathcal{CS}(x_0)$.

There are PWA systems for which no Caratheodory solution exists (or where solution stop to exist in finite time) for some initial value. In the context of stability analysis this is undesirable and this problem can be circumvented by "convexifying" the problem and consider Filippov solutions:

*Definition 2.* (Filippov solution). We call $\xi : [t_0, T) \to \mathbb{R}^n$, $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, a *Filippov solution* of the PWA system (2) iff

(i) $\xi$ is absolutely continuous and
(ii) for almost all $t \in [t_0, T)$:

$$\dot{\xi}(t) \in \operatorname{conv}\left\{\ A_s \xi(t) + b_s \ \middle|\ s \in \Sigma^{\xi(t)}\ \right\}. \tag{4}$$

The set of maximal Filippov solutions with intitial condition $x(0) = x_0 \in \mathbb{R}^n$ is denoted by $\mathcal{FS}(x_0)$.

*Definition 3.* (Sliding solution). A Filippov solution $\xi : [t_0, T) \to \mathbb{R}^n$ is called *sliding solution* iff it is not a Caratheodory solution on any subinterval of $[t_0, T)$ and there exists an index set $\Sigma^{\xi(\cdot)}_{\text{slide}} \subseteq \Sigma$ such that $\Sigma^{\xi(\cdot)}_{\text{slide}} = \Sigma^{\xi(t)}$ for all $t \in (t_0, T)$, $\dot{\xi}(t) \in \operatorname{conv}\left\{\ A_s\xi(t) + b_s \ \middle|\ s \in \Sigma^{\xi(\cdot)}_{\text{slide}}\ \right\}$ for almost all $t \in [t_0, T)$.

Clearly, a Caratheodory solution is a Filippov solution, but a sliding solution is not a Caratheodory solution.

The more general class of Filippov solutions allows $\mathcal{FS}(x_0) \neq \emptyset$ *for all* initial values $x_0 \in \mathbb{R}^n$ and it can also be shown that all Filippov solutions of (2) exists globally. However, the convexification of the vector fields even on boundaries where already Caratheodory solution exists may introduce "unnecessary" sliding solutions. These unnecessary sliding solutions are usually not physically feasible, because they cannot be obtained as a limit of a chattering solution (obtained by introducing a small dwell time) and they also lead to conservative stability conditions. Therefore, we want to restrict our attention to *feasible Filippov solutions* defined as follows.

*Definition 4.* (Feasible Filippov solutions). A sliding solution $\xi : [t_0, T) \to \mathbb{R}^n$ of (2) is said to exhibit *unnecessary sliding* iff $\mathcal{CS}(\xi(t)) \neq \emptyset$ for some $t \in (t_0, T)$, i.e. iff somewhere along the trajectory it is possible to continue the trajectory with a Caratheodory solution instead of a sliding solution. We now call a Filippov solution $\xi : [t_0, T) \to \mathbb{R}^n$ *feasible* iff there is no subinterval on which $\xi$ is unnecessarily sliding. Or, in other words, a Filippov solution is called *infeasible* iff it contains unnecessary sliding.

Let the set of all (maximal) feasible solutions starting in $x_0 \in \mathbb{R}^n$ be denoted by:

$$\mathcal{FS}^f(x_0) := \{\ \xi \in \mathcal{FS}(x_0) \mid \xi \text{ is feasible}\ \}.$$

We will now make certain assumptions on the (Filippov) solution behavior of the PWA system (2). We believe that *all* PWA systems of the form (2) satisfy these assumptions, however, as of now, we are not able to formally prove these properties.

*Assumptions.*

(A1) The PWA system (2) has for all initial values global *feasible* Filippov solutions.
(A2) Let $\xi : [0, \infty) \to \mathbb{R}^n$ be any Filippov solution of the PWA system (2). Then for all $t \geq 0$ there is an $\varepsilon > 0$ such that exactly one of the three cases holds:
   1) $\xi\big|_{[t,t+\varepsilon)}$ is a *single-mode Caratheodory solution.*
   2) $\xi\big|_{[t,t+\varepsilon)}$ is a *sliding solution.*
   3) $\xi\big|_{[t,t+\varepsilon)}$ is a *forward Zeno solution*, i.e. it is neither a single-mode Caratheodory nor a sliding solution and there exists a sequence of positive and strictly decreasing numbers $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_0 = \varepsilon$, $\varepsilon_k \to 0$ as $k \to \infty$ and for each $k \in \mathbb{N}$ the piece $\xi|_{[t+\varepsilon_{k+1}, t+\varepsilon_k)}$ is either a single-mode Caratheodory or sliding solution.

## 3. POINTWISE MODE CLASSIFICATIONS

In addition to the current modes $\Sigma^x$ of a point $x \in \mathbb{R}^n$ it is useful for the forthcoming stability analysis to introduce also backward and forward modes. Towards this end we first introduce the set of forward and backward feasible Filippov solutions as follows:

$$\mathcal{FS}_+^f(x_0) := \left\{ \xi : [0,\infty) \to \mathbb{R}^n \,\middle|\, \begin{array}{l} \xi \text{ is a feasible} \\ \text{Filippov sol. of (2)} \\ \text{with } \xi(0) = x_0 \end{array} \right\},$$

$$\mathcal{FS}_-^f(x_0) := \left\{ \xi : [-\omega, 0) \to \mathbb{R}^n \,\middle|\, \begin{array}{l} \xi \text{ is a feasible} \\ \text{Filippov sol. of (2)} \\ \text{with } \xi(0^-) = x_0 \\ \text{and maximal } \omega > 0 \end{array} \right\}.$$

*Definition 5.* (Forward and strict forward mode). We call $s \in \Sigma$ a *forward mode* for $x$ with respect to the PWA system (2) if there exists a solution $\xi \in \mathcal{FS}_+^f(x)$ such that $\xi(t) \in X_s$ for infinitely many small $t > 0$, or, more formally, the set of all forward modes for $x$ is

$$\Sigma_+^x := \bigcup_{\xi \in \mathcal{FS}_+^f(x)} \bigcap_{\varepsilon > 0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(\tau)}.$$

We call $s \in \Sigma$ a *strict forward mode* for $x \in \mathbb{R}^n$ if there exists a single-mode Caratheodory solution $\xi : [0, \varepsilon) \to \mathbb{R}^n$, $\varepsilon > 0$, such that $\xi(t) \in \operatorname{int} X_s$ for all $t \in (0, \varepsilon)$; the set of all strict forward modes for $x$ is denoted by $\Sigma_{++}^x$.

*Definition 6.* (Backward and strict backward mode). We call $s \in \Sigma$ a *backward mode* of $x$ with respect to the PWA system (2) if there exists a solution $\xi \in \mathcal{FS}_-^f(x)$ such that $\xi(-t) \in X_s$ for infinitely many small $t > 0$, or, more formally, the set of all backwards modes for $x$ is

$$\Sigma_-^x := \bigcup_{\xi \in \mathcal{FS}_-^f(x)} \bigcap_{\varepsilon > 0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(-\tau)}.$$

A mode $s \in \Sigma$ is a *strict backward mode* for $x$ if it is a strict forward mode for the time-reversed PWA system (2), i.e. if there exists a single-mode Caratheodory solution $\xi : (-\varepsilon, 0] \to \mathbb{R}^n$, $\varepsilon > 0$ with $\xi(t) \in \operatorname{int} X_s$ for all $t \in (-\varepsilon, 0)$; the set of all strict backward modes for $x$ is denoted by $\Sigma_{--}^x$.

*Definition 7.* (Sliding mode). We call $s \in \Sigma$ a sliding mode for $x \in \mathbb{R}^n$ with respect to the PWA system (2) if there is a (feasible) sliding solution $\xi : [t_0, T) \to \mathbb{R}^n$ with $\xi(t_0) = x$ and $s \in \Sigma_{\text{slide}}^{\xi(\cdot)}$, with $\Sigma_{\text{slide}}^{\xi(\cdot)}$ as in Definition 3.

It is clear, that strict forward/backward modes are always forward/backward modes, i.e. $\Sigma_{++}^x \subseteq \Sigma_+^x$ and $\Sigma_{--}^x \subseteq \Sigma_-^x$. Furthermore, if some point $x \in \mathbb{R}^n$ is not reachable via a feasible Filippov solution (i.e. $\mathcal{FS}_-^f(x) = \emptyset$) then there are no backwards mode for $x$, i.e. $\Sigma_-^x = \emptyset$.

## 4. STABILITY AND PIECEWISE LYAPUNOV FUNCTIONS

We will now study stability of the PWA system (2) with (feasible) Filippov solutions.

*Definition 8.* The PWA (2) is called *stable* iff

(S1) $\mathcal{FS}_+^f(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{R}^n$ and all (feasible Filippov) solutions are defined on $[0, \infty)$.

(S2) The origin is stable, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all solutions $\xi \in \mathcal{FS}_+^f(x_0)$ the following implication holds:

$$\|\xi(0)\| < \delta \implies \|\xi(t)\| < \varepsilon \quad \forall t \geq 0.$$

It is called *globally asymptotically stable* if additionally the origin is globally attractive, i.e.

(S3) $\xi(t) \to 0$ as $t \to \infty$ for all $\xi \in \mathcal{FS}_+^f(x_0)$ and all $x_0 \in \mathbb{R}^n$.

Our goal is to prove stability of the PWA system (2) via a piecewisely defined Lyapunov function. For this we first define "local" Lyaypunov functions.

*Definition 9.* (Local Lyapunov function). Consider the system (2). We call $V_s : \mathbb{R}^n \to \mathbb{R}$ a *local Lyapunov function for mode* $s \in \Sigma$ iff

(L1) $V_s$ is continuous on $\mathbb{R}^n$ and continuously differentiable on $X_s$.

(L2) $V_s$ is positive definite on $X_s$, i.e. $V_s(x) > 0$ for all $x \in X_s \setminus \{0\}$ and if $0 \in X_s$ then $V_s(0) = 0$.

(L3) $V_s$ is radially unbounded in the following sense:

$$\forall \overline{v} \in V_s(X_s) \subseteq \mathbb{R}^n : V_s^{-1}([0, \overline{v}]) \cap X_s \text{ is compact},$$

(L4) $V_s$ is decreasing along "classical" solutions within $X_s$ in the following sense

$$\nabla V_s(x)(A_s x + b_s) < 0 \quad \forall x \in X_s \setminus \{0\},$$

The challenge is to formulate suitable compatibility conditions for this Lyapunov function on the boundaries. The simplest case (but also most restrictive case) is the assumption that there is a common Lyapunov function for all modes, then stability is obviously guaranteed. It is common to assume continuity of the local Lyapunov functions across the boundaries, then asymptotic stability is guaranteed if no sliding and no Zeno-behavior occur. However, requiring continuity is neither necessary nor sufficient for proving stability; for the latter see e.g. Example 4.9 in Johansson (2003).

Our main result will not impose continuity of the local Lyapunov functions across the boundaries, but we will now present weaker suitable compatibility conditions which, if satisfied, ensure stability of the PWA system (2) with feasible Filippov solutions; including sliding and Zeno behaviors as well as non-unique solutions.

*Theorem 10.* Consider the PWA system (2) satisfying Assumptions (A1) and (A2). Assume that for each mode $s \in \Sigma$ there is a local Lyapunov-function $V_s : \mathbb{R}^n \to \mathbb{R}$ as in Definition 9. Furthermore, assume that the different Lyapunov functions are compatible in the following sense:

(B1) $\forall x \in \mathbb{R}^n \; \forall (i,j) \in \Sigma_-^x \times \Sigma_+^x : \quad V_i(x) \geq V_j(x)$.

(B2) $\exists \mu > 0 \; \forall x \in \mathbb{R}^n$ with $\Sigma_{\text{slide}}^x \neq \emptyset \; \exists i_x \in \Sigma_{\text{slide}}^x$ :

$$\nabla V_{i_x}(x)(A_j x + b_j) \leq -\mu \|x\| \quad \forall j \in \Sigma_{\text{slide}}^x. \tag{5}$$

Then (2) is globally asymptotically stable.

*Remarks 11.* (i) Conditions (B1) and (B2) are trivially satisfied for all $x$ in the interior of some $X_s$, hence it needs only to be checked for points $x$ on the boundaries. Furthermore, (B1) is also trivially satisfied for those $x$ with $\Sigma_-^x = \emptyset$.

(ii) We do not explicitly require equality of the Lyapunov function values at sliding points. However, for a sliding solution $\xi : [0, \omega) \to \mathbb{R}^n$ it turns out that for almost all $t \in [0, \omega)$ the equality $\Sigma_-^{\xi(t)} = \Sigma_+^{\xi(t)}$ holds; consequently, (B1) implicitly implies equality of the Lyapunov function values.

(iii) Condition (B2) is satisfied if the in general stronger conditions $\nabla V_i(x) = \nabla V_j(x)$ for all $i, j \in \Sigma^x_{\text{slide}}$ holds. Note that, similar as in Johansson (2003), we are *not* requiring (5) to hold for *all pairs* $(i, j) \in \Sigma^x_{\text{slide}} \times \Sigma^x_{\text{slide}}$, this is in contrast to other recent approaches, see e.g. Hajiahmadi et al. (2016).

(iv) It is straightforward to extend Definition 8 to PWA systems (2) with general Filippov solutions (i.e. not restricting the solution space to *feasible* Filippov solutions). Then Assumption (A1) can be dropped in the formulation of Theorem 10. However, in that case $\Sigma^x_-$ will never be empty, so that jump condition (B1) have to be satisfied on *all* boundaries; in particular, for "splitting" boundaries an "unnecessary" sliding can occur, which in turn enforces an "unnecessary" continuity requirement of the Lyapunov function on that boundary. ◀

## 5. PIECEWISE-QUADRATIC LYAPUNOV FUNCTIONS AND LMIS

We now assume that the sets $\Sigma^x_+$, $\Sigma^x_-$ and $\Sigma^x_{\text{slide}}$ are constant on (the relative interior of) each boundary $X_{\mathcal{B}} := \bigcap_{s\in\mathcal{B}} X_s$, $\mathcal{B} \subseteq \Sigma$ and that the local Lyapunov functions have the form

$$V_s(x) = x^\top P_s x + 2\nu_s^\top x + \omega_s \tag{6}$$

with $P_s \in \mathbb{R}^{n\times n}$ symmetric matrix, $\nu_s \in \mathbb{R}^n$, $\omega_s \in \mathbb{R}$. It is then possible to utilize the cone-copositive approach (Sponsel et al., 2012) to express the conditions in Definition 9 and Theorem 10 in terms of LMIs, which are less conservative than the ones obtained for other approaches. A solution of this set of LMIs will then provide a, in general discontinuous, PWQ Lyapynov function which guarantees asymptotic stability of the PWA system (2) in the presence of (feasible) sliding and Zeno solutions. For details we refer to the full paper Iervolino et al. (2020).

The number of LMIs is governed by the number of regions (two sets of LMIs for each local Lyapunov equation) together with the number of nonempty boundaries $X_{\mathcal{B}}$ (one LMI for each such boundary). Note that in our approach also boundaries of dimension smaller than $n-1$ have to be considered, so that in high dimensional spaces one can easily run into a combinatorical explosion for the number of boundaries and the standard LMIs-solvers (like e.g. Matlab) will not be usuable anymore due to the large number of LMIs. Nevertheless for moderately sized examples ($n = 2$ and $S = 5$) we had no trouble finding a solution.

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