On stabilizability of switched differential algebraic equations

Paul Wijnbergen [*] **Mark Jeeninga** [*,**] **Stephan Trenn** [*]

[*] *Bernoulli Institute for Maths, CS and AI, University of Groningen, 9747 AG, Groningen, The Netherlands (e-mail: p.wijnbergen@rug.nl, s.trenn@rug.nl).*
[**] *Engineering and Technology Institute Groningen (ENTEG), University of Groningen, 9747 AG, Groningen, The Netherlands (e-mail: m.jeeninga@rug.nl).*

Abstract: This paper considers stabilizability of switched differential algebraic equations (DAEs). We first introduce the notion of interval stabilizability and show that under a certain uniformity assumption, stabilizability can be concluded from interval stabilizability. A geometric approach is taken to find necessary and sufficient conditions for interval stabilizability. This geometric approach can also be utilized to derive a novel characterization of controllability.

Keywords: Switched systems, Differential Algebraic Equations, Stabilizability, Linear systems.

1. INTRODUCTION

In this note we consider *switched differential algebraic equations* (switched DAEs) of the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p, n, m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in Trenn (2009).

Differential algebraic equations (DAEs) arise naturally when modeling physical systems with certain algebraic constraints on the state variables. Examples of applications of DAEs in electrical circuits (with distributional solutions) can be found e.g. in Tolsa and Salichs (1993). The algebraic constraints are often eliminated such that the system is described by ordinary differential equations (ODEs). However, in the case of switched systems, the elimination process of the constraints is in general different for each individual mode. Therefore, in general, there does not exist a description as a switched ODE with a common state variable for every mode. This problem can be overcome by studying switched DAEs directly.

We study stabilizability of (1), i.e. the property that for all consistent initial values there exists an input such that the state x converges to zero as time goes to infinity. Apart from the obvious relevance to investigate stabilizability in its own right, it is also important in the context of optimal control, where in the non-switched case, stabilizability is necessary for the existence of a finite quadratic cost (Cobb (1983); Bender and Laub (1985); Reis and Voigt (2012)).

We would like to highlight, that we assume the switching signal to be fixed and known, i.e. (1) is viewed as a time-varying linear system. In particular, the switching signal is *not* considered to be an (additional) control input.

Several other structural properties of (switched) DAEs have been studied recently. Among those are controllability (Küsters et al., 2015), stability (Liberzon and Trenn, 2009) and observability (Küsters et al., 2017). However, stabilizability has thus far only been studied in the non-switched case in Cobb (1984); Lewis (1992); Berger and Reis (2013) and, to the best of the authors knowledge, there are no results yet for the switched case.

An obvious sufficient condition for stabilizability is to demand the last mode to be stabilizable. However, determining what the last mode is of a switched system poses a problem as time tends to infinity. To overcome this problem, we define a notion of stabilizability of a switched system on a bounded interval. Then under certain uniformity assumptions (which are automatically satisfied e.g. for periodic systems) we can prove that the system is stabilizable if there exists a partition of the time axis such that on each subinterval the system is *interval stabilizable.* Furthermore, we present necessary and sufficient conditions for a DAE to be interval stabilizable. The approach for obtaining these results is then utilized to derive novel results on controllability as well.

The outline of the paper is as follows: notations and results for non-switched DAEs are presented in Section II. The main results on stabilizability and interval stabilizability are presented in Section III, followed by a brief discussion on the interpretation of the results. Conclusions and discussions on future work are given in Section IV.

2. MATHEMATICAL PRELIMINARIES

2.1 Properties and definitions for regular matrix pairs

In the following, we consider *regular* matrix pairs (E, A), i.e. for which the polynomial $\det(sE - A)$ is not the

⋆ This work was supported by the NWO Vidi-grant 639.032.733.

zero polynomial. Recall the following result on the *quasi-Weierstrass form* (Berger et al., 2012).

Proposition 1. A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.

The matrices S and T can be calculated by using the so-called *Wong sequences* (Berger et al., 2012; Wong, 1974):

$$\begin{array}{ll} \mathcal{V}_0 := \mathbb{R}^n, & \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i),\ i = 0, 1, ... \\ \mathcal{W}_0 := \{0\}, & \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i),\ i = 0, 1, ... \end{array} \tag{3}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup_i \mathcal{W}_i. \tag{4}$$

If (E, A) is regular, then $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$ and $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$ (see Berger et al. (2012)); in particular, for any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (2) holds.

Based on the Wong sequences we define the following projectors and selectors.

Definition 2. Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (2). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential selector* is given by

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

and the *impulse selector* is given by

$$\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$\begin{array}{ll} A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, & E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E, \\ B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B, & B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B. \end{array}$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). An important feature for DAEs is the so called consistency space, defined as follows for the DAE

$$E\dot{x} = Ax + Bu. \tag{5}$$

Definition 3. Consider the DAE (5), then the *consistency space* is defined as

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } x \text{ of (5)} \\ \text{with } u = 0 \text{ and } x(0) = x_0 \end{array} \right\},$$

and the *augmented consistency space* is defined as

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solutions } (x, u) \text{ of (5)} \\ \text{with } x(0) = x_0 \end{array} \right\}.$$

In order to express (augmented) consistency spaces in terms of the Wong limits we introduce the following notation for matrices A, B of conformable sizes:

$$\langle A \mid B \rangle := \operatorname{im}[B, AB, \dots, A^{n-1}B].$$

Proposition 4. (Berger and Trenn (2014)). Consider the regular DAE (5), then $\mathcal{V}_{(E,A)} = \mathcal{V}^* = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\text{diff}}_{(E,A)}$ and

$$\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle.$$

For studying impulsive solutions of (5), we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from Trenn (2009) as the solution space. That is, we seek a solution $(x, u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+m}$ to the following initial-trajectory problem (ITP) associated to (5):

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{6a}$$
$$(E\dot{x})_{[0,\infty)} = (Ax)_{[0,\infty)} + (Bu)_{[0,\infty)}, \tag{6b}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In Trenn (2009) it is shown that the ITP (6) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. As a direct consequence, the switched DAE (1) with regular matrix pairs is also uniquely solvable (with piecewise-smooth distributional solutions) for any switching signal with locally finitely many switches.

Recall the following definitions and characterization of (impulse) controllability (Berger and Trenn, 2014).

Proposition 5. The reachable space of the regular DAE (5), defined as

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0\ \exists \text{ smooth solution } (x, u) \text{ of (5)} \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\},$$

satisfies $\mathcal{R} = \langle A^{\text{diff}} \mid B^{\text{diff}} \rangle \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

It is easily seen that the reachable space for (5) coincides with the (null-)controllable space, i.e.

$$\mathcal{R} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0\ \exists \text{ smooth solution } (x, u) \text{ of (5)} \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Corollary 6. The augmented consistency space of (5) satisfies $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}}, B^{\text{imp}} \rangle$.

According to Trenn (2012) if the input $u(\cdot)$ is sufficiently smooth, trajectories of (5) on $(0, \infty)$ are continuous and given by

$$\begin{aligned} x(t) &= x_u(t, t_0; x_0) = e^{A^{\text{diff}}(t-t_0)} \Pi_{(E,A)} x_0 \\ &+ \int_{t_0}^{t} e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s)\, ds - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \end{aligned} \tag{7}$$

In particular, all trajectories can be written as the sum of an autonomous part $x_{\text{aut}}(t, t_0; x_0) = e^{A^{\text{diff}}(t-t_0)} \Pi_{(E,A)} x_0 \in \mathcal{V}_{(E,A)}$ and a controllable part $x_u(t, t_0) \in \mathcal{R}$ as follows:

$$x_u(t, t_0; x_0) = x_{\text{aut}}(t, t_0; x_0) + x_u(t, t_0).$$

With some adjustment in notation, this decomposition remains valid also for switched DAEs, in particular, x_0 is the (possible inconsistent) initial value at t_0^-.

2.2 Stabilizability notions

The concepts introduced in the previous section are now utilized to investigate stabilizability of switched DAEs. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals from the following class

$$\Sigma := \left\{ \sigma : \mathbb{R} \to \mathbb{N} \;\middle|\; \begin{array}{l} \sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps} \\ \text{and constant in the past} \end{array} \right\}.$$

Since we are concerned with a single switching signal, we can assume (by relabeling the corresponding matrices accordingly) that at time t_k we switch to mode k, i.e.

$$\sigma(t) = k, \qquad \text{for } t_k \leqslant t < t_{k+1}. \tag{8}$$

Since we do not allow infinitely many switches in the past we may assume for the first switching instant t_1 that $t_1 > t_0 := 0$. Denote with $\tau_k := t_{k+1} - t_k$ the duration of mode k.

Roughly speaking, in classical literature on non-switched systems, a control system is called stabilizable if every trajectory can be steered towards zero as time tends to infinity. We will define stabilizability for switched DAEs in a similar fashion as follows.

Definition 7. (Stabilizability). The switched DAE (1) with switching signal (8) is stabilizable if the corresponding solution behavior $\mathfrak{B}_\sigma$ is stabilizable in the behavioral sense on the interval $[0, \infty)$, i.e.

$$\begin{aligned} &\forall (x, u) \in \mathfrak{B}_\sigma \; \exists (x^*, u^*) \in \mathfrak{B}_\sigma : \\ &(x^*, u^*)_{(-\infty,0)} = (x, u)_{(-\infty,0)}, \\ &\text{and } \lim_{t\to\infty} (x^*(t^+), u^*(t^+)) = 0. \end{aligned}$$

In constrast to previous works on stability of switched DAEs (Liberzon and Trenn, 2009, 2012) we adopt the viewpoint as in Tanwani and Trenn (2015) (cf. Def. 6 and Prop. 7 therein) and do not require impulse-free solutions for asymptotic stability. Simultaneously stabilizing and eliminating impulses is a topic of future research.

Since stabilizability is an asymptotic property, i.e. $t \to \infty$, it is reasonable to assume that there are an infinite amount of switching instances. This poses a problem when it comes to verifying conditions for stabilizability in a finite amount of steps. To overcome this problem, we investigate stabilizability on a bounded interval. To that extent we introduce the following definition of interval stabilizability (cf. Def. 5 in Tanwani and Trenn (2017)).

Definition 8. (Interval-stabilizabilty). Consider the switched DAE (1) with a switching signal given by (8). Then (1) is called *interval-stabilizable* on the finite inteval $[\underline{t}, \bar{t}) \subseteq [0, \infty)$, if there exists a class $\mathcal{KL}$ function[1] $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with

$$\beta(r, \bar{t} - \underline{t}) < r, \quad \forall r > 0,$$

and for any (possibly inconsistent) initial value $x_0 \in \mathbb{R}^n$ there exist a local solution (x, u) of (1) on $[\underline{t}, \bar{t})$ with $x(\underline{t}^-) = x_0$ such that

$$|x(t^+)| \leqslant \beta(|x_0|, t - \underline{t}), \quad \forall t \in [\underline{t}, \bar{t}),$$

where $|\cdot|$ denotes the Euclidian norm on $\mathbb{R}^n$.

One should note that a solution on some interval is not necessarily a part of a solution on a larger interval. Consequently, stabilizability does not always imply interval stabilizability. The switched system $0 = x$ on $[0, t_1)$ and $\dot{x} = 0$ on $[t_1, \infty)$ is obviously stabilizable, since the only global solution is the zero solution. However, on the interval $[t_1, s)$ there are nonzero solutions which do not converge towards zero. Furthermore, we would like to emphasize that in general the interval $[\underline{t}, \bar{t})$ contains multiple switches, i.e. it is not assumed that the individual modes of the switched system are stabilizable.

We need some uniformity assumption to conclude that interval stabilizability on each interval of a partition of $[0, \infty)$ implies stabilizability.

Assumption 1. (Uniform interval-stabilizability). Consider the switched system (1) with switching signal σ and switching times t_k, $k \in \mathbb{N}$. Assume that there exists a strictly increasing sequence $(q_i)_{i=0}^\infty$ with $q_0 > 0 =: q_{-1}$ such that for $p_i = q_{i-1}$ the system is $[t_{p_i}, t_{q_i})$-stabilizable with $\mathcal{KL}$ function β_i for which additionally it holds that

$$\begin{aligned} \beta_i(r, t_{q_i} - t_{p_i}) &\leqslant \alpha r, \quad \forall r > 0, \forall i \in \mathbb{N} \\ \beta_i(r, 0) &\leqslant M r, \quad \forall r > 0, \forall i \in \mathbb{N}, \end{aligned}$$

for some uniform $\alpha \in (0, 1)$ and $M \geqslant 1$.

We now present the following result.

Proposition 9. If the switched system (1) is uniformly interval-stabilizable in the sense of Assumption 1 then (1) is stabilizable.

The proof of Proposition 9 is along the same lines as the proof of Proposition 8 in Tanwani and Trenn (2019) and therefore omitted.

3. INTERVAL STABILIZABILITY FOR (1)

In the following we will derive conditions under which a switched system (1) is interval stabilizable. Without loss of generality, we consider the switched DAE on the interval $[0, t_f)$ for some $t_f > 0$ and a switching signal of the form (8). By assumption, there are only finitely many switching instants in $(0, t_f)$, say $t_1 < t_2 < \ldots < t_{\mathtt{n}}$ for some $\mathtt{n} \in \mathbb{N}$; for notational convenience we let $t_0 := 0$ and $t_{\mathtt{n}+1} := t_f$. Furthermore, we denote in the following with Π_i, A_i^{diff}, B_i^{diff}, E_i^{imp}, B_i^{imp} the corresponding matrices related to (E_i, A_i) for $i = 0, 1, \ldots, \mathtt{n}$.

In order to verify whether the system is interval stabilizable, we need to compute the minimum norm of the state at the end of the interval. To do so, we first consider the (orthogonal) projector $\Pi_{\mathcal{R}_i^\perp}$ onto the orthogonal complement of the reachable space $\mathcal{R}_i$ of mode i. An important property of these projectors is that their restriction to the corresponding augmented consistency space is well defined:

Lemma 10. Consider the DAE (1) with switching signal (8). For any $i \in \{0, 1, \ldots, \mathtt{n}\}$ let $\xi \in \mathcal{V}_{(E_i, A_i, B_i)}$, then

$$\Pi_{\mathcal{R}_i^\perp} \xi \in \mathcal{V}_{(E_i, A_i, B_i)}.$$

Proof. From $\xi \in \mathcal{V}_{(E_i, A_i, B_i)}$ and $\Pi_{\mathcal{R}_i^\perp} + (I - \Pi_{\mathcal{R}_i^\perp}) = I$, it follows that

$$\Pi_{\mathcal{R}_i^\perp} \xi + (I - \Pi_{\mathcal{R}_i^\perp}) \xi \in \mathcal{V}_{(E_i, A_i, B_i)}.$$

[1] A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is called a class $\mathcal{KL}$ function if 1) for each $t \geqslant 0$, $\beta(\cdot, t)$ is continuous, strictly increasing, with $\beta(0, t) = 0$; 2) for each $r \geqslant 0$, $\beta(r, \cdot)$ is decreasing and converging to zero as $t \to \infty$.

Since $\operatorname{im}(I-\Pi_{\mathcal{R}_i^\perp}) = \mathcal{R}_i$ and $\mathcal{R}_i \subseteq \mathcal{V}_{(E_i,A_i,B_i)}$ we obtain

$$\Pi_{\mathcal{R}_i^\perp}\xi \in \mathcal{V}_{(E_i,A_i,B_i)} - (I-\Pi_{\mathcal{R}_i^\perp})\xi \subseteq \mathcal{V}_{(E_i,A_i,B_i)}.$$

as was to be shown. ■

Given Lemma 10 we are ready to conclude the following lemma.

Lemma 11. Consider the system (1) with switching signal (8). Then we have that

$$\min_u |x_u(t_{i+1}^-, t_0; x_0)| = \min_u |\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|.$$

Furthermore, the minimization on the right hand side does not depend on the choice of u on $[t_i, t_{i+1})$.

Proof. It follows that for any input u

$$x_u(t_{i+1}^-, t_0; x_0) = (\Pi_{\mathcal{R}_i^\perp} + (I-\Pi_{\mathcal{R}_i^\perp}))x_u(t_{i+1}^-, t_0; x_0)$$

and since $\operatorname{im}(I-\Pi_{\mathcal{R}_i^\perp})$ and $\operatorname{im}\Pi_{\mathcal{R}_i^\perp}$ are orthogonal subspaces, we have by Pythagoras' Theorem

$$\begin{aligned}|x_u(t_{i+1}^-, t_0; x_0)|^2 &= |\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|^2 \\ &\quad + |(I-\Pi_{\mathcal{R}_i^\perp})x_u(t_{i+1}^-, t_0; x_0)|^2.\end{aligned}$$

Invoking $(I-\Pi_{\mathcal{R}_i^\perp})x_u(t_{i+1}^-, t_0; x_0) \in \mathcal{R}_i$ we can choose our input on $[t_i, t_{i+1})$ such that $|(I-\Pi_{\mathcal{R}_i^\perp})x_u(t_{i+1}^-, t_0; x_0)| = 0$, regardless of the input on $[0, t_i)$. What remains to minimize is $|\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|$. This component is however not dependent on u on $[t_i, t_{i+1})$, because any effect of a non-zero input will evolve in $\mathcal{R}_i$ and is therefore annihilated by $\Pi_{\mathcal{R}_i^\perp}$. ■

In order to investigate the state at the end of an interval $[0, t_f)$ we introduce the following sequence of subspaces and show that they correspond to the reachable spaces at the end of the corresponding switching intervals.

Proposition 12. Consider the system (1) with switching signal (8) and let

$$\begin{aligned}\mathcal{S}_0 &= \mathcal{R}_0, \\ \mathcal{S}_i &= e^{A_i^{\text{diff}}\tau_i}\Pi_i\mathcal{S}_{i-1} + \mathcal{R}_i, \quad i = 1, 2, \ldots, \mathtt{n}.\end{aligned}$$

Then $\mathcal{S}_i$ is the reachable space of (1) at t_{i+1}, i.e.

$$\mathcal{S}_i = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l}\exists \text{ solution } (x,u) \text{ of (1) on } [0, t_{i+1}) \\ \text{with } x(0^-) = 0 \text{ and } x(t_{i+1}^-) = \xi\end{array} \right\}.$$

Proof. Since no switch occurs in the interval $(0, t_1)$ the statement is true by definition for $i = 0$.

We now show the statement by induction and therefore assume that the statement holds for some $i-1 \geqslant 0$. Let (x, u) be a solution of (1) with $x(0^-) = 0$ and $x(t_{i+1}^-) = \xi_i$. Utilizing (7) on the interval (t_i, t_{i+1}) we have

$$\xi_i = e^{A_i^{\text{diff}}\tau_i}\Pi_i x(t_i^-) + x_u(t_{i+1}^-, t_i)$$

with $x_u(t_{i+1}^-, t_i) \in \mathcal{R}_i$ and, by induction, $x(t_i^-) \in \mathcal{S}_{i-1}$. This shows that $\xi_i \in \mathcal{S}_i$.

Conversely, assume that $\xi_i \in \mathcal{S}_i$, then there exists $\xi_{i-1} \in \mathcal{S}_{i-1}$ and $\eta_i \in \mathcal{R}_i$ such that

$$\xi_i = e^{A_i^{\text{diff}}\tau_i}\Pi_i\xi_{i-1} + \eta_i.$$

By induction, there exist a solution (x, u) on $[0, t_i)$ with $x(t_i^-) = \xi_{i-1}$. Furthermore, by the definition of the reachable space $\mathcal{R}_i$, the input u can be extended on the interval $[t_i, t_{i+1})$ such that $x_u(t_{i+1}^-, t_i) = \eta_i$. Hence (7) considered on (t_i, t_{i+1}) implies that ξ_i is reachable by (1) on the interval $[0, t_{i+1})$. ■

Due to Lemma 11, we are interested on how we can influence $\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)$ and therefore we define the following subspace.

Definition 13. Consider the system (1) with switching signal (8). The *reachable mode-i-uncontrollable space* is defined by

$$\widetilde{\mathcal{S}}_i := (\mathcal{R}_i)^\perp \cap \mathcal{S}_i.$$

Lemma 14. Consider the system (1) with switching signal (8). Then $\widetilde{\mathcal{S}}_i = \Pi_{\mathcal{R}_i^\perp}\mathcal{S}_i$.

Proof. The inclusion $\widetilde{\mathcal{S}}_i \subseteq \Pi_{\mathcal{R}_i^\perp}\mathcal{S}_i$ holds trivially.

Conversely, consider $\zeta \in \Pi_{\mathcal{R}_i^\perp}\mathcal{S}_i$, then $\zeta = \Pi_{\mathcal{R}_i^\perp}\theta$ for some $\theta \in \mathcal{S}_i$. Invoking Proposition 12 choose u_0 such that $x_{u_0}(t_{i+1}^-, t_0; 0) = \theta$. Then since $\operatorname{im}(I-\Pi_{\mathcal{R}_i^\perp}) \subseteq \mathcal{R}_i$ there exists a u_1 such that

$$\begin{aligned}\zeta = \Pi_{\mathcal{R}_i^\perp}\theta &= \theta - (I-\Pi_{\mathcal{R}_i^\perp})\theta, \\ &= \theta - x_{u_1}(t_{i+1}, t_i).\end{aligned}$$

By linearity of solutions there thus exists an input $\bar{u}$ such that $x_{\bar{u}}(t_{i+1}^-, t_0; 0) = x_{u_0}(t_{i+1}^-, t_0; 0) - x_{u_1}(t_{i+1}, t_i) = \zeta$ and thus $\zeta \in \widetilde{\mathcal{S}}_i$. Hence $\Pi_{\mathcal{R}_i^\perp}\mathcal{S}_i \subseteq \widetilde{\mathcal{S}}_i$. ■

As will turn out, the state projected to $\mathcal{R}_i^\perp$ at $t = t_{i+1}$ can be decomposed into a reachable component and a component resulting from the initial condition. To that extent we define the x_0-uncontrollable orthogonal component.

Definition 15. Consider the system (1) with switching signal (8). The *x_0-uncontrollable orthogonal component* $\xi_i(x_0)$ is defined by the following sequence

$$\begin{aligned}\xi_0(x_0) &= \Pi_{\mathcal{R}_0^\perp} e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0, \\ \xi_{i+1}(x_0) &= \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}}(t_{i+2}-t_{i+1})}\Pi_{i+1}\xi_i(x_0).\end{aligned}$$

Lemma 16. Consider the switched system (1) with switching signal (8). Then for all $i \in \{0, ..., \mathtt{n}\}$ we have

$$\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0) - \xi_i(x_0) \in \widetilde{\mathcal{S}}_i.$$

Furthermore, for all intitial values x_0 and any $\widetilde{\xi}_i \in \widetilde{\mathcal{S}}_i$, there exists an input u such that

$$\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0) - \xi_i(x_0) = \widetilde{\xi}_i. \tag{9}$$

Proof. Let (x, u) be a solution of (1) with $x(0^-) = x_0$. Then for $i = 0$ we have that

$$\begin{aligned}\Pi_{\mathcal{R}_0^\perp} x(t_1^-) &= \Pi_{\mathcal{R}_0^\perp}(e^{A_0^{\text{diff}}\tau_1}\Pi_0 x_0 + x_u(t_1^-, t_0)) \\ &= \Pi_{\mathcal{R}_0^\perp} e^{A_0^{\text{diff}}\tau_1}\Pi_0 x_0 \\ &= \xi_0(x_0),\end{aligned}$$

hence the statement holds, because $\widetilde{\mathcal{S}}_0 = \{0\}$.

We will now proceed inductively from i to $i+1$. Similarly as for $i = 0$ we first observe that

$$\Pi_{\mathcal{R}_{i+1}^\perp} x(t_{i+2}^-) = \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}}\tau_{i+1}}\Pi_{i+1} x(t_{i+1}^-).$$

According to the induction hypothesis we have $\widetilde{\xi}_i := \Pi_{\mathcal{R}_i^\perp} x(t_{i+1}^-) - \xi_i(x_0) \in \widetilde{\mathcal{S}}_i$ and hence

$$x(t_{i+1}^-) = \Pi_{\mathcal{R}_i^\perp} x(t_{i+1}^-) + (I - \Pi_{\mathcal{R}_i^\perp}) x(t_{i+1}^-) = \xi_i(x_0) + \widetilde{\xi}_i + \eta_i,$$

where $\eta_i := (I - \Pi_{\mathcal{R}_i^\perp}) x(t_{i+1}^-) \in \mathcal{R}_i$. Consequently,

$$\Pi_{\mathcal{R}_{i+1}^\perp} x(t_{i+2}^-) = \xi_{i+1}(x_0) + \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} \widehat{\xi}_i,$$

where $\widehat{\xi}_i := \widetilde{\xi}_i + \eta_i \in \mathcal{S}_i$ (because $\widetilde{\mathcal{S}}_i \subseteq \mathcal{S}_i$ and $\mathcal{R}_i \subseteq \mathcal{S}_i$). Invoking Lemma 14, this concludes the proof of the first part because

$$\Pi_{\mathcal{R}_{i+1}^\perp} x(t_{i+2}^-) - \xi_{i+1}(x_0) = \Pi_{\mathcal{R}_{i+1}^\perp} \widehat{\xi}_{i+1}$$

with $\widehat{\xi}_{i+1} := e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} \widehat{\xi}_i \in \mathcal{S}_{i+1}$.

It remains to be shown, that for each $\widetilde{\xi}_{i+1} \in \widetilde{\mathcal{S}}_{i+1}$ there exists an input u such that $\Pi_{\mathcal{R}_{i+1}^\perp} x_u(t_{i+2}^-, t_0; x_0) - \xi_{i+1}(x_0) = \widetilde{\xi}_{i+1}$. For given $\widetilde{\xi}_{i+1}$ we can (by invoking Lemma 14) choose $\widehat{\xi}_{i+1} \in \mathcal{S}_{i+1}$ as well as $\widehat{\xi}_i \in \mathcal{S}_i$ and $\eta_{i+1} \in \mathcal{R}_{i+1}$ such that

$$\begin{aligned}\widetilde{\xi}_{i+1} &= \Pi_{\mathcal{R}_{i+1}^\perp} \widehat{\xi}_{i+1} = \Pi_{\mathcal{R}_{i+1}^\perp} (e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} \widehat{\xi}_i + \eta_{i+1}) \\ &= \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} (\widetilde{\xi}_i + \eta_i),\end{aligned}$$

where $\widetilde{\xi}_i := \Pi_{\mathcal{R}_i^\perp} \widehat{\xi}_i \in \widetilde{\mathcal{S}}_i$ and $\eta_i := (I - \Pi_{\mathcal{R}_i^\perp}) \widehat{\xi}_i \in \mathcal{R}_i$. By the induction hypothesis we can now choose an input such that (9) holds. Due to the projection the value of $\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)$ does not depend on the choice of u on $[t_i, t_{i+1})$ and we can alter u on this interval such that $x_u(t_{i+1}^-, t_i) = \eta_i$. Consequently, $x_u(t_{i+1}^-, t_0; x_0) = \widetilde{\xi}_i + \xi(x_0) + \eta_i$. With this input (arbitrarily extended on the interval $[t_{i+1}, t_{i+1})$ we now have

$$\begin{aligned}&\Pi_{\mathcal{R}_{i+1}^\perp} x_u(t_{i+2}^-, t_0; x_0) - \xi_{i+1}(x_0) \\ &= \Pi_{\mathcal{R}_{i+1}^\perp} \big(e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} x_u(t_{i+1}^-, t_0; x_0) + x_u(t_{i+2}^-, t_{i+1})\big) \\ &\quad - \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} \xi_i(x_0) \\ &= \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} \big(x_u(t_{i+1}^-, t_0; x_0) - \xi_i(x_0)\big) \\ &= \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}} \tau_{i+1}} \Pi_{i+1} \big(\widetilde{\xi}_i + \xi(x_0) + \eta_i - \xi_i(x_0)\big) = \widetilde{\xi}_{i+1}.\end{aligned}$$

■

Lemma 17. Consider the system (1) with switching signal (8). Then for all $x_0 \in \mathbb{R}^n$ we have that for all $i \in \{0, ..., \mathtt{n}\}$

$$\min_u |x_u(t_{i+1}^-, t_0; x_0)| = \operatorname{dist}(\xi_i(x_0), \widetilde{\mathcal{S}}_i). \tag{10}$$

Proof. We have

$$\begin{aligned}\min_u |x_u(t_{i+1}^-, t_0; x_0)|^2 &\overset{\text{Lem. 11}}{=} \min_u |\Pi_{\mathcal{R}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|^2 \\ &\overset{\text{Lem. 16}}{=} \min_{\eta_i \in \widetilde{\mathcal{S}}_i} |\xi_i(x_0) + \eta_i|^2 = \operatorname{dist}(\xi_i(x_0), \widetilde{\mathcal{S}}_i)^2\end{aligned}$$

■

This leads us to the main theorem on interval stabilizability of switched DAEs.

Theorem 18. Consider the switched DAE (1) with switching signal (8) having $\mathtt{n} \in \mathbb{N}$ switches in the finite interval $[0, t_f)$. For $x_0 \in \mathbb{R}^n$ let $\xi_{\mathtt{n}}(x_0)$ be given as in Definition 15 and let $\widetilde{\mathcal{S}}_{\mathtt{n}}$ be given as in Definition 13. Then (1) is interval stabilizable on $[0, t_f)$ if and only if for all $x_0 \in \mathbb{R}^n$

$$\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \widetilde{\mathcal{S}}_{\mathtt{n}}) < |x_0|.$$

Proof. Assume that the system is interval stabilizable and that interval stabilizability is achieved by $\tilde{u}$. Then it follows that

$$\begin{aligned}\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \widetilde{\mathcal{S}}_{\mathtt{n}}) &= \min_u |(x_u(t_f^-, t_0; x_0)| \\ &\leqslant |x_{\tilde{u}}(t_f^-, t_0; x_0)|, \\ &\leqslant \beta(|x_0|, t_f) < |x_0|\end{aligned}$$

Conversely if $\operatorname{dist}(\xi_{\mathtt{n}}(x_0)) = \min_u |(x_u(t_f^-, t_0; x_0)| < |x_0|$ then obviously for the u that attains this minimum there exists a class $\mathcal{KL}$ function β such that $\beta(r, t_f) < r$ and $|x_u(t^+, 0; x_0)| \leqslant \beta(|x_u(t^+, 0; x_0)|, t)$ for all $t \in [0, t_f)$. ■

Remark 19. The conditions stated in Theorem 18 need to be valid for an infinite amount of points, however, it is sufficient to just check it for any orthogonal basis of $\mathbb{R}^n$. This is a consequence of $\xi_{\mathtt{n}}(x_0)$ being linear in x_0 together with the semi-norm property of the distance-from-a-subspace functions and Pythagoras' Theorem. In fact, let $x_0 = \sum_{i=1}^n \alpha_i b_i$ for some orthogonal basis $b_1, \ldots, b_n \in \mathbb{R}^n$ and some coordinates $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, then (under the assumption that (11) holds for each of the n basis vectors $b_1, \ldots, b_n$)

$$\begin{aligned}\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \widetilde{\mathcal{S}}_{\mathtt{n}})^2 &\leqslant \left(\sum_{i=1}^n |\alpha_i| \operatorname{dist}(b_i, \widetilde{\mathcal{S}}_i)\right)^2 \\ &\leqslant \sum_{i=1}^n |\alpha_i b_i|^2 = |x_0|^2\end{aligned}$$

Example 20. Consider the following switched DAE defined on the interval $[0, t_f)$ with $t_f := 2\ln(2)$ and a switch at $t_1 = \ln(2)$.

$$\Sigma_\sigma \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), & 0 \leqslant t < t_1, \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(t) & t_1 \leqslant t \leqslant t_f. \end{cases}$$

Note that neither of the two modes of the switched system is stabilizable. In order to show that the system is interval stabilizable, use the Wong sequences to compute A_0^{diff}, A_1^{diff} and Π_0 and Π_1. We have that

$$\operatorname{im} \Pi_0 = \operatorname{im} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \mathcal{V}_{(E_0, A_0)}. \tag{11}$$

Furthermore, we compute

$$\mathcal{R}_0 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}, \ \Pi_{\mathcal{R}_0^\perp} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ \ \Pi_{\mathcal{R}_1^\perp} = I,$$

In view of Remark 19, we only need to verify the conditions of Lemma 18 for set of orthogonal base vectors of $\mathbb{R}^3$. Hence we consider the basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that

$$\xi_1(v_1) = 0, \ \xi_1(v_2) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{3}{5} \end{bmatrix}^\top, \ \xi_1(v_3) = 0.$$

Computing the time t_1 reachable uncontrollable space yields that $\widetilde{\mathcal{S}}_1 = \operatorname{span}\{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top\}$ from which we calculate that

$$\begin{aligned}\operatorname{dist}(\xi_1(v_1), \tilde{\mathcal{S}}_1) &= 0 < 1 = |v_1|, \\ \operatorname{dist}(\xi_1(v_2), \tilde{\mathcal{S}}_1) &= \frac{1}{2} < 1 = |v_2|, \\ \operatorname{dist}(\xi_1(v_3), \tilde{\mathcal{S}}_1) &= 0 < 1 = |v_3|.\end{aligned}$$

Hence we can conclude that the system is interval stabilizable. Note that a nontrivial input on $[0, t_1)$ is necessary for achieving interval stability.

3.1 Controllability of switched DAEs

The approach taken in the previous section does not only lead to results on stabilizability, but can also be used to find conditions for controllability of switched DAEs. To see this, we first state the following lemma.

Lemma 21. Consider the switched DAE (1) with switching signal (8). The initial condition $x_0 \in \mathbb{R}^n$ of the switched system is controllable if and only if $\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \widetilde{\mathcal{S}}_{\mathtt{n}}) = 0$.

Proof. ($\Rightarrow$) Assume that x_0 is a controllable initial value. Then there exists an input u such that $x_u(t_f^-, t_0; x_0) = 0$ and hence $\min_u |x_u(t_f, t_0; x_0)| = 0$. Then it follows from Lemma 17 that $\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \widetilde{\mathcal{S}}_{\mathtt{n}}) = 0$.

($\Leftarrow$) Assume that $\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \widetilde{\mathcal{S}}_{\mathtt{n}}) = 0$. Then by Lemma 17 we have that $\min_u |x_u(t_f, t_0; x_0)| = 0$. The input attaining this minimum controls the initial value to 0 and hence x_0 is controllable. ■

Defining the following subspaces

$$\Psi_0 = \operatorname{im} \Pi_{\mathcal{R}_0^\perp} e^{A_0^{\mathrm{diff}}(t_1 - t_0)} \Pi_0,$$
$$\Psi_{i+1} = \operatorname{im} \Pi_{\mathcal{R}_{i+1}^\perp} e^{A_{i+1}^{\mathrm{diff}}(t_{i+1} - t_i)} \Pi_{i+1} \Psi_i.$$

we can utilize Lemma 21 and Remark 19 to arrive at the following novel controllability characterization for switched DAEs.

Corollary 22. Consider the switched DAE (1) with switching signal (8). The system is controllable if and only if

$$\Psi_{\mathtt{n}} \subseteq \widetilde{\mathcal{S}}_{\mathtt{n}}.$$

Remark 23. The result of Corollary 22 gives a condition for controllability that only require computations that run forward in time. This is in contrast to the result of Küsters et al. (2015), where the last mode is considered first and the computation runs backwards in time.

Remark 24. All results in this paper can be applied to switched ordinary differential equations (ODEs) without difficulty. In the case of an ODE we have $E = I$, $\Pi = I$, $B^{\mathrm{diff}} = B$ and $A^{\mathrm{diff}} = A$. Plugging this into the conditions in this paper yields the result for switched ODEs.

4. CONCLUSION

This paper considered stabilizability of switched DAEs. We introduced the notion of interval stabilizability. Moreover, we showed that –under a uniformity assumption– interval stabilizability implies stabilizability. Necessary and sufficient conditions for interval stabilizability of switched DAEs are given. In addition, the method to analyse the interval stabilizability was used to obtain a necessary and sufficient condition for controllability.

As a future direction of research, a natural extension is to obtain results on impulse free stabilization. However, already simple examples show that impulse controllability together with stabilizability is not sufficient for impulse-free stabilizability in general and investigating this behaviors is ongoing research.

REFERENCES

Bender, D.J. and Laub, A.J. (1985). The linear-quadratic optimal regulator for descriptor systems. In *Proc. 24th IEEE Conf. Decis. Control, Ft. Lauderdale, FL*, 957–962. doi:10.1109/CDC.1985.268642.

Berger, T., Ilchmann, A., and Trenn, S. (2012). The quasi-Weierstraß form for regular matrix pencils. *Linear Algebra Appl.*, 436(10), 4052–4069. doi:10.1016/j.laa.2009.12.036.

Berger, T. and Reis, T. (2013). Controllability of linear differential-algebraic systems - a survey. In A. Ilchmann and T. Reis (eds.), *Surveys in Differential-Algebraic Equations I*, Differential-Algebraic Equations Forum, 1–61. Springer-Verlag, Berlin-Heidelberg. doi:10.1007/978-3-642-34928-7_1.

Berger, T. and Trenn, S. (2014). Kalman controllability decompositions for differential-algebraic systems. *Syst. Control Lett.*, 71, 54–61. doi:10.1016/j.sysconle.2014.06.004.

Cobb, J.D. (1983). Descriptor variable systems and optimal state regulation. *IEEE Trans. Autom. Control*, 28, 601–611. doi:10.1109/TAC.1983.1103283.

Cobb, J.D. (1984). Controllability, observability and duality in singular systems. *IEEE Trans. Autom. Control*, 29, 1076–1082. doi:10.1109/TAC.1984.1103451.

Küsters, F., Ruppert, M.G.M., and Trenn, S. (2015). Controllability of switched differential-algebraic equations. *Syst. Control Lett.*, 78(0), 32 – 39. doi:10.1016/j.sysconle.2015.01.011.

Küsters, F., Trenn, S., and Wirsen, A. (2017). Switch observability for homogeneous switched DAEs. In *Proc. of the 20th IFAC World Congress, Toulouse, France*, 9355–9360. doi:10.1016/j.ifacol.2017.08.1434. IFAC-PapersOnLine 50 (1).

Lewis, F.L. (1992). A tutorial on the geometric analysis of linear time-invariant implicit systems. *Automatica*, 28(1), 119–137. doi:10.1016/0005-1098(92)90012-5.

Liberzon, D. and Trenn, S. (2009). On stability of linear switched differential algebraic equations. In *Proc. IEEE 48th Conf. on Decision and Control*, 2156–2161. doi:10.1109/CDC.2009.5400076.

Liberzon, D. and Trenn, S. (2012). Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability. *Automatica*, 48(5), 954–963. doi:10.1016/j.automatica.2012.02.041.

Reis, T. and Voigt, M. (2012). Linear-quadratic infinite time horizon optimal control for differential-algebraic equations - a new algebraic criterion. In *Proceedings of MTNS-2012*.

Tanwani, A. and Trenn, S. (2015). On detectability of switched linear differential-algebraic equations. In *Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan*, 2957–2962. doi:10.1109/CDC.2015.7402666.

Tanwani, A. and Trenn, S. (2017). Observer design for detectable switched differential-algebraic equations. In *Proceedings of the 20th IFAC World Congress*, 2953–2958. doi:10.1016/j.ifacol.2017.08.659.

Tanwani, A. and Trenn, S. (2019). Detectability and observer design for switched differential algebraic equations. *Automatica*, 99, 289–300. doi:10.1016/j.automatica.2018.10.043.

Tolsa, J. and Salichs, M. (1993). Analysis of linear networks with inconsistent initial conditions. *IEEE Trans. Circuits Syst.*, 40(12), 885 – 894. doi:10.1109/81.269029.

Trenn, S. (2009). *Distributional differential algebraic equations*. Ph.D. thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany. URL `http://www.db-thueringen.de/servlets/DocumentServlet?id=13581`.

Trenn, S. (2012). Switched differential algebraic equations. In F. Vasca and L. Iannelli (eds.), *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, chapter 6, 189–216. Springer-Verlag, London. doi:10.1007/978-1-4471-2885-4_6.

Wong, K.T. (1974). The eigenvalue problem $\lambda Tx + Sx$. *J. Diff. Eqns.*, 16, 270–280. doi:10.1016/0022-0396(74)90014-X.