On stabilizability of switched Differential Algebraic Equations

Paul Wijnbergen * **Mark Jeeninga** ** **Stephan Trenn** *

* *Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, 9747 AG, Groningen, The Netherlands (e-mail: p.wijnbergen@rug.nl, s.trenn@rug.nl).*
** *Engineering and Technology Institute Groningen and Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, 9747 AG, Groningen, The Netherlands (e-mail: m.jeeninga@rug.nl).*

Abstract: This paper considers stabilizability of switched differential algebraic equations (DAEs). We first introduce the notion of interval stabilizability and show that under a certain uniformity assumption, stabilizability can be concluded from interval stabilizability. A geometric approach is taken to find necessary and sufficient conditions for interval stabilizability. Then the analysis is extended resulting in a characterization of controllability.

Keywords: Switched systems, Differential Algebraic Equations, Stabilizability, Linear systems.

1. INTRODUCTION

In this note we consider *switched differential algebraic equations* (switched DAEs) of the following form:

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p, n, m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in Trenn (2009). We study stabilizability of (1) where (1) is stabilizable if for all consistent initial values there exists an input such that $\lim_{t\to\infty} x(t) = 0$.

Differential algebraic equations (DAEs) arise naturally when modeling physical systems with certain algebraic constraints on the state variables. These constraints are often eliminated such that the system is described by ordinary differential equations (ODEs). Examples of applications of DAEs in electrical circuits (with distributional solutions) can be found in Tolsa and Salichs (1993). However, in the case of switched systems, the elimination process of the constraints is in general different for each individual mode. Therefore there does not exist a description as a switched ODE with a common state variable for every mode in general. This problem can be overcome by studying switched DAEs directly.

Ever since control systems have been considered, the question whether the control objective can be achieved with minimal (quadratic) cost has been of great interest. In the non switched case, optimal control of DAEs has been studies in e.g. Cobb (1983); Bender and Laub (1985); Reis and Voigt (2012). It is proven in both Cobb (1983) and Bender and Laub (1985) that stabilizability is a necessary condition for the existence of finite (quadratic) cost regardless of the initial condition, whenever a infinite horizon is considered. For if the state does not tend to zero one can not expect to have finite cost over an infinite time interval. This argument is independent of the underlying system model and hence the state of a switched DAE needs to converge to zero as well in order to achieve finite quadratic cost. Therefore, there is a need for a characterization of all switched DAEs that are stabilizable. Note that here we assume that the switching signal is fixed (i.e. (1) is viewed as a time-varying linear system), in particular, the switching signal is *not* considered to be an (additional) control input.

Several other structural properties of (switched) DAEs have been studied recently. Among those are controllability (Küsters et al., 2015), stability (Liberzon and Trenn, 2009) and observability (Küsters et al., 2017). However, stabilizability has thus far only been studied in the non-switched case in Cobb (1984); Lewis (1992); Berger and Reis (2013) and, to the best of the authors knowledge, there are no results yet for the switched case.

An obvious sufficient condition for stabilizability is to demand the last mode to be stabilizable. However, determining what the last mode is of a switched system poses a problem as time tends to infinity. To overcome this problem, we define a notion of stabilizability of a switched system on a bounded interval. Then under a certain uniformity assumption we can prove that the system is stabilizable if there exists a partition of the time axis such that on each subinterval the system is *interval stabilizable*. Furthermore, we present necessary and sufficient conditions for a DAE to be interval stabilizable. The approach taken in obtaining these results is then utilized to obtain results on controllability as well.

The outline of the paper is as follows: notations and results for non-switched DAEs are presented in Section II. The main results on stabilizability and interval stabilizability are presented in Section III, followed by a brief discussion on the interpretation of the results. Conclusions and discussions on future work are given in Section IV.

2. MATHEMATICAL PRELIMINARIES

2.1 Properties and definitions for regular matrix pairs

In the following, we consider *regular* matrix pairs (E, A), i.e. for which the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form* (Berger et al., 2012).

Proposition 1. A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2}$$

where $j \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$, is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_1 := n - n_1$, is a nilpotent matrix.

The matrices S and T can be calculated by using the so-called *Wong sequences* (Berger et al., 2012; Wong, 1974):

$$\begin{array}{l} \mathcal{V}_0 := \mathbb{R}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \ i = 0, 1, ... \\ \mathcal{W}_0 := \{0\}, \ \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \ i = 0, 1, ... \end{array} \tag{3}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup \mathcal{W}_i. \tag{4}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (2) holds.

Based on the Wong sequences we define the following projectors and selectors.

Definition 2. Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (2). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential selector* is given by

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S,$$

and the *impulse selector* is given by

$$\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$\begin{array}{l} A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, \ E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E, \\ B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B, \ B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B. \end{array}$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). An important feature for DAEs is the so called consistency space, defined as follows:

Definition 3. Consider the DAE $E\dot{x}(t) = Ax(t) + Bu(t)$, then the *consistency space* is defined as

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solution } x \text{ of} \\ E\dot{x} = Ax, \text{ with } x(0) = x_0 \end{array} \right\},$$

and the *augmented consistency space* is defined as

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ smooth solutions } (x, u) \text{ of} \\ E\dot{x} = Ax + Bu \text{ and } x(0) = x_0 \end{array} \right\}.$$

In order to express (augmented) consistency spaces in terms of the Wong limits we introduce the following notation for matrices A,B of conformable sizes:

$$\langle A \mid B \rangle := \operatorname{im}[B, AB, \ldots, A^{n-1}B].$$

Proposition 4. (Berger and Trenn (2014)). Consider the regular DAE $E\dot{x} = Ax + Bu$, then $\mathcal{V}_{(E,A)} = \mathcal{V} = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\text{diff}}_{(E,A)}$ and $\mathcal{V}_{(E,A,B)} = \mathcal{V} \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

For studying impulse solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from Trenn (2009) as the solution space. That is, we seek a solution $(x, u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+m}$ to the following initial-trajectory problem (ITP):

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{5a}$$

$$(E\dot{x})_{[0,\infty)} = (Ax)_{[0,\infty)} + (Bu)_{[0,\infty)}, \tag{5b}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_{\mathcal{I}}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In Trenn (2009) it is shown that the ITP (5) has a unique solution for any initial trajectory if, and only if, the matrix pair (E, A) is regular. As a direct consequence, the switched DAE (1) with regular matrix pairs is also uniquely solvable (with piecewise-smooth distributional solutions) for any switching signal with locally finitely many switches.

2.2 Properties of DAE's

For the rest of this section we are considering the DAE

$$E\dot{x} = Ax + Bu. \tag{6}$$

Recall the following definitions and characterization of (impulse) controllability (Berger and Trenn, 2014).

Proposition 5. The reachable space of the regular DAE (6) defined as

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \ \exists \text{ smooth solution } (x, u) \text{ of } (6) \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\}$$

satisfies $\mathcal{R} = \langle A^{\text{diff}} \mid B^{\text{diff}} \rangle \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

It is easily seen that the reachable space for (6) coincides with the (null-)controllable space, i.e. $\mathcal{R} = \mathcal{C}$ where

$$\mathcal{C} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \ \exists \text{ smooth solution } (x, u) \text{ of } (6) \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Corollary 6. The augmented consistency space of (6) satisfies $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}}, B^{\text{imp}} \rangle$.

According to Trenn (2012) if the input $u(\cdot)$ is sufficiently smooth, trajectories of (6) are continuous and given by

$$x(t) = x_u(t, t_0; x_0) = e^{A^{\text{diff}}(t-t_0)} \Pi_{(E,A)} x_0 + \int_{t_0}^{t} e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s)\, ds - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \tag{7}$$

In particular, all trajectories can be written as the sum of an autonomous part $x_{\mathrm{aut}}(t,t_0;x_0) = e^{A^{\mathrm{diff}}t}\Pi_{(E,A)}x_0$ and a controllable part $x_u(t,t_0)$ as follows:

$$x_u(t,t_0;x_0) = x_{\mathrm{aut}}(t,t_0;x_0) + x_u(t,t_0).$$

This decomposition remains valid for switched DAEs when evaluated at the initial condition at time t_0^-.

2.3 Stabilizability notions

The concepts introduced in the previous section are now utilized to investigate stabilizability of switched DAEs. In order to use the piecewise-smooth distributional solution framework and to avoid technical difficulties in general, we only consider switching signals from the following class, for $\mathtt{n} \in \mathbb{N}$,

$$\Sigma_{\mathtt{n}} := \left\{ \sigma : \mathbb{R} \to \{0,1,...,\mathtt{n}\};\, \middle|\, \begin{array}{l}\sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps}\end{array} \right\},$$

i.e. we exclude an accumulation of switching times (see Trenn (2009)). In fact, we restrict ourselves to the case where at time t_k we switch to mode k, i.e.

$$\sigma(t) = k, \qquad \text{for } t_k \leqslant t < t_{k+1}. \tag{8}$$

In particular, we do not allow infinitely many switches in the past and may assume for the first switching instant t_1 that $t_1 > t_0 := 0$.

Roughly speaking, in classical literature on non-switched systems, a dynamical system is called stabilizable if every trajectory can be steered towards zero as time tends to infinity. We will define stabilizability for switched DAEs in a similar fashion as follows.

Definition 7. (Stabilizability). The switched DAE (1) with switching signal (8) is stabilizable if the corresponding solution behavior $\mathfrak{B}_\sigma$ is stabilizable in the behavioral sense on the interval $[0,\infty)$, i.e.

$$\begin{aligned} &\forall (x,u) \in \mathfrak{B}_\sigma \ \exists (x^*,u^*) \in \mathfrak{B}_\sigma : \\ &(x^*,u^*)_{(-\infty,0)} = (x,u)_{(-\infty,0)}, \\ &\text{and } \lim_{t\to\infty} (x^*(t^+),u^*(t^+)) = 0. \end{aligned}$$

In constrast to previous works on stability of switched DAEs (Liberzon and Trenn, 2009, 2012) we adopt the viewpoint as in Tanwani and Trenn (2015) (cf. Def. 6 and Prop. 7 therein) and do not require impulse-free solutions for asymptotic stability. Simultaneously stabilizing and eliminating impulses is a topic of future research.

Since stabilizability is an asymptotic property, i.e. $t \to \infty$, it is reasonable to assume that there are an infinite amount of switching instances. This poses a problem when it comes to verifying conditions for stabilizability in a finite amount of steps. To overcome this problem, we investigate stabilizability on a bounded interval. To that extent we introduce the following definition of interval stabilizability (cf. Def. 5 in Tanwani and Trenn (2017)).

Definition 8. (Interval-stabilizabilty). The switched DAE (1) is called $[t_p,t_q)$-*stabilizable* for a given switching signal σ, if there exists a class $\mathcal{KL}$ function[1] $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with

$$\beta(r, t_q - t_p) < r, \quad \forall r > 0,$$

and for any (possibly inconsistent) initial value $x_p \in \mathbb{R}^n$ there exist a local solution (x,u) of (1) on $[t_p,t_q)$ with $x(t_p^-) = x_p$ such that

$$|x(t^+)| \leqslant \beta(|x_p|, t-t_p), \quad \forall t \in [t_p,t_q).$$

One should note that a solution on some interval is not necessarily a part of a solution on a larger interval. Consequently, stabilizability does not always imply interval stabilizability. The switched system $0 = x$ on $[0,t_1)$ and $\dot{x} = 0$ on $[t_1,\infty)$ is obviously stabilizable, since the only global solution is the zero solution. However, on the interval $[t_1,s)$ there are nonzero solutions which do not converge towards zero. Furthermore, we would like to emphasize that in general the interval $[t_p,t_q)$ contains multiple switches, i.e. it is not assumed that the individual modes of the switched system are stabilizable.

We need some uniformity assumption to conclude that interval stabilizability on each interal of a partition of $[0,\infty)$ implies stabilizability.

Assumption 1. (Uniform interval-stabilizability). Consider the switched system (1) with switching signal σ and switching times t_k, $k \in \mathbb{N}$. Assume that there exists a strictly increasing sequence $(q_i)_{i=0}^\infty$ with $q_0 > 0 =: q_{-1}$ such that for $p_i = q_{i-1}$ the system is $[t_{p_i}, t_{q_i})$-stabilizable with $\mathcal{KL}$ function β_i for which additionally it holds that

$$\begin{aligned} \beta_i(r, t_{q_i} - t_{p_i}) &\leqslant \alpha r, \quad \forall r > 0, \forall i \in \mathbb{N} \\ \beta_i(r,0) &\leqslant Mr, \quad \forall r > 0, \forall i \in \mathbb{N}, \end{aligned}$$

for some uniform $\alpha \in (0,1)$ and $M \geqslant 1$.

We now present the following result.

Proposition 9. If the switched system (1) is uniformly interval-stabilizable in the sense of Assumption 1 then (1) is stabilizable.

The proof of Proposition 9 is along the same lines as the proof of Proposition 8 in Tanwani and Trenn (2019).

3. INTERVAL STABILIZABILITY FOR SWITCHED DAES

In the following we will derive conditions under which a switched system (1) is interval stabilizable. Since interval stabilizability is defined for a finite interval, we consider the switched DAE on the interval $[t_0,t_f)$ and a switching signal of the form (8).

In order to verify whether the system is interval stabilizable, we need to compute the minimum norm of the state at the end of the interval. In order to do so, we first introduce certain projectors for a given switched DAE. Let $\mathcal{C}_i$ be the controllable space of the i^{th} mode. Then $\Pi_{\mathcal{C}_i^\perp}$ is the projector onto $\mathcal{C}_i^\perp$ along $\mathcal{C}_i$. These projectors project solutions on the interval (t_i,t_{i+1}) to elements of the augmented consistency space of the i^{th} mode. This is formalized in the next lemma.

Lemma 10. Consider the DAE (1) with switching signal (8). Let $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$, Then

$$\Pi_{\mathcal{C}_i^\perp}\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$$

Proof. Since $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$ and $\Pi_{\mathcal{C}_i^\perp} + (I - \Pi_{\mathcal{C}_i^\perp}) = I$, it follows that

[1] A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is called a class $\mathcal{KL}$ function if 1) for each $t \geqslant 0$, $\beta(\cdot,t)$ is continuous, strictly increasing, with $\beta(0,t) = 0$; 2) for each $r \geqslant 0$, $\beta(r,\cdot)$ is decreasing and converging to zero as $t \to \infty$.

$$\Pi_{\mathcal{C}_i^\perp}\xi + (I - \Pi_{\mathcal{C}_i^\perp})\xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Since $\operatorname{im}(I - \Pi_{\mathcal{C}_i^\perp}) = \mathcal{C}_i$ and $\mathcal{C}_i \subseteq \mathcal{V}_{(E_i,A_i,B_i)}$ we obtain

$$\Pi_{\mathcal{C}_i^\perp}\xi \in \mathcal{V}_{(E_i,A_i,B_i)} - (I - \Pi_{\mathcal{C}_i^\perp})\xi \subseteq \mathcal{V}_{(E_i,A_i,B_i)}.$$

as was to be shown. ■

Given Lemma 10 we are ready to conclude the following lemma.

Lemma 11. Consider the system (1) with switching signal (8). Then we have that $\min_u |x_u(t_{i+1}^-, t_0; x_0)| = \min_{u, u_{[t_i,t_{i+1})}=0} |\Pi_{\mathcal{C}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|$.

Proof. It follows that for any input u

$$x_u(t_{i+1}^-, t_0; x_0) = (\Pi_{\mathcal{C}_i^\perp} + (I - \Pi_{\mathcal{C}_i^\perp}))x_u(t_{i+1}^-, t_0; x_0)$$

and since $\operatorname{im}(I - \Pi_{\mathcal{C}_i^\perp})$ and $\operatorname{im}\Pi_{\mathcal{C}_i^\perp}$ are orthogonal subspaces, we have by Pythagoras' Theorem

$$|x_u(t_{i+1}^-, t_0; x_0)| = |\Pi_{\mathcal{C}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)| + |(I - \Pi_{\mathcal{C}_i^\perp})x_u(t_{i+1}^-, t_0; x_0)|.$$

Invoking $(I - \Pi_{\mathcal{C}_i^\perp})x_u(t_{i+1}^-, t_0; x_0) \in \mathcal{C}_i$ we can choose our input on $[t_i, t_{i+1})$ such that $|(I - \Pi_{\mathcal{C}_i^\perp})x_u(t_{i+1}^-, t_0; x_0)| = 0$, regardless of the input on $[t_0, t_i)$. What remains to minimize is $|\Pi_{\mathcal{C}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|$. This component is however not dependent on u on $[t_i, t_{i+1})$, because any effect of a non-zero input has will evolve in $\mathcal{C}_i$ and is therefore annihilated by $\Pi_{\mathcal{C}_i^\perp}$. It is only dependent on u on $[t_0, t_i)$ and thus we need to minimize $|\Pi_{\mathcal{C}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)|$ over u on $[t_0, t_i)$ and can assume that $u_{[t_i,t_{i+1})}$ is zero in the minimization. ■

In order to investigate the state at the end of an interval $[t_0, t_f)$ we introduce the time t_i reachable space.

Definition 12. Consider the system (1) with switching signal (8). The *time t_i reachable space* is defined by

$$\mathcal{R}^{[0,t_i]} = \{x \in \mathbb{R}^n \mid \exists u \text{ s.t. } x_u(t_i, 0) = x\}.$$

Proposition 13. Consider the following sequence of sets:

$$\mathcal{S}_0 = \mathcal{C}_0,$$
$$\mathcal{S}_{i+1} = e^{A_{i+1}^{\text{diff}}(t_{i+2}-t_{i+1})}\Pi_{i+1}\mathcal{S}_i + \mathcal{C}_{i+1},$$

then we have $\mathcal{S}_i = \mathcal{R}^{[0,t_{i+1}]}$ for $i \in \{0, 1, ..., \mathtt{n}\}$.

Proof. The controllable space and the reachable space coincide for $i = 0$ and hence for $i = 0$ the statement holds.

We now show the statement by induction. Hence assume that the statement holds for i. Then let $x_{t_{i+2}} \in \mathcal{R}^{[0,t_{i+2}]}$ and denote $\tau_{i+1} = t_{i+2} - t_{i+1}$. Then

$$x_{t_{i+2}} = e^{A_{i+1}^{\text{diff}}\tau_{i+1}}\Pi_{i+1}x_{u_i}(t_{i+1}^-, t_0, 0) + x_{u_{i+1}}(t_{i+2}^-, t_i).$$

We have that $x_{u_i}(t_{i+1}^-, t_0; 0) \in \mathcal{R}^{[0,t_{i+1}]} = \mathcal{S}_i$ by the inductivity assumption and furthermore $x_{u_{i+1}}(t_{i+2}^-, t_i) \in \mathcal{C}_{i+1}$ and hence $x_{t_{i+2}} \in e^{A_{i+1}^{\text{diff}}\tau_{i+1}}\Pi_{i+1}\mathcal{S}_i + \mathcal{C}_{i+1} = \mathcal{S}_{i+1}$.

Conversely, assume that $x_{t_{i+2}} \in \mathcal{S}_{i+1}$, then there exists $\eta_i \in \mathcal{S}_i$ and $\eta_{i+1} \in \mathcal{C}_{i+1}$ such that

$$x_{t_{i+2}} = e^{A_{i+1}^{\text{diff}}\tau_{i+1}}\Pi_{i+1}\eta_i + \eta_{i+1}.$$

Since $\eta_i \in \mathcal{S}_i = \mathcal{R}^{[0,t_{i+1}]}$ and $\eta_{i+1} \in \mathcal{C}_{i+1}$ we can choose u_i on $[0, t_{i+1})$ such that $x_{u_i}(t_{i+1}^-, t_0; 0) = \eta_i$ and u_{i+1} on $[t_{i+2}, t_{i+1})$ such that $x_{u_{i+1}}(t_{i+2}^-, t_{i+1}) = \eta_{i+1}$. Then we have that

$$x_{t_{i+2}} = e^{A_{i+1}^{\text{diff}}\tau_{i+1}}\Pi_{i+1}x_{u_{i+1}}(t_{i+1}^-, t_0; 0) + x_{u_{i+2}}(t_{i+2}^-, t_{i+1}),$$

and thus $x_{t_{i+2}} \in \mathcal{R}^{[0,t_{i+2}]}$ which concludes the proof. ■

Due to Lemma 11, we are interested on how we can influence $\Pi_{\mathcal{C}_i^\perp} x_u(t_{i+1}^-, t_0; x_0)$ and therefore we define the following subspace.

Definition 14. Consider the system (1) with switching signal (8). The *time t_i reachable uncontrollable space* is defined by

$$\widetilde{S}_i := \{x_0 \in (\mathcal{C}_i)^\perp \mid \exists u \text{ s.t. } x_u(t_{i+1}^-, 0) = x_0\}$$

Lemma 15. Consider the system (1) with switching signal (8). Then $\widetilde{S}_i = \Pi_{\mathcal{C}_i^\perp}\mathcal{S}_i$.

Proof. Consider $\zeta \in \widetilde{\mathcal{S}}_i$, then there exists an input u such that $x_u(t_{i+1}, t_0; 0) = \zeta$ and thus $\zeta \in \mathcal{S}_i$. Furthermore, $\zeta \in \mathcal{C}_i^\perp$ and thus $\Pi_{\mathcal{C}_i^\perp}\zeta = \zeta$ which shows $\widetilde{\mathcal{S}}_i \subseteq \Pi_{\mathcal{C}_i^\perp}\mathcal{S}_i$.

Conversely, consider $\zeta \in \Pi_{\mathcal{C}_i^\perp}\mathcal{S}_i$, then $\zeta = \Pi_{\mathcal{C}_i^\perp}\theta$ for some $\theta \in \mathcal{S}_i$. Invoking Proposition 13 choose u_0 such that $x_{u_0}(t_{i+1}^-, t_0; 0) = \theta$. Then since $\operatorname{im}(I - \Pi_{\mathcal{C}_i^\perp}) \subseteq \mathcal{C}_i$ there exists a u_1 such that

$$\zeta = \Pi_{\mathcal{C}_i^\perp}\theta = \theta - (I - \Pi_{\mathcal{C}_i^\perp})\theta,$$
$$= \theta - x_{u_1}(t_{i+1}, t_i).$$

By linearity of solutions there thus exists an input $\bar{u}$ such that $x_{\bar{u}}(t_{i+1}^-, t_0; 0) = x_{u_0}(t_{i+1}^-, t_0; 0) - x_{u_1}(t_{i+1}, t_i) = \zeta$ and thus $\zeta \in \widetilde{\mathcal{S}}_i$. Hence $\Pi_{\mathcal{C}_i^\perp}\mathcal{S}_i \subseteq \widetilde{\mathcal{S}}_i$, completing the proof. ■

As will turn out, the state projected to $\mathcal{C}_i^\perp$ at $t = t_{i+1}$ can be decomposed into a reachable component and a component resulting from the initial condition. To that extent we define the x_0-uncontrollable orthogonal component.

Definition 16. Consider the system (1) with switching signal (8). The *x_0-uncontrollable orthogonal component* $\xi_i(x_0)$ is defined by the following sequence

$$\xi_0(x_0) = \Pi_{\mathcal{C}_0^\perp} e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0,$$
$$\xi_{i+1}(x_0) = \Pi_{\mathcal{C}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}}(t_{i+2}-t_{i+1})}\Pi_{i+1}\xi_i(x_0).$$

For convenience we write $x_u(t_i, x_0)$ as a shorthand notation for $x_u(t_i, t_0; x_0)$.

Lemma 17. Consider the switched system (1) with switching signal (8). Then for all $i \in \{0, ..., \mathtt{n}\}$ we have

$$\Pi_{\mathcal{C}_i^\perp} x_u(t_{i+1}^-, x_0) - \xi_i(x_0) \in \widetilde{\mathcal{S}}_i.$$

Proof. For $i = 0$ we have that

$$\Pi_{\mathcal{C}_0^\perp} x_u(t_1^-, x_0) = \Pi_{\mathcal{C}_0^\perp}(e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + x_u(t_1, t_0)),$$
$$= \Pi_{\mathcal{C}_0^\perp} e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0,$$
$$= \xi_0(x_0).$$

Hence the statement holds. Now assume that the claim holds for i, then for $i + 1$. If we denote $\tau_{i+1} = t_{i+2} - t_{i+1}$ we obtain for some $\eta_i \in \tilde{\mathcal{S}}_i$ and $\eta_{i+1} \in \widetilde{\mathcal{S}}_{i+1}$

$$
\begin{aligned}
\Pi_{\mathcal{C}_{i+1}^\perp} x(t_{i+2}^-, x_0) &= \Pi_{\mathcal{C}_{i+1}^\perp}(e^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}}\Pi_{i+1}x(t_{i+1}^-, x_0) \\
&\quad + x_{u_i}(t_{i+2}^-, t_{i+1})) \\
&= \Pi_{\mathcal{C}_{i+1}^\perp} e^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}}\Pi_{i+1}(\Pi_{\mathcal{C}_i^\perp} x(t_{i+1}^-, x_0) \\
&\quad + (I - \Pi_{\mathcal{C}_i^\perp})x(t_{i+1}^-, x_0)) \\
&= \Pi_{\mathcal{C}_{i+1}^\perp} e^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}}\Pi_{i+1}(\xi_i(x_0) + \eta_i) \\
&\quad + x_{u_i}(t_i^-, t_{i-1})) \\
&= \Pi_{\mathcal{C}_{i+1}^\perp}(e^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}}\Pi_{i+1}\xi_i(x_0) \\
&\quad + e^{A_{i+1}^{\mathrm{diff}}\tau_{i+1}}\Pi_{i+1}(\eta_i + x_{u_i}(t_i^-, t_{i-1}))) \\
&= \xi_{i+1}(x_0) + \eta_{i+1},
\end{aligned}
$$

which proves the desired result. ■

Lemma 18. Consider the system (1) with switching signal (8). Then for all $x_0 \in \mathbb{R}^n$ we have that for all $i \in \{1, ..., \mathtt{n}\}$

$$\min_u |x_u(t_i^-, x_0)| = \operatorname{dist}(\xi_i(x_0), \tilde{\mathcal{S}}_i). \tag{9}$$

Proof. Due to the orthogonality of $\operatorname{im}\Pi_{\mathcal{C}_i^\perp}$ and $\operatorname{im}(I - \Pi_{\mathcal{C}_i^\perp})$ we obtain

$$
\begin{aligned}
\min_u |x_u(t_i^-, x_0)|^2 &= \min_u |(\Pi_{\mathcal{C}_i^\perp} + (I - \Pi_{\mathcal{C}_i^\perp}))x_u(t_i^-, x_0)|^2, \\
&= \min_u |\Pi_{\mathcal{C}_i^\perp} x_u(t_i^-, x_0)|^2 \\
&\quad + |(I - \Pi_{\mathcal{C}_i^\perp})x_u(t_i^-, x_0)|^2, \\
&\overset{*}{=} \min_u |\Pi_{\mathcal{C}_i^\perp} x_u(t_i^-, x_0)|^2, \\
&\overset{**}{=} \min_{\eta \in \tilde{\mathcal{S}}_i} |\xi_i(x_0) + \eta_i|^2, \\
&= \operatorname{dist}(\xi_i(x_0), \tilde{\mathcal{S}}_i)^2
\end{aligned}
$$

where equation $*$ follows with similar arguments as in Lemma 11 and equation $**$ is implied by Lemma 17. ■

This leads us to the main theorem on interval stabilizability of switched DAEs.

Theorem 19. The system (1) with switching signal (8) is interval stabilizable if and only if for all $x_0 \in \mathbb{R}^n$

$$\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \tilde{\mathcal{S}}_{\mathtt{n}}) < |x_0|$$

Proof. Assume that the system is interval stabilizable and that interval stabilizability is achieved by $\tilde{u}$. Then it follows that there exists a u such that

$$
\begin{aligned}
\operatorname{dist}(\xi_{\mathtt{n}}(x_0)) &= \min_u |(x_u(t_f^-, x_0)| \\
&\leqslant |x_{\tilde{u}}(t_f^-, x_0)|, \\
&\leqslant \beta(|x_0|, t_f) < x_0
\end{aligned}
$$

Conversely if $\operatorname{dist}(\xi_{\mathtt{n}}(x_0)) = \min_u |(x_u(t_f^-, x_0)| < |x_0|$ then obviously for the u that attains this minimum there exists a class $\mathcal{KL}$ function $\beta(r, t)$ such that $\beta(r, t_f) < r$ and $|x_u(t^+, x_0; t_p)| \leqslant \beta(|x_u(t_p^-, x_0; t_p)|, t - t_p), \quad \forall t \in [t_p, t_q)$. This concludes the proof. ■

Remark 20. The conditions stated in Theorem 19 need to be valid for an infinite amount of points, but only need to be verified for an finite amount of vectors. If the basis for $\mathbb{R}^n$ is orthogonal, we only need to verify the conditions for each base vector. This follows from the fact that if x_0 and y_0 are orthogonal that for $z_0 = x_0 + y_0$

$$
\begin{aligned}
\operatorname{dist}(\xi_{\mathtt{n}}(z_0), \mathcal{S}_{\mathtt{n}})^2 &\leqslant \operatorname{dist}(\xi_{\mathtt{n}}(x_0), \mathcal{S}_{\mathtt{n}})^2 + \operatorname{dist}(\xi_{\mathtt{n}}(y_0), \mathcal{S}_{\mathtt{n}})^2, \\
&< |x_0|^2 + |y_0|^2, \\
&= |x_0 + y_0|^2, \\
&= |z_0|^2.
\end{aligned}
$$

Example 21. Consider the following switched DAE defined on the interval $[0, 2\ln(2)]$ with a switch at $t = \ln(2)$.

$$
\Sigma_\sigma \begin{cases} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1&0&0\\0&2&0\\1&-1&1 \end{bmatrix} x(t) + \begin{bmatrix} 1\\0\\0 \end{bmatrix} u(t), & 0 \leqslant t < t_1, \\ \begin{bmatrix} 0&0&1\\0&1&0\\0&0&0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1&1&1\\0&-1&0\\1&0&0 \end{bmatrix} x(t) & t_1 \leqslant t \leqslant t_f. \end{cases}
$$

Note that neither of the two modes of the switched system is stabilizable. In order to show that the system is interval stabilizable, use the Wong sequences to compute A_0^{diff}, A_1^{diff} and Π_0 and Π_1. We have that

$$\operatorname{im}\Pi_0 = \operatorname{im}\begin{bmatrix} 1&1&0\\0&2&0\\-1&1&0 \end{bmatrix} = \mathcal{V}_{(E_0, A_0)}. \tag{10}$$

Furthermore, we compute

$$\mathcal{C}_0 = \operatorname{span}\left\{\begin{bmatrix} 1\\0\\-1 \end{bmatrix}\right\},\ \Pi_{\mathcal{C}_0^\perp} = \frac{1}{2}\begin{bmatrix} 1&0&-1\\0&2&0\\-1&0&1 \end{bmatrix},\ \Pi_{\mathcal{C}_1^\perp} = I,$$

Since the column vectors of Π_0 are orthogonal, we only need to verify the the conditions of Lemma 18 for the column vectors of Π_0, and a basis vector complementing $\operatorname{im}\Pi_0$ to a basis of $\mathbb{R}^3$ in an orthogonal way. Hence we consider the basis

$$v_1 = \begin{bmatrix} 1\\0\\-1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1\\2\\1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1\\-1\\1 \end{bmatrix}.$$

It follows that

$$\xi_1(v_1) = 0,\ \xi_1(v_2) = \begin{bmatrix} 0 & 4 & 10 \end{bmatrix}^\top,\ \xi_1(v_3) = \begin{bmatrix} 0 & -2 & -11 \end{bmatrix}^\top.$$

Computing the time t_1 reachable uncontrollable space yields that $\tilde{\mathcal{S}}_1 = \operatorname{span}\left\{\begin{bmatrix} 0 & 0 & 2 \end{bmatrix}^\top\right\}$ from which we calculate that

$$
\begin{aligned}
\operatorname{dist}(\xi_1(v_1), \tilde{\mathcal{S}}_1) &= 0 < |v_1|, \\
\operatorname{dist}(\xi_1(v_2), \tilde{\mathcal{S}}_1) &= 4 < |v_2|, \\
\operatorname{dist}(\xi_1(v_3), \tilde{\mathcal{S}}_1) &= 2 < |v_3|.
\end{aligned}
$$

Hence we can conclude that the system is interval stabilizable. Note that the input on $[0, t_s]$ is necessary for achieving interval stability.

3.1 Controllability of switched DAEs

The approach taken in the previous section does not only lead to results on stabilizability, but can also be used to find conditions for controllability of switched DAEs. To see this, we first state the following lemma.

Lemma 22. Consider the switched DAE (1) with switching signal (8). The initial condition $x_0 \in \mathcal{R}$ of the switched system is controllable if and only if $\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \tilde{S}_{\mathtt{n}}) = 0$.

Proof. ($\Rightarrow$) Assume that x_0 is a controllable initial value. Then there exists an input u such that $x_u(t_f^-, x_0) = 0$ and hence $\min_u |x_u(t_f, x_0)| = 0$. Then it follows from Lemma 18 that $\operatorname{dist}(\xi_{\mathtt{n}}, \tilde{S}_{\mathtt{n}}) = 0$.

($\Leftarrow$) Assume that $\operatorname{dist}(\xi_{\mathtt{n}}, \tilde{S}_{\mathtt{n}}) = 0$. Then by Lemma 18 we have that $\min_u |x_u(t_f, x_0)| = 0$. The input attaining this minimum controls the initial value to 0 and hence x_0 is controllable. ■

In order to obtain results on controllability of switched DAEs we define the following sequence of subspaces

$$\Psi_0 = \operatorname{im} \Pi_{\mathcal{C}_0^\perp} e^{A_0^{\text{diff}}(t_1 - t_0)} \Pi_0,$$

$$\Psi_{i+1} = \operatorname{im} \Pi_{\mathcal{C}_{i+1}^\perp} e^{A_{i+1}^{\text{diff}}(t_{i+1} - t_i)} \Pi_{i+1} \Psi_i.$$

Theorem 23. Consider the switched DAE (1) with switching signal (8). The system is controllable if and only if

$$\Psi_{\mathtt{n}} \subseteq \tilde{\mathcal{S}}_{\mathtt{n}}.$$

Proof. ($\Rightarrow$) Assume that the system is controllable. Then for all $x_0 \in \mathbb{R}^n$ we have that $\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \tilde{\mathcal{S}}_{\mathtt{n}}) = 0$. Furthermore, for all $\psi \in \Psi_{\mathtt{n}}$ there exists an x_0 such that $\psi = \xi_{\mathtt{n}}(x_0)$. Hence for all $\psi \in \Psi_{\mathtt{n}}$ we have that $\operatorname{dist}(\psi, \tilde{\mathcal{S}}_{\mathtt{n}}) = 0$, which implies that $\psi \in \tilde{\mathcal{S}}_{\mathtt{n}}$ and thus $\Psi_{\mathtt{n}} \subseteq \tilde{\mathcal{S}}_{\mathtt{n}}$.

($\Leftarrow$) Assume that $\Psi_{\mathtt{n}} \subseteq \tilde{\mathcal{S}}_{\mathtt{n}}$. Then since for all x_0 and thus for all $\xi_{\mathtt{n}}(x_0)$ there exists a $\psi \in \Psi_{\mathtt{n}}$ such that $\xi_{\mathtt{n}}(x_0) = \psi$. Hence $\xi_{\mathtt{n}}(x_0) \subseteq \tilde{\mathcal{S}}_{\mathtt{n}}$, which implies that for all x_0 we have that $\operatorname{dist}(\xi_{\mathtt{n}}(x_0), \tilde{\mathcal{S}}_{\mathtt{n}}) = 0$. ■

Remark 24. The result of Theorem 23 gives a condition for controllability that only require computations that run forward in time. This is in contrast to the result of Küsters et al. (2015), where the last mode is considered first and the computation runs backwards in time.

Remark 25. All results in this paper can be applied to switched ordinary differential equations (ODEs) without difficulty. In the case of an ODE we have $E = I$, $\Pi = I$, $B^{\text{diff}} = B$ and $A^{\text{diff}} = A$. Plugging this into the conditions in this paper yields the result for switched ODEs.

4. CONCLUSION

This paper considered stabilizability of switched DAEs. We introduced the notion of interval stabilizability. Moreover, we showed that –under a uniformity assumption– interval stabilizability implies stabilizability. Necessary and sufficient conditions for interval stabilizability of switched DAEs are given. In addition, the method to analyse the interval stabilizability was used to obtain a necessary and sufficient condition for controllability.

As a future direction of research, a natural extension is to obtain results on impulse free stabilization. Impulse controllability of the system is an obvious necessary condition for impulse free stabilizability. However, necessary and sufficient conditions for impulse free stabilizability of non autonomous switched DAEs are yet to be formulated.

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