A time-varying Gramian based model reduction approach for Linear Switched Systems

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Abstract: We propose a model reduction approach for switched linear system based on a balanced truncation reduction method for linear time-varying systems. The key idea is to approximate the piecewise-constant coefficient matrices with continuous time-varying coefficients and then apply available balance truncation methods for (continuous) time-varying systems. The proposed method is illustrated with a low dimensional academic example.

Keywords: Balanced truncation, model reduction, time-varying, differential Lyapunov equations, Gramians.

1. INTRODUCTION

Model order reduction (MOR) turned out to be a important tool in the context of the simulation of various applications and problems over the last decades, see e.g. Antoulas (2005). The main purpose of MOR is to find a lower order approximation of a dynamical system which can be used in simulation and optimization instead of the original (large) system.

In the last decades, switched systems gained much interest as a modelling framework with applications in mechanical and aeronautical systems, power converters and automotive industry, for details we refer to Liberzon (2003), Sun and Ge (2005). Clearly, MOR of switched systems is highly relevant for large scale applications in the systems and control community.

So far, many works have been done on MOR of switched systems, we refer to the following contributions: Monshizadeh et al. (2012), Papadopoulus and Prandini (2016), Schulze and Unger (2018), Gosea et al. (2018b), Pontes Duff et al. (2020), Gosea et al. (2018a), Petreczky et al. (2013). In contrast to these references, we view here the switched system as a special *time-varying* system and aim for a (time-varying) model-reduction depending on a given (known) switching signal.

In this paper, we study linear switched system described by the equations:

$$S_\sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad x(0) = \mathbf{0}, \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{1}$$

where $\sigma : \mathbb{R} \to M = \{0, 1, 2, \cdots, f\}$ is a given piecewise constant function with finitely many switching times $0 < t_1 < t_2 < \cdots < t_f$ in the bounded interval $[0, t_{f+1})$ of interest. The system matrices are $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, where $i \in M$ and n is the number of state variables, called the order of the subsystems. We can assume that the i-th mode is active in the interval $[t_i, t_{i+1})$, for $i = 0, 1, \cdots, f$ (where $t_0 := 0$) and so the duration of i-th mode is $\tau_i = t_{i+1} - t_i$. The input function u is assumed to be piecewise continuous and bounded.

The notion of balanced realization of state space systems was introduced extensively in Moore (1981) and connections to the Hankel-norm approximations were drawn in Glover (1984) and Antoulas (2005). The concept was extended to time-varying systems in Shokoohi et al. (1983) and Verriest and Kailath (1983) where input-output balancing was discussed for various Gramians. In particular, model reduction is considered for exponentially stable and uniformly completely controllable and observable systems. Uniform and infinite-interval balanced realization are also investigated in Shokoohi et al. (1983). Stability of the reduced order model is studied in Shokoohi et al. (1984). In Sandberg and Rantzer (2004), error bounds are given in the finite interval case.

In this paper, we propose a balanced truncation method for model reduction of linear switched system (1). To the authors' knowledge, this is the first attempt to solve the model reduction for switched system by considering it as a time-varying system. It is known, that the balancing procedure involves computing the controllability and observability Gramians from the pair of differential Lyapunov equations, for details we refer Shokoohi et al. (1983); Lang et al. (2016).

The paper is organized as follows. The problem formulation and some preliminaries are discussed in Section 2. Section 3 reviews the steps of the computation of the balanced realization by constructing balancing coordinates. In Section 4, we study the notion of time-varying balanced truncation for switched systems, so that a reduced order model can be constructed. Finally, some numerical results are presented in Section 5.

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2. PROBLEM STATEMENT AND PRELIMINARIES

In this section, we present examples for motivation and some preliminaries are given.

The basic idea of model reduction is to represent a complex dynamical system by a much simpler one. This may refer to many different techniques. In this paper, we focus only on projection-based method, in particular on balanced truncation (BT) which is the most prominent projection-based model reduction methods.

It can be shown that the states of a realization that are either uncontrollable or unobservable do not appear in the corresponding transfer function matrix. Furthermore, states that are almost uncontrollable or unobservable can be omitted from the realization with little effect on the input-output behavior of the system.

In projection-based technique, the original state variables x is approximated by $\widehat{T}\widehat{x}$ for some $\widehat{T} \in \mathbb{R}^{n\times r}$ and where $\widehat{x} : \mathbb{R} \to \mathbb{R}^r$ is the solution of the reduced order model

$$\hat{S}_\sigma : \begin{cases} \dot{\widehat{x}}(t) = \hat{A}_\sigma \widehat{x}(t) + \hat{B}_\sigma u(t), \quad \widehat{x}(0) = \mathbf{0}, \\ \hat{y}(t) = \hat{C}_\sigma \widehat{x}(t), \end{cases} \tag{2}$$

with reduced system matrices $\hat{A}_i \in \mathbb{R}^{r\times r}$, $\hat{B}_i \in \mathbb{R}^{r\times m}$, $\hat{C}_i \in \mathbb{R}^{p\times r}$, for $i \in M$ and $r \ll n$.

A question arises: Is the balanced truncation approach applicable to switched systems by reducing each individual subsystems independently?

The following simple example shows that this naive idea doesn't work in general.

Example 1. Consider the switched system (1) with $A_0 = A_1 = \left[\begin{smallmatrix} -0.5 & 0.01 \\ 0.01 & -0.5 \end{smallmatrix}\right]$, $B_0 = C_0^\top = \left[\begin{smallmatrix} 0.001 \\ 1 \end{smallmatrix}\right]$ and $B_1 = C_1^\top = \left[\begin{smallmatrix} 1 \\ 0.001 \end{smallmatrix}\right]$. The switching signal is given by $\sigma(t) = 0$ on $t \in [0, 1)$ and $\sigma(t) = 1$ on $[1, 2)$. The input for this example is $u(t) = (\sin(5t) + 0.05)e^{-.5t}$.

Applying balanced truncation on the two intervals $[0, 1)$ and $[1, 2)$ separately, we get the same one-dimensional reduced order model for each mode:

$$\dot{\widehat{x}}(t) = -0.5\,\widehat{x}(t) + u(t),$$
$$\widehat{y}(t) = \widehat{x}(t).$$

Figure 1 shows the output of the original and reduced system. Clearly, the output of the reduced model does not match the original output after the switch, although each individual mode is approximated sufficiently well with a small (known) error bound. ◇

The above example shows, that a piecewise balanced truncation method will not result in good approximations of a switched system in general. The underlying problem is the time-varying nature of the switched system, so we propose to use a time-varying balanced truncation method for the switched system.

2.1 Approximate time-varying system

The available balanced truncation methods for time-varying system (as discussed in the Introduction) assume that the coefficient matrices are at least continuous. However, the switched system (1) seen as a time-varying system has *discontinuous* coefficient matrices. In order to

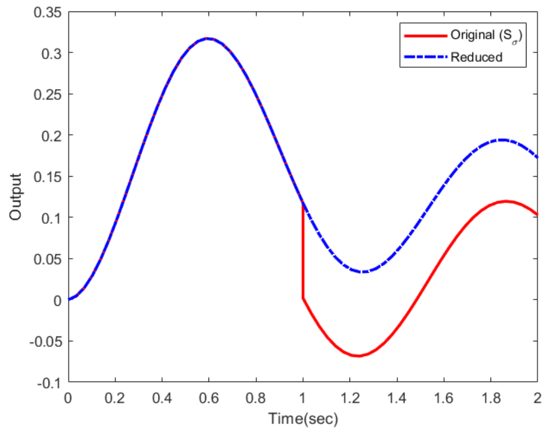

Fig. 1. Outputs of Example 1 of original system and 1st order reduced system.

still be able to use the existing methods we propose the following approximation of the switched system (1) by the following (continuously) time-varying systems:

$$S_\varepsilon : \begin{cases} \dot{x}(t) = A_\varepsilon(t)x(t) + B_\varepsilon(t)u(t), \quad x(0) = \mathbf{0}, \\ y(t) = C_\varepsilon(t)x(t), \end{cases} \tag{3}$$

where $A_\varepsilon(\cdot)$, $B_\varepsilon(\cdot)$ and $C_\varepsilon(\cdot)$ are defined as follows:

$$(A_\varepsilon(t), B_\varepsilon(t), C_\varepsilon(t)) := (A_0, B_0, C_0), \quad : t \in [0, t_1),$$

$$A_\varepsilon(t) = \begin{cases} A_{i-1} + \dfrac{t-t_i}{\varepsilon}(A_i - A_{i-1}) & : t \in [t_i, t_i + \varepsilon), \\ A_i & : t \in [t_i + \varepsilon, t_{i+1}) \end{cases}$$

$$B_\varepsilon(t) = \begin{cases} B_{i-1} + \dfrac{t-t_i}{\varepsilon}(B_i - B_{i-1}) & : t \in [t_i, t_i + \varepsilon), \\ B_i & : t \in [t_i + \varepsilon, t_{i+1}) \end{cases}$$

$$C_\varepsilon(t) = \begin{cases} C_{i-1} + \dfrac{t-t_i}{\varepsilon}(C_i - C_{i-1}) & : t \in [t_i, t_i + \varepsilon), \\ C_i & : t \in [t_i + \varepsilon, t_{i+1}) \end{cases}$$

where $i = 1, \cdots, f$.

Remark 2. It is clear that the coefficient matrices in time-varying system (3) are bounded and continuous even for small ε, since the term $\frac{t-t_i}{\varepsilon} \in (0, 1)$, $\forall\, t \in (t_i, t_i + \varepsilon)$. Furthermore, the coefficient matrices are differentiable almost everywhere, however, the derivatives grow proportional to $1/\varepsilon$.

Remark 3. Consider (1) and (3) with the same input u and denote with x_σ and x_ε the corresponding solutions. Using an inductive argument, it is easy to see that there exist $\overline{\varepsilon} > 0$ and a constant $c > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon})$

$$\|x_\varepsilon(t) - x_\sigma(t)\| < c\,\varepsilon \quad \forall t \in [0, t_{f+1}). \tag{4}$$

The constant c depends on the input u, on the length of the interval $[0, t_{f+1})$ and on the (magnitude of the) matrices A_i and B_i. A similar bound for the output does *not* hold for *all* $t \in [0, t_{f+1})$ in general because the output of (1) is discontinuous (because the C-matrix switches) while the output of (3) is continuous. Nevertheless, away from the switching times (where $C_\varepsilon(t) = C_{\sigma(t)}$) the error bound (4) trivially carries over to a corresponding error bound for the outputs.

Example 4. We consider again the switched system from Example 1 and approximate it by (3) with $\varepsilon = 0.1$ and $\varepsilon = 10^{-3}$. Figure 2 shows the output of the approximation compared to the output of the original switched system

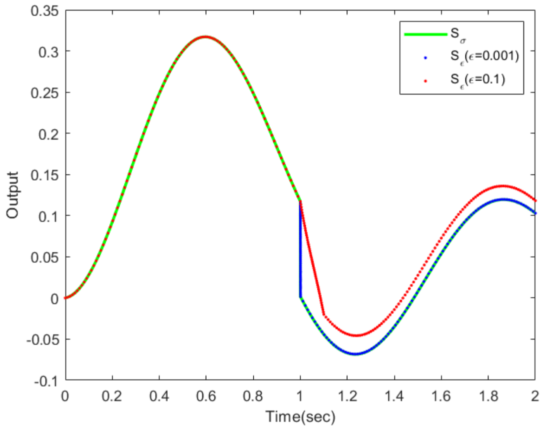

Fig. 2. Comparison for Example 1 between the output of the original system and proposed approximation.

and it is clearly visible that indeed (3) is a good approximation of (1) for sufficiently small ε. $\diamond$

Our goal is now to derive a reduced model for the approximated system (3) of the form

$$\hat{S}_\varepsilon : \begin{cases} \dot{\widehat{x}}(t) = \widehat{A}_\varepsilon(t)\widehat{x}(t) + \widehat{B}_\varepsilon(t)u(t), & \widehat{x}(0) = \mathbf{0}, \\ \hat{y}(t) = \widehat{C}_\varepsilon(t)\widehat{x}(t), \end{cases} \tag{5}$$

We conclude this section by recalling that the solution of the time-varying state equations in system (3) is

$$x(t) = \Phi_\varepsilon(t,0)x(0) + \int_0^t \Phi_\varepsilon(t,\tau)B_\varepsilon(\tau)u(\tau)d\tau$$

where $\Phi_\varepsilon(t,\tau)$, known as state transition matrix, is the unique solution of

$$\frac{\partial}{\partial t}\Phi_\varepsilon(t,0) = A_\varepsilon(t)\Phi_\varepsilon(t,0)$$

with the initial condition $\Phi_\varepsilon(0,0) = I$.

3. BALANCED REALIZATIONS

In this section, we discuss the balanced realization of time-varying system (3) and establish the conditions for the existence of a balancing transformation. Further, we show how the system can be reduced to obtain a lower order model in the input-output description.

It is well known that controllability and observability are essential concepts for balancing theory. Here, we define the controllability and observability Gramians which characterize the controllability and observability subspaces. In particular, we are interested in the input-output balancing of the system over finite time interval $[0, t_{f+1}]$ with respect to the controllability and observability Gramians.

3.1 Controllability and observability Gramians

Inspired by the Shokoohi et al. (1983); Verriest and Kailath (1983); Sandberg and Rantzer (2004) we define the time-varying controllability and observability Gramians as follows.

Definition 5. The time-varying controllability and observability Gramians at $t \in [0, t_{f+1}]$ are defined as

$$P_\varepsilon(t) := P_0 + \int_0^t \Phi_\varepsilon(t,\tau)B_\varepsilon(\tau)B_\varepsilon^\top(\tau)\Phi_\varepsilon^\top(t,\tau)d\tau, \tag{6}$$

$$Q_\varepsilon(t) := Q_f + \int_t^{t_{f+1}} \Phi_\varepsilon^\top(\tau,t)C_\varepsilon^\top(\tau)C_\varepsilon(\tau)\Phi_\varepsilon(\tau,t)d\tau, \tag{7}$$

where $P_0, Q_f \in \mathbb{R}^{n\times n}$ are some symmetric positive definite matrices.

Remark 6. If the dynamical system is considered on the whole time axis (as is usually the case for the time-invariant case, then P_0 and Q_f are chosen such that $P_\varepsilon(-\infty) = 0$ and $Q_\varepsilon(\infty) = 0$. This however makes it necessary to assume that the linear system is asymptotically stable. In our situation, where we are only interested in the behavior on the finite interval $[0, t_{f+1})$, we can basically arbitrarily assign the values for P_0 and Q_f; in other words, we can choose arbitrarily how the system behaves outside the interval of interest as long as it is exponentially stable.

Now we define certain controllable and observable notions for a time-varying system.

Definition 7. (i) The pair $(A_\varepsilon(\cdot), B_\varepsilon(\cdot))$ is called boundedly completely controllable on $[0, t_{f+1}]$ if there exist two constants $0 < \alpha < \bar{\alpha} < \infty$ and $P_\varepsilon(t) = P_\varepsilon^\top(t)$ given by (6) such that

$$0 < \alpha I \leq P_\varepsilon(t) \leq \bar{\alpha} I < \infty, \quad \forall\, t \in [0, t_{f+1}].$$

(ii) The pair $(A_\varepsilon(\cdot), C_\varepsilon(\cdot))$ is called boundedly completely observable on $[0, t_{f+1}]$ if there exist $0 < \beta < \bar{\beta} < \infty$ and $Q_\varepsilon(t) = Q_\varepsilon^\top(t)$ given by (7) such that

$$0 < \beta I \leq Q_\varepsilon(t) \leq \bar{\beta} I < \infty, \quad \forall\, t \in [0, t_{f+1}].$$

It can be shown that these Gramians are the solution of the following differential Lyapunov equations:

$$\dot{P}_\varepsilon(t) = A_\varepsilon(t)P_\varepsilon(t) + P_\varepsilon(t)A_\varepsilon^\top(t) + B_\varepsilon(t)B_\varepsilon^\top(t) \tag{8}$$

$$\dot{Q}_\varepsilon(t) = -(A_\varepsilon^\top(t)Q_\varepsilon(t) + Q_\varepsilon(t)A_\varepsilon(t) + C_\varepsilon^\top(t)C_\varepsilon(t)) \tag{9}$$

with initial conditions $P_\varepsilon(0) = P_0$, $Q_\varepsilon(t_{f+1}) = Q_f$.

3.2 Computational aspects of Gramians

The Gramians can be computed either by solving the differential Lyapunov equations (8), (9) (see Behr et al. (2019)), or by explicitly computing the transition matrix and integration. For the latter approach the following property of the Gramians can be exploited.

Lemma 8. Assume the time steps $s_0 < s_1 < \cdots < s_k$, $k \in \mathbb{N}$ then the Gramians defined in (6) and (7) at s_i can be calculated recursively as follows

$$\begin{aligned} P_\varepsilon(s_i) &= \Phi_\varepsilon(s_i, s_{i-1})P_\varepsilon(s_{i-1})\Phi_\varepsilon^\top(s_i, s_{i-1}) \\ &\quad + \int_{s_{i-1}}^{s_i} \Phi_\varepsilon(s_i,\tau)B_\varepsilon(\tau)B_\varepsilon^\top(\tau)\Phi_\varepsilon^\top(s_i,\tau)d\tau \\ Q_\varepsilon(s_i) &= \Phi_\varepsilon^\top(s_{i+1}, s_i)Q_\varepsilon(s_{i+1})\Phi_\varepsilon(s_{i+1}, s_i) \\ &\quad + \int_{s_i}^{s_{i+1}} \Phi_\varepsilon^\top(\tau, s_i)C_\varepsilon^\top(\tau)C_\varepsilon(\tau)\Phi_\varepsilon(\tau, s_i)d\tau \end{aligned}$$

for $i = 1, 2, \cdots, k$.

Proof. This is simple consequence from the definition and the property of the transition matrix, in particular,

$$\Phi_\varepsilon(s_i,\tau) = \Phi_\varepsilon(s_i, s_{i-1})\Phi_\varepsilon(s_{i-1},\tau)$$

for any $i = 1, 2, \ldots, k$ and $\tau \in \mathbb{R}$. $\blacksquare$

The Gramians will be used in the next subsection to obtain a coordinate transformation which results in a balanced system.

3.3 Balancing of time-varying system

The balancing of a system is accomplished by a transformation of the state vector. In time-varying systems, such transformation is also time-varying and we need to restrict the class of allowed coordinate transformations to so called Lyapunov transformations.

Definition 9. The mapping $T : [0, t_{f+1}] \to \mathbb{R}^{n\times n}$ is called a Lyapunov transformation (or, short, Lyapunov) iff $T(t)$, $T(t)^{-1}$ and $\dot{T}(t)$ are well defined and bounded on $[0, t_{f+1}]$.

We now highlight that bounded controllability/observability ensures that the Gramians $P_\varepsilon(t)$ and $Q_\varepsilon(t)$ are Lyapunov transformations.

Lemma 10. Assume that the system (3) is boundedly completely controllable and observable on $[0, t_{f+1}]$ then $P_\varepsilon(t)$ and $Q_\varepsilon(t)$ are Lyapunov.

Proof. Boundedness of $P_\varepsilon, Q_\varepsilon$ and its inverses on $[0, t_{f+1}]$ is direct consequence from the definition of bounded complete controllability/observability. Furthermore, P_ε and Q_ε are clearly countinously differentiable (as the integral of a continuous function), hence $\dot{P}_\varepsilon$ and $\dot{Q}_\varepsilon$ are bounded, because they are continuous on $[0, t_{f+1}]$. ■

The idea of balanced truncation is to find a coordinate transformation such that the controllability and observability Gramians are the same and diagonal, so states can be identified which are simultaneously difficult to control and difficult to observe. We now proceed to study the conditions under which controllability and observability Gramians can be diagonalized by Lyapunov transformations.

Let $T(t)$ be a Lyapunov transformations. Then the state equations in system (3) can be transformed under the transformation

$$x(t) = T(t)\bar{x}(t)$$

resulting in

$$\dot{\bar{x}}(t) = \bar{A}_\varepsilon(t)\bar{x}(t) + \bar{B}_\varepsilon(t)u(t), \quad \bar{x}(0) = \mathbf{0},$$
$$y(t) = \bar{C}_\varepsilon(t)\bar{x}(t)$$

with

$$\bar{A}_\varepsilon(t) := T(t)^{-1}(A_\varepsilon(t)T(t) - \dot{T}(t)),$$
$$\bar{B}_\varepsilon(t) := T(t)^{-1}B_\varepsilon(t),$$
$$\bar{C}_\varepsilon(t) := C_\varepsilon(t)T(t).$$

In general, when there is a Lyapunov transformation such that one system can be transformed to another as above, we call both systems equivalent. For two equivalent systems it is easily seen that the controllability and observability Gramians transform to

$$\bar{P}_\varepsilon(t) = T(t)^{-1}P_\varepsilon(t)T(t)^{-\top},$$
$$\bar{Q}_\varepsilon(t) = T(t)^\top Q_\varepsilon(t)T(t).$$

In particular,

$$\bar{P}_\varepsilon(t)\bar{Q}_\varepsilon(t) = T(t)^{-1}P_\varepsilon(t)Q_\varepsilon(t)T(t)$$

which implies that the eigenvalues of the product of Gramians are invariant under such transformation.

It is well known (Shokoohi et al., 1983) that the transformation matrices in an eigenvalue decomposition of $P_\varepsilon(t)Q_\varepsilon(t)$ may not be Lyapunov, although the Gramians are Lyapunov. Therefore, we need an additional assumption to ensure that the forthcoming coordinate transformations will be Lyapunov.

Assumption 1. Consider the time-varying system (3) with corresponding controllability and observability Gramians P_ε and Q_ε. Assume that the eigenvalues of $P_\varepsilon(t)Q_\varepsilon(t)$ only cross each other at isolated points, and they do not have common derivatives at their crossing points. ◇

In the following, we compute balancing transformation for the system (3) using the Gramians.

Lemma 11. (Lang et al. (2016)). Assume the time-varying system (3) is boundedly completely controllable and observable on $[0, t_{f+1}]$ and the controllability/observability Gramians $P_\varepsilon, Q_\varepsilon$ satisfy Assumption 1. Then there exists a Lyapunov transformation $T_\varepsilon : [0, t_{f+1}] \to \mathbb{R}^{n\times n}$ such that

$$T_\varepsilon(t)^{-1}P_\varepsilon(t)T_\varepsilon(t)^{-\top} = T_\varepsilon(t)^\top Q_\varepsilon(t)T_\varepsilon(t) = \Pi_\varepsilon(t),$$

for all $t \in [0, t_{f+1}]$, for a diagonal matrix $\Pi_\varepsilon(t)$. In fact,

$$T_\varepsilon(t) = R_\varepsilon(t)U_\varepsilon(t)\Pi_\varepsilon(t)^{-1/2},$$
$$T_\varepsilon(t)^{-1} = \Pi_\varepsilon(t)^{-1/2}V_\varepsilon(t)^\top L_\varepsilon(t)^\top,$$

where $U_\varepsilon(t)\Pi_\varepsilon(t)V_\varepsilon(t)^\top$ is the singular value decomposition of $R_\varepsilon(t)^\top L_\varepsilon(t)$ and where $R_\varepsilon(t)R_\varepsilon(t)^\top = P_\varepsilon(t)$ and $L_\varepsilon(t)L_\varepsilon(t)^\top = Q_\varepsilon(t)$ are the Cholesky decompositions of P_ε and Q_ε, respectively.

Proof. First observe bounded complete controllability / observability, ensures that $P_\varepsilon(t)Q_\varepsilon(t)$ is invertible, hence $\Pi_\varepsilon(t)^{-1/2}$ is well defined. Furthermore, it can be shown that Assumption 1 ensures that all involved matrices to define T_ε are Lyapunov (for details see Lang et al. (2016). Now simple calculations show

$$\bar{P}_\varepsilon(t) := T_\varepsilon(t)^{-1}P_\varepsilon(t)T_\varepsilon(t)^{-\top} = \Pi_\varepsilon(t),$$
$$\bar{Q}_\varepsilon(t) := T_\varepsilon(t)^\top Q_\varepsilon(t)T_\varepsilon(t) = \Pi_\varepsilon(t).$$

■

The diagonal entries of $\Pi_\varepsilon(t)$ are called in the following *Hankel singular values.*

4. MODEL REDUCTION

In this section, we show a way to find a reduced model of a balanced realization over a finite interval $[0, t_{f+1}]$. In balanced coordinates, for each coordinate direction the degrees of controllability and observability Gramians are equal.

The main idea for MOR is to eliminate that part corresponding to relatively small singular values. Therefore, we make to following assumption.

Assumption 2. Using the notation of Lemma 11, assume that the Hankel singular values in $\Pi_\varepsilon(\cdot)$ can be divided into two parts

$$R_\varepsilon^\top(t)L_\varepsilon(t) := \begin{bmatrix}\widehat{U}_\varepsilon(t) & \widetilde{U}_\varepsilon(t)\end{bmatrix}\begin{bmatrix}\widehat{\Pi}_\varepsilon(t) & 0\\ 0 & \widetilde{\Pi}_\varepsilon(t)\end{bmatrix}\begin{bmatrix}\widehat{V}_\varepsilon(t) & \widetilde{V}_\varepsilon(t)\end{bmatrix}^\top,$$

such that all Hankel singular values in $\widehat{\Pi}_\varepsilon(t)$ are larger than all Hankel singular values in $\widetilde{\Pi}_\varepsilon(t)$. ◇

According to the partition from Assumption 2, the approximation (3) of the original switched system (1) is equivalent to the balanced system

$$\begin{bmatrix}\dot{\widehat{x}}(t)\\ \dot{\widetilde{x}}(t)\end{bmatrix} = \begin{bmatrix}\widehat{A}_\varepsilon(t) & \bar{A}_{12}(t)\\ \bar{A}_{21}(t) & \widetilde{A}_\varepsilon(t)\end{bmatrix}\begin{bmatrix}\widehat{x}(t)\\ \widetilde{x}(t)\end{bmatrix} + \begin{bmatrix}\widehat{B}_\varepsilon(t)\\ \widetilde{B}_\varepsilon(t)\end{bmatrix}u(t),$$
$$y(t) = \begin{bmatrix}\widehat{C}_\varepsilon(t) & \widetilde{C}_\varepsilon(t)\end{bmatrix}\begin{bmatrix}\widehat{x}(t)\\ \widetilde{x}(t)\end{bmatrix}, \tag{10}$$

where $\widehat{x}(t) \in \mathbb{R}^r$, $\widehat{A}_\varepsilon(t) \in \mathbb{R}^{r\times r}$, $\widehat{B}_\varepsilon \in \mathbb{R}^{r\times m}$, $\widehat{C}_\varepsilon \in \mathbb{R}^{p\times r}$ and where $r < n$ is the dimension of $\widehat{\Pi}_\varepsilon$.

In the balanced system (10) the states $\widetilde{x}$ corresponding to $\widetilde{\Pi}_\varepsilon(\cdot)$ are nearly uncontrollable and unobservable (since the corresponding part in the controllability and observability Gramian given by $\widetilde{\Pi}_\varepsilon$ is small), and therefore don't play an important role in the input-output behavior and approximating them by zero results in a small error in the output. This motivates to consider the following reduced (and balanced) model

$$\dot{\widehat{x}}(t) = \widehat{A}_\varepsilon(t)\widehat{x}(t) + \widehat{B}_\varepsilon(t)u(t), \quad \widehat{x}(0) = \mathbf{0},$$
$$\hat{y}(t) = \widehat{C}_\varepsilon(t)\widehat{x}(t). \tag{11}$$

The ROM (11) can be calculated without first calculating the full balanced transformation, by observing that

$$\widehat{A}_\varepsilon(t) := \widehat{T}_\varepsilon^{-1}(t)(A_\varepsilon(t)\widehat{T}_\varepsilon(t) - \dot{\widehat{T}}_\varepsilon(t)),$$
$$\widehat{B}_\varepsilon(t) := \widehat{T}_\varepsilon^{-1}(t)B_\varepsilon(t),$$
$$\widehat{C}_\varepsilon(t) := C_\varepsilon(t)\widehat{T}_\varepsilon(t),$$

where (using the notation from Lemma 11)

$$\widehat{T}_\varepsilon(t) = R_\varepsilon(t)\widehat{U}_\varepsilon(t)\widehat{\Pi}_\varepsilon(t)^{-1/2} \in \mathbb{R}^{n\times r},$$
$$\widehat{T}_\varepsilon^{-1}(t) = \widehat{\Pi}_\varepsilon(t)^{-1/2}\widehat{V}_\varepsilon(t)^\top L_\varepsilon(t)^\top \in \mathbb{R}^{r\times n}.$$

Remark 12. (Shokoohi et al. (1983)).
The triple $(\widehat{A}_\varepsilon(t), \widehat{B}_\varepsilon(t), \widehat{C}_\varepsilon(t))$ in (11) satisfies the differential Lyapunov equations

$$\widehat{A}_\varepsilon(t)\widehat{\Pi}_\varepsilon(t) + \widehat{\Pi}_\varepsilon(t)\widehat{A}_\varepsilon^\top(t) + \widehat{B}_\varepsilon(t)\widehat{B}_\varepsilon^\top(t) = \dot{\widehat{\Pi}}_\varepsilon(t), \tag{12}$$

$$\widehat{A}_\varepsilon^\top(t)\widehat{\Pi}_\varepsilon(t) + \widehat{\Pi}_\varepsilon(t)\widehat{A}_\varepsilon(t) + \widehat{C}_\varepsilon^\top(t)\widehat{C}_\varepsilon(t) = -\dot{\widehat{\Pi}}_\varepsilon(t). \tag{13}$$

We conclude this section by providing an error bound for above proposed time-varying balanced truncation method. For this we need some additional assumption on the time-behavior of the Hankel singular values. Here we provide the simplest, but most conservative assumption.

Assumption 3 Consider the partition from Assumption 2. Assume that each Hankel singular value $\sigma_i(\cdot)$, $i = r+1, \cdots, n$ in $\widetilde{\Pi}_\varepsilon(\cdot)$ is either nonincreasing or nondecreasing over time. ◇

Theorem 13. (Sandberg and Rantzer (2004)). Consider the approximation system (3) of the switched system (1) satisfying Assumptions 1,2 and 3 and with corresponding reduced system (11). Then for every input $u : [0, t_{f+1}] \to \mathbb{R}^m$ the error between the output $y_\varepsilon : [0, t_{f+1}] \to \mathbb{R}^p$ of (3) and the output $\widehat{y} : [0, t_{f+1}] \to \mathbb{R}^p$ of (11) satisfies

$$\|y_\varepsilon - \hat{y}\|_{L_2} \le 2\sum_{i=r+1}^{n} \sup_{t\in[0,t_{f+1}]} \sigma_i(t) \quad \|u\|_{L_2}$$

5. NUMERICAL RESULTS

In this section, we consider two examples to illustrate the proposed method.

Example 14. Consider the switched systems of Example 1 with a single switch and the corresponding approximation (3) with $\varepsilon = 10^{-3}$. After applying the proposed reduction technique, we can compute a first order reduce system.

Figure 3 shows that for the given input (see Example 1) the output of the first order reduced system is a good approximation of the output of the switched system.

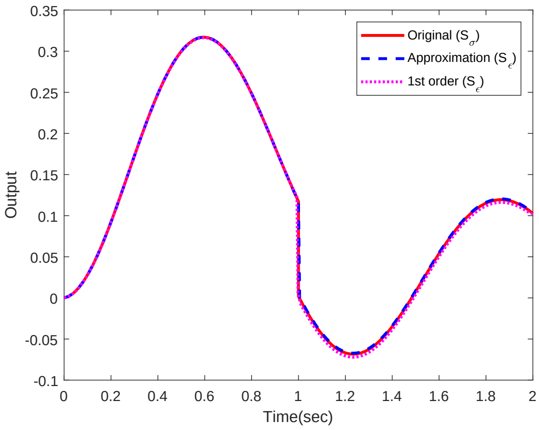

Fig. 3. Comparison for Example 1 between the output of original system (in solid red), approximation (in dashed blue) and 1st order reduced system (in dashed magenta).

Example 15. Consider a randomly generated SISO switched linear system with $x_0 := \mathbf{0}$:

$$A_1 = \begin{bmatrix} -0.74 & 0.3 & 0.2 & -0.01 & -0.06\\ 0.965 & -1.43 & -0.5 & 0.8 & -0.26\\ 0.922 & -0.0487 & -0.44 & 0.03 & 0.054\\ -0.98 & 0.28 & 0.31 & -0.764 & 0.07\\ -0.634 & -1.26 & 0.534 & 0.662 & -0.48 \end{bmatrix},$$

$B_1 = \begin{bmatrix} 2 & 1.4 & 1.1 & -0.06 & 0.08 \end{bmatrix}^T, C_1 = \begin{bmatrix} 2.5 & 2 & 1.6 & 0.02 & -0.03 \end{bmatrix}$

$A_2 = A_1 - 0.5 * I_5$(I denotes identity matrix),

$B_2 = \begin{bmatrix} 2.5 & 1.8 & .3 & 0.6 & -1 \end{bmatrix}^T, C_2 = \begin{bmatrix} 1.5 & 1.4 & .7 & 0.1 & 0.2 \end{bmatrix}, \varepsilon = 10^{-3}.$

We derive the system with $u(t) = (\sin(5t) + 0.05)e^{-0.5t}$ and switching signal

$$\sigma : [0, 6] \to \{1, 2\} : \sigma(t) = \begin{cases} 1 & : t \in [0, 1) \cup [2, 4),\\ 2 & : t \in [1, 2) \cup [4, 6] \end{cases}$$

In Figure 4, we show that the first singular value is significantly larger than the others (by taking $P_\varepsilon(0) = Q_\varepsilon(6) = 0.2I$). So we can truncate the small four singular values to obtain a 1st order reduced model. Figure 5 displays the output of original switched system (S_σ), proposed time-varying approximate system (S_ε) and 1st order reduced system ($\hat{S}_\varepsilon(t)$) which shows that they are nicely matching.

Note that from Figure 4 it is apparent that the truncated Hankel singular values do not satisfy Assumption 3, hence we are currently unable to make a general statement about the error bounds for this example.

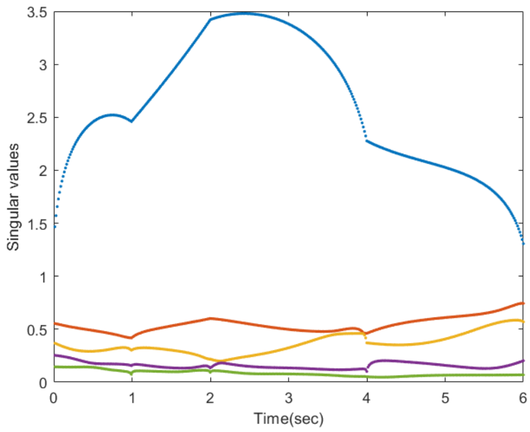

Fig. 4. Hankel singular values of pointwise Gramians.

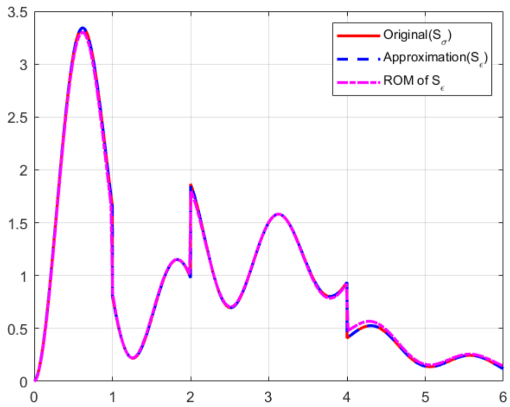

Fig. 5. Comparison for Example 15 between the output of original system, proposed approximation and 1st order approximation.

6. CONCLUSION

In this paper, we have presented a time-varying approach for proposing a reduced order approximation of switched linear system. The key idea is to approximate the discontinuous switched system by a continuously time-varying system and use available balanced truncation methods time-varying linear systems. We also propose some error bounds. Two numerical examples illustrate the applicability and good performance.

The overall error bound is composed of two error bounds: One between the switched system and its (full order) approximation and another error bound coming from the time-varying balanced truncation. For the former we do not have quantitative bounds yet and for the latter it is not clear yet, how smaller values for ε effect the error bound; these questions are ongoing research.

Furthermore, calculation the time-varying Gramians may be computationally infeasible for higher-order systems and it has to be invested whether efficient approximations methods can be derived.

Finally, we want to stress that the reduced model is not a simple switched system anymore. Hence the model complexity significantly increases and we are currently investigating alternative approaches which preserve the simple piecewise-constant structure.

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