A time-varying Gramian based model reduction approach for Linear Switched Systems

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Abstract: Switched system can be considered as a special class of piecewise constantly time-varying systems. In this paper, a model reduction approach is proposed for piecewise constantly switched systems, based on balancing based model order reduction (MOR) method for linear time-varying systems. Time-varying controllability and observability Gramians are computed in finite time interval and then developed balancing theory for the linear time-varying systems. A low dimensional approximate system is computed by applying projection based balanced truncation (BT) method. Finally, the proposed approach is illustrated numerically.

Keywords: Balanced truncation, model reduction, time-varying, differential Lyapunov equations, Gramians.

1. INTRODUCTION

MOR turned out to be a necessary tool in the context of the simulation of various applications and problems over the last decades, (see details Antoulas (2005)). The main purpose of MOR is to find a lower order approximation of a dynamical system which can be used in simulation and optimization instead of the original (large) system. In the context of control design, dynamical systems sometimes may be switched between operating modes, known as switched systems. Such systems usually come from the applications in control of mechanical and aeronautical systems, power converters and auto-motive industry, for details we refer to Liberzon (2003), Sun and Ge (2005). In large scale setting, MOR of switched systems is of constant interest in the system and control community.

Recently, many works have been done on MOR of switched systems, we refer to the following contributions: Monshizadeh et al. (2012), Papadopoulus and Prandini (2016), Schulze and Unger (2018), Gosea et al. (2018b), Duff et al. (2018), Gosea et al. (2018a), Petreczky et al. (2018). In contrast to these references, we view here the switched system as a special *time-varying* system and aim for a (time-varying) model-reduction depending on a given (known) switching signal.

In this paper, we study linear switched system described by the equations:

$$S_\sigma : \begin{cases} \dot{x}(t) = A_\sigma x(t) + B_\sigma u(t), \quad x(t_0) = \mathbf{0}, \\ y(t) = C_\sigma x(t), \end{cases} \tag{1}$$

where σ is a given piecewise constant function of time t, i.e., $\sigma : \mathbb{R} \to M = \{0, 1, 2, \cdots, m\}$ with finitely many switching times $t_0 < t_1 < t_2 < t_3 < \cdots < t_m$. The system matrices are $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, $D_i \in \mathbb{R}^{n\times m}$, where $i \in M$ and n is the number of state variables, called the order of the subsystems. Here, the i-th mode is active in the interval $[t_i, t_{i+1})$, for $i = 1, \cdots, m$ and so the duration of i-th mode is $\tau_i = t_{i+1} - t_i$. The input function $u(t)$ is assumed to be piecewise continuous and bounded.

The notion of balanced realization of state space systems was introduced extensively in Moore (1981) and connections to the Hankel-norm approximations were drawn in Glover (1984), Antoulas (2005). The concept was extended to time-varying systems in Shokoohi et al. (1983) and, Verriest and Kailath (1983) gives the possibility of input-output balancing for various Gramians with necessary and sufficient conditions for the balancing transformation. Uniform and infinite-interval balanced realization are investigated in Shokoohi et al. (1983). In Verriest and Kailath (1983) model reduction is considered for exponentially stable and uniformly completely controllable and observable systems. For these realization, stability of the reduced order model is studied in Shokoohi et al. (1984). In Sandberg and Rantzer (2004), error bounds is given in the finite interval case.

In this paper, we propose balanced truncation method for model reduction of piecewise constantly time-varying switched systems. To the authors' knowledge, this is the first attempt to solve the model reduction for switched system by considering as a time-varying system. It is known, that the balancing procedure involves computing the controllability and observability Gramians from the pair of differential Lyapunov equations, for details we refer Shokoohi et al. (1983); Lang et al. (2016).

The paper is organized as follows. The problem formulation and some preliminaries are discussed in Section 2. Section 3 reviews the steps of the computation of the balanced realization by constructing balancing coordinates. In Section 4, we study the notion of time-varying balanced truncation for switched systems, so that a reduced order

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model can be constructed. Finally, some numerical results are presented in Section 5.

2. PROBLEM STATEMENT AND PRELIMINARIES

In this section, we formulate some problem with examples and then some preliminaries are given. We consider the piecewise constant switched system (1).

The basic idea of model reduction is to represent a complex dynamical system by a much simpler one. This may refer to many different techniques. In this paper, we focus only on projection-based method, known as balanced truncation (BT) which is the most prominent projection-based model reduction methods.

In projection-based technique, the original state variables $x(t)$ is approximated by a vector $\hat{x}(t)$ so that the system (1) can be approximated by the reduced order model

$$\hat{S}_\sigma : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_\sigma \hat{x}(t) + \hat{B}_\sigma u(t), \quad \hat{x}(t_0) = \mathbf{0}, \\ \hat{y}(t) = \hat{C}_\sigma \hat{x}(t), \end{cases} \tag{2}$$

where the system matrices are $\hat{A}_i \in \mathbb{R}^{r\times r}$, $\hat{B}_i \in \mathbb{R}^{r\times m}$, $\hat{C}_i \in \mathbb{R}^{p\times r}$, $D_i \in \mathbb{R}^{r\times m}$, for $i \in M$ and $r \ll n$.

In order to determine the quality of model reduction, it is essential to find a bound $k \geq 0$ which estimates the error between y and $\hat{y}$ such that the approximation errors are

$$\sup_{t\geq 0} \|y(t) - \hat{y}(t)\| \leq k\|u\|,$$

for all admissible input signals u, and suitable norms. However, we have no plan for such error bounds here.

It can be shown that the states of a realization that are either uncontrollable or unobservable donot appear in the corresponding transfer function matrix. Therefore, the states that are almost uncontrollable or unobservable can be omitted from the realization with little effect on the input-output behavior of the system.

For convenience, we assume that the switched system (1) is controllable and observable so is minimal. A question arises: Is the balanced truncation approach applicable to reduce the subsystems independently? Another problem of the switched system is that it losses the differentiability of the coefficient system matrices, so needs to be incorporated.

The following simple example shows that the balanced truncation method could not be applicable in a naive way.

Example 1. Consider a parameter dependent systems:

$$\dot{x}(t) := \begin{bmatrix} -0.5 & 0.01 \\ 0.01 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} \delta_1(t) \\ \delta_2(t) \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} \delta_1(t) & \delta_2(t) \end{bmatrix} x(t),$$

with time-varying parameters $\delta_1(t), \delta_2(t)$ and $x_0 := \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

We now consider a switched system by letting $\delta_1(t) = 0.001 \ll \delta_2(t) = 1$, for $t \in [0, 1)$ and $\delta_1(t) = 1 \gg \delta_2(t) = 0.001$, for $t \in [1, 2]$. The input $u(t) = (\sin(5t) + 0.05)e^{-.5t}$.

Applying balanced truncation on the two intervals $[0, 1]$ and $[1, 2]$ separately, we get a one-dimensional reduced order model

$$\dot{x}(t) = -0.5x(t) + \max\{\delta_1, \delta_2\} \cdot u(t),$$
$$y(t) = \max\{\delta_1, \delta_2\} \cdot x(t)$$

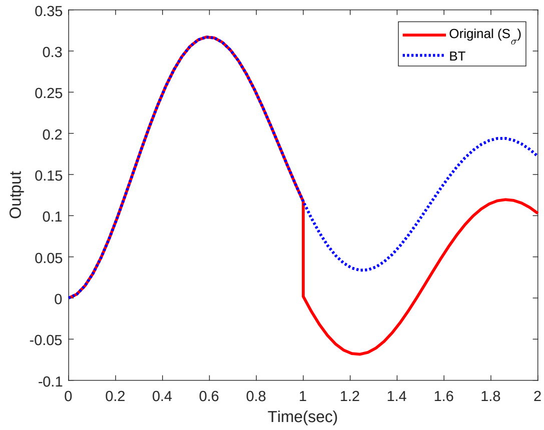

Fig. 1. Comparison for Example 1 between the output of original system (in red) and its reduced 1st order system (in dashed blue line)

Figure 1 shows the output of the original and reduced system. Clearly, the output of the reduced model does not match the original output after the switch, although each individual mode is approximated sufficiently well with a small (known) error bound.

The above example shows, that a piecewise balanced truncation method will not result in good approximations of a switched system in general. The underlying problem is the time-varying nature of the switched system, so we propose to use a time-varying balanced truncation method for the switched system.

2.1 Approximate time-varying system

The available balanced truncation methods for time-varying system (as discussed in the Introduction) assume that the coefficient matrices are at least continuous. However, the switched system (1) seen as a time-varying system has *discontinuous* coefficient matrices. In order to still be able to use the existing methods we propose the following approximation of the switched system (1) by the following (continuously) time-varying systems:

$$S_\varepsilon : \begin{cases} \dot{x}(t) = A_\varepsilon(t)x(t) + B_\varepsilon(t)u(t), \quad x(t_0) = \mathbf{0}, \\ y(t) = C_\varepsilon(t)x(t), \end{cases} \tag{3}$$

where $A_\varepsilon(.)$, $B_\varepsilon(.)$ and $C_\varepsilon(.)$ are defined as follows:

$$A_\varepsilon(t) = \begin{cases} A_i + \dfrac{t-t_i}{\varepsilon}(A_{i+1} - A_i) & : t \in [t_i, t_i + \varepsilon], \\ A_{i+1} & : t \in (t_i + \varepsilon, t_{i+1}] \end{cases}$$

$$B_\varepsilon(t) = \begin{cases} B_i + \dfrac{t-t_i}{\varepsilon}(B_{i+1} - B_i) & : t \in [t_i, t_i + \varepsilon], \\ B_{i+1} & : t \in (t_i + \varepsilon, t_{i+1}] \end{cases}$$

$$C_\varepsilon(t) = \begin{cases} C_i + \dfrac{t-t_i}{\varepsilon}(C_{i+1} - C_i) & : t \in [t_i, t_i + \varepsilon], \\ C_{i+1} & : t \in (t_i + \varepsilon, t_{i+1}] \end{cases}$$

where $i = 0, \cdots, m-1$.

Remark 2. It is clear that the coefficient matrices in time-varying system (3) are bounded and continuous even for small ε, since the term $\frac{t-t_i}{\varepsilon} \in (0, 1)$, $\forall\, t \in (t_i, t_i + \varepsilon)$.

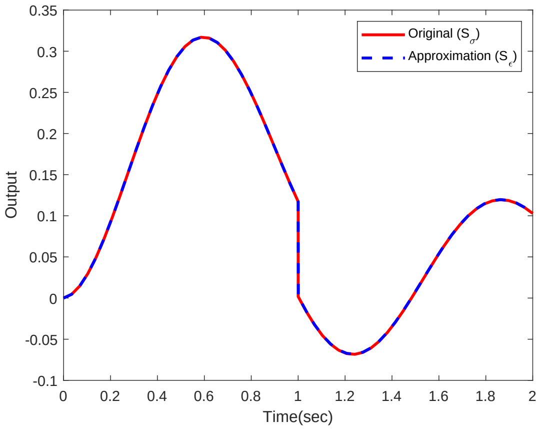

Fig. 2. Comparison for Example 1 between the output of the original system (in red) and proposed approximation (in dashed blue).

Remark 3. The stability of the coefficient matrices of the switched system (1) does not imply the stability of the coefficient matrices in $[t_i, t_i + \varepsilon]$ in general.

Assumption 4. We consider the switched system on $[t_0, t_f]$ and assume that on $(-\infty, t_0)$ and on (t_f, ∞) asymptotically stable systems S_0 and S_{m+1}, resp., are active.

Example 5. We consider again the switched system from Example 1 and approximate it by (3) with $\varepsilon = 10^{-3}$.

Figure 2 shows the output of the approximation compared to the output of the original switched system and it is clearly visible that indeed (3) is a good approximation of (1).

Our goal is now to derive a reduce model for the approximated system (3) of the form

$$\hat{S}_\varepsilon : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_\varepsilon(t)\hat{x}(t) + \hat{B}_\varepsilon(t)u(t), & \hat{x}(t_0) = \mathbf{0}, \\ \hat{y}(t) = \hat{C}_\varepsilon(t)\hat{x}(t). \end{cases} \tag{4}$$

Some basic concepts are recalled first.

The solution of the time-varying state equations in system (3) is

$$x(t) = \Phi_\varepsilon(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi_\varepsilon(t, \tau)B_\varepsilon(\tau)u(\tau)d\tau$$

where $\Phi_\varepsilon(t, \tau)$, known as state transition matrix, is the unique solution of

$$\frac{\partial}{\partial t}\Phi_\varepsilon(t, t_0) = A_\varepsilon(t)\Phi_\varepsilon(t, t_0)$$

with the initial condition $\Phi_\varepsilon(t_0, t_0) = I$.

The state-space system (3) with zero initial value can be viewed as a bounded operator Γ_{yu}: $L_2^m[t, \infty) \to L_2^p[t, \infty)$ given by

$$u \mapsto y = \Gamma_{yu}(u) := \int_{t_0}^{t} C_\varepsilon(t)\Phi_\varepsilon(t, \tau)B_\varepsilon(\tau)u(\tau)d\tau, \quad t \geq t_0.$$

The L_2 induced norm of the operator is defined as follows

$$\|\Gamma_{yu}\| := \sup_{u \neq 0} \frac{||y||}{||u||},$$

where the L_2 norm is defined by

$$||u||_2 = \left(\int_{-\infty}^{\infty} ||u(t)||_2^2 dt \right)^{1/2}.$$

3. BALANCED REALIZATIONS

In this section, we discuss the balanced realization of time-varying system (3) and establish the conditions for the existence of a balancing transformation. Further, we show how the system can be reduced to obtain a lower order model in the input-output description.

It is well known that controllability and observability are essential concepts for balancing theory. Here, we define the controllability and observability Gramians which characterize the controllability and observability subspaces. In particular, we are interested in the input-output balancing of the system over finite time interval $[t_0, t_f]$ with respect to the controllability and observability Gramians.

3.1 Controllability and observability Gramians

For time-varying systems, some balanced realizations are discussed in Shokoohi et al. (1983); Verriest and Kailath (1983) and Sandberg and Rantzer (2004) based on controllability and observability Gramians.

Definition 6. The time-varying controllability and observability Gramians at $t \in [t_0, t_f]$ are defined as follows

$$P(t) := \int_{-\infty}^{t} \Phi_\varepsilon(t, \tau)B_\varepsilon(\tau)B_\varepsilon^\top(\tau)\Phi_\varepsilon^\top(t, \tau)d\tau, \tag{5}$$

$$Q(t) := \int_{t}^{\infty} \Phi_\varepsilon^\top(\tau, t)C_\varepsilon^\top(\tau)C_\varepsilon(\tau)\Phi_\varepsilon(\tau, t)d\tau, \tag{6}$$

respectively.

Note that both Gramians are finite for each $t \in [t_0, t_f]$ due to Assumption 4.

Now we define certain controllable and observable notions for a time-varying system.

Definition 7. (i) The pair $(A_\varepsilon(\cdot), B_\varepsilon(\cdot))$ is called boundedly completely controllable on $[t_0, t_f]$ if there exist two constants $0 < \alpha_n < \alpha_M < \infty$ and $P(t) = P^\top(t)$ given by (5) satisfies

$$0 < \alpha_n I \leq P(t) \leq \alpha_M I < \infty, \quad \forall\, t \in [t_0, t_f].$$

(ii) The pair $(A_\varepsilon(\cdot), C_\varepsilon(\cdot))$ is called boundedly completely observable on $[t_0, t_f]$ if there exist $0 < \beta_n < \beta_M < \infty$ and $Q(t) = Q^\top(t)$ given by (6) satisfies

$$0 < \beta_n I \leq Q(t) \leq \beta_M I < \infty, \quad \forall\, t \in [t_0, t_f].$$

It can be shown that these Gramians are the solution of the following differential Lyapunov equations:

$$A_\varepsilon(t)P(t) + P(t)A_\varepsilon^\top(t) + B_\varepsilon(t)B_\varepsilon^\top(t) = \dot{P}(t) \tag{7}$$

$$A_\varepsilon^\top(t)Q(t) + Q(t)A_\varepsilon(t) + C_\varepsilon^\top(t)C_\varepsilon(t) = -\dot{Q}(t) \tag{8}$$

where initial conditions $P(t_0)$, $Q(t_f)$ are given by (5) and (6), respectively. Note that $P(t_0)$ and $Q(t_f)$ are completely determined by the systems dynamics outside our interval of interest $[t_0, t_f]$, so for implementational issues, we can choose these initial/final values arbitrarily.

3.2 Computational aspects of Gramians

The Gramians can be computed either by solving the differential Lyapunov equations (7), (8) (see Behr et al. (2018)), or by explicitly computing the transition matrix and integration. For the latter approach the following property of the Gramians can be exploited.

Lemma 8. Assume the time steps $s_0 < s_1 < \cdots < s_i < \cdots < s_f$, then the Gramians defined in (5) and (6) are equivalent to the relation:

$$P(s_i) = \Phi(s_i, s_{i-1})P(s_{i-1})\Phi^\top(s_i, s_{i-1}) + \int_{s_{i-1}}^{s_i} \Phi(s_i,\tau)B_\varepsilon(\tau)B_\varepsilon^\top(\tau)\Phi^\top(s_i,\tau)d\tau$$

$$Q(s_i) = \Phi^\top(s_{i+1}, s_i)Q(s_{i+1})\Phi(s_{i+1}, s_i) + \int_{s_i}^{s_{i+1}} \Phi^\top(\tau, s_i)C_\varepsilon^\top(\tau)C_\varepsilon(\tau)\Phi(\tau, s_i)d\tau$$

where $i = 1, 2, \cdots, f$.

Proof. This is simple consequence from the defintion and the property of the transition matrix satisfies $\Phi(s_i,\tau) = \Phi(s_i, s_{i-1})\Phi(s_{i-1},\tau)$ for any $i = 1, 2, \ldots, f$ and $\tau \in (-\infty, s_{i-1})$.

Using the Gramians, we now construct a balancing system.

3.3 Balancing of time-varying system

The balancing of a system is accomplished by a transformation of the state vector. In time-varying systems, such transformation is also time-varying and we need to restrict the class of allowed coordinate transformations to so called Lyapunov transformations:

Definition 9. The mapping $T : [t_0, t_f] \to \mathbb{R}^{n\times n}$ is called a Lyapunov transformation (or, short, Lyapunov) iff $T(t)$, $T(t)^{-1}$ and $\dot{T}(t)$ are well defined and bounded on $[t_0, t_f]$.

The importance of Lyapunov transformations is the preservation of stability. We now show that the Gramians $P(t)$ and $Q(t)$ are Lyapunov transformations.

Lemma 10. Assume that the system (3) is boundedly completely controllable and observable on $[t_0, t_f]$ then $P(t)$ and $Q(t)$ are Lyapunov.

Proof. Boundedness of P, Q and its inverses on $[t_0, t_f]$ is direct consequence from the definition of bounded complete controllability/observability. Furthermore, P and Q are clearly countinously differentiable (as the integral of a continuous function), hence $\dot{P}$ and $\dot{Q}$ are bounded, because they are continuous on $[t_0, t_f]$. ■

The idea of balanced truncation is to find a coordinate transformation such that the controllability and observability Gramians are same and diagonal, so states can be identified which are simultaneously difficult to control and difficult to observe. We now proceed to study the conditions under which controllability and observability Gramians can be diagonalized by Lyapunov transformations.

Let $T(t)$ be a non-singular, bounded and differentiable Lyapunov transformations. Then the state equations in system (3) can be transformed under the transformation

$$x(t) = T(t)\bar{x}(t)$$

resulting in

$$\dot{\bar{x}}(t) = \bar{A}_\varepsilon(t)\bar{x}(t) + \bar{B}_\varepsilon(t)u(t), \quad \bar{x}(t_0) = \mathbf{0},$$
$$y(t) = \bar{C}_\varepsilon(t)\bar{x}(t)$$

with

$$\bar{A}_\varepsilon(t) := T(t)^{-1}(A_\varepsilon(t)T(t) - \dot{T}(t)),$$
$$\bar{B}_\varepsilon(t) := T(t)^{-1}B_\varepsilon(t),$$
$$\bar{C}_\varepsilon(t) := C_\varepsilon(t)T(t).$$

Then it is easily seen that the controllability and observability Gramians transform to

$$\bar{P}(t) = T(t)^{-1}P(t)T(t)^{-\top},$$
$$\bar{Q}(t) = T(t)^\top Q(t)T(t).$$

In particular,

$$\bar{P}(t)\bar{Q}(t) = T(t)^{-1}P(t)Q(t)T(t)$$

which implies that the eigenvalues of the product of Gramians are invariant under such transformation.

We can formulate the definition of equivalence system.

Definition 11. Two systems are said to be equivalent if they can be transformed into the other by Lyapunov transformations.

It is well known (Shokoohi et al., 1983) that the terms in eigenvalue decomposition of $P(t)Q(t)$ may not be Lyapunov, although the Gramians are Lyapunov. Therefore, we provide a conditions for the existence of a decomposition of $P(t)Q(t)$.

Assumption 12. Each eigenvalues of $P(t)Q(t)$ only cross other at isolated point, and they donot have common derivatives at their crossing points.

In the following, we compute balancing transformation for the system (3) using the Gramians.

Lemma 13. (Lang et al. (2016)). Assume the time-varying system (3) is boundedly completely controllable and observable on $[t_0, t_f]$. Then there exists a time-varying coordinate transformation T such that

$$T(t)^{-1}P(t)T(t)^{-\top} = T(t)^\top Q(t)T(t) = \Pi(t),$$

for all $t \in [t_0, t_f]$, for a diagonal matrix $\Pi(t)$. In fact,

$$T(t) = R(t)U(t)\Pi(t)^{-1/2},$$
$$T(t)^{-1} = \Pi(t)^{-1/2}V(t)^\top L(t)^\top,$$

where $U(t)\Pi(t)V(t)^\top$ is the singular value decomposition of $R(t)^\top L(t)$ and where $R(t)R(t)^\top = P(t)$ and $L(t)L(t)^\top = Q(t)$ are the Cholesky decompositions of P and Q, respectively.

Proof. First observe bounded complete controllability / observability, ensures that $P(t)Q(t)$ is invertible, hence $\Pi(t)^{-1/2}$ is well defined. Now simple calculations show

$$\bar{P}(t) := T(t)^{-1}P(t)T(t)^{-\top} = \Pi(t),$$
$$\bar{Q}(t) := T(t)^\top Q(t)T(t) = \Pi(t).$$

■

4. MODEL REDUCTION

In this section, we show the way to find a reduced model of a balanced realization over a finite interval $[t_0, t_f]$. In balanced coordinates, for each coordinate direction the degrees of controllability and observability Gramians are equal.

The main idea for MOR is to eliminate that part corresponding to relatively small singular values. Assume that the singular values in $\Pi(.)$ can be divided into two parts in order decreasing large and small in the singular value decomposition

$$R^{\top}(t)L(t) := [U_1(t)\ U_2(t)] \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & \Pi_2(t) \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}^{\top},$$

where $\Pi_1(t) \gg \Pi_2(t) > 0$.

As the singular values change over time, the ordering may be changed. Here we assume that the order only changes within the two groups and no two singular values move between Π_1 to Π_2.

In the balanced framework, the states corresponding to $\Pi_1(.)$ are very controllable and observable. On the other hand, since $\Pi_2(.)$ is small, the states corresponding to $\Pi_2(.)$ are nearly uncontrollable and unobservable, and therefore removable. Then we can compute the balancing transformations (using the notation from Lemma 13):

$$T(t) = R(t)U_1(t)\Pi_1(t)^{-1/2} \in \mathbb{R}^{n\times r},$$
$$T(t)^{-1} = \Pi_1(t)^{-1/2}V_1(t)^{\top}L_1(t)^{\top} \in \mathbb{R}^{n\times r},$$

where $r(\ll n)$ be the dimension of Π_1. Then

$$\hat{P}(t) = \hat{Q}(t) = \Pi_1(t).$$

Once consider a balanced realization and divide the state vector $\bar{x}(t)$ into components to be retained and components to be discard as follows

$$\bar{x}(t) = \begin{bmatrix} \hat{x}(t) \\ z(t) \end{bmatrix},$$

where $z(t)$ be small in some sense and a reduced order system is obtained by truncating the corresponding states.

Such partition gives the following system

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11}(t) & \mathcal{A}_{12}(t) \\ \mathcal{A}_{21}(t) & \mathcal{A}_{22}(t) \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1(t) \\ \mathcal{B}_2(t) \end{bmatrix} u(t), \quad y(t) = [\mathcal{C}_1(t)\ \mathcal{C}_2(t)] \begin{bmatrix} \hat{x}(t) \\ z(t) \end{bmatrix}, \tag{9}$$

where $\hat{x}(t) \in \mathbb{R}^r$, $\mathcal{A}_{11} \in \mathbb{R}^{r\times r}$, $\mathcal{B}_1 \in \mathbb{R}^{r\times m}$, $\mathcal{C}_1 \in \mathbb{R}^{p\times r}$.

By deleting states $z(t)$, we obtain the reduced system

$$\dot{\hat{x}}(t) = \mathcal{A}_{11}(t)\hat{x}(t) + \mathcal{B}_1(t)u(t), \quad \hat{y}(t) = \mathcal{C}_1(t)\hat{x}(t) \tag{10}$$

Now we characterize the balanced system in terms of Lyapunov equations. The following theorem implies the stability of the subsystem.

Theorem 14. (Shokoohi et al. (1983)). Let the triplet $(A_\varepsilon(t), B_\varepsilon(t), C_\varepsilon(t))$ be bounded and differentiable on $[t_0, t_f]$. Let $(\mathcal{A}_{11}(t), \mathcal{B}_1(t), \mathcal{C}_1(t))$ be a subsystem of an equivalent balanced realization with $\hat{P}(t) = \hat{Q}(t) = \Pi_1(t)$. Then the subsystem $(\mathcal{A}_{11}(t), \mathcal{B}_1(t), \mathcal{C}_1(t))$ satisfies the differential Lyapunov equations

$$\mathcal{A}_{11}(t)\Pi_1(t) + \Pi_1(t)\mathcal{A}_{11}^{\top}(t) + \mathcal{B}_1(t)\mathcal{B}_1^{\top}(t) = \dot{\Pi}_1(t) \tag{11}$$
$$\mathcal{A}_{11}^{\top}(t)\Pi_1(t) + \Pi_1(t)\mathcal{A}_{11}(t) + \mathcal{C}_1^{\top}(t)\mathcal{C}_1(t) = -\dot{\Pi}_1(t) \tag{12}$$

Proof. Follows directly from substitution and simplification (Shokoohi et al. (1983)). ■

5. NUMERICAL RESULTS

In this section, we consider two examples to illustrate the proposed method.

Example 15. Consider the switched systems of example 1 with single switch. Taking $\varepsilon = 10^{-3}$, then we can construct a time-varying approximation.

After applying proposed technique, we can compute first order reduce system.

Figure 15 shows that the first order reduced system gives good approximation of the original system.

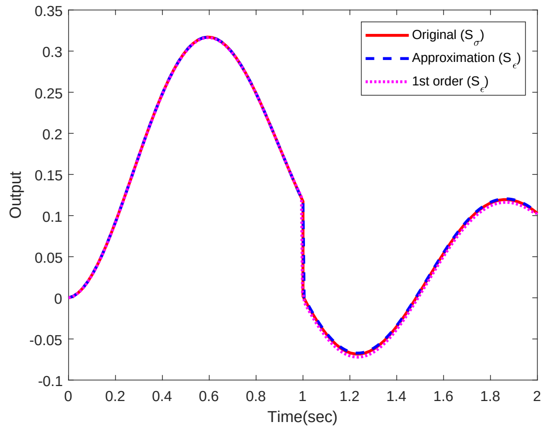

Fig. 3. Comparison for Example 1 between the output of original system (in line red), approximations (in dashed blue) and 1st order reduced system (in dashed red).

Example 16. Consider a randomly generated SISO switched linear system

$$A_1 = \begin{bmatrix} -0.74 & 0.3 & 0.2 & -0.01 & -0.06 \\ 0.965 & -1.43 & -0.5 & 0.8 & -0.26 \\ 0.922 & -0.0487 & -0.44 & 0.03 & 0.054 \\ -0.98 & 0.28 & 0.31 & -0.764 & 0.07 \\ -0.634 & -1.26 & 0.534 & 0.662 & -0.48 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 2 & 1.4 & 1.1 & -0.06 & 0.08 \end{bmatrix}^T, C_1 = \begin{bmatrix} 2.5 & 2 & 1.6 & 0.02 & -0.03 \end{bmatrix}$$
$$A_2 = A_1 - 0.5 * I_5, B_2 = \begin{bmatrix} 2.5 & 1.8 & .3 & 0.6 & -1 \end{bmatrix}^T,$$
$$C_2 = \begin{bmatrix} 1.5 & 1.4 & .7 & 0.1 & 0.2 \end{bmatrix}, \varepsilon = 10^{-3}.$$

Consider $x_0 := \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $u(t) = (\sin(5t) + 0.05)e^{-0.5t}$.

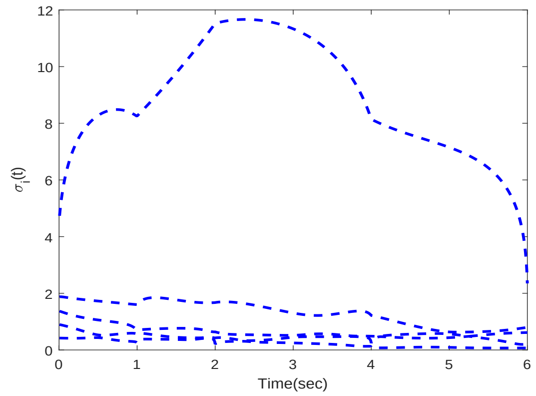

Fig. 4. Singular values of pointwise Gramians.

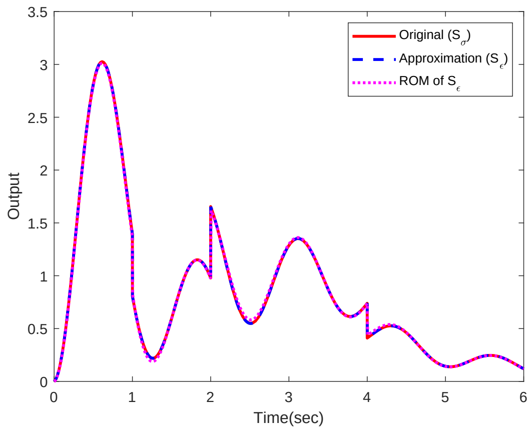

Fig. 5. Comparison for Example 16 between the output of original system, proposed approximation and 1st order approximation.

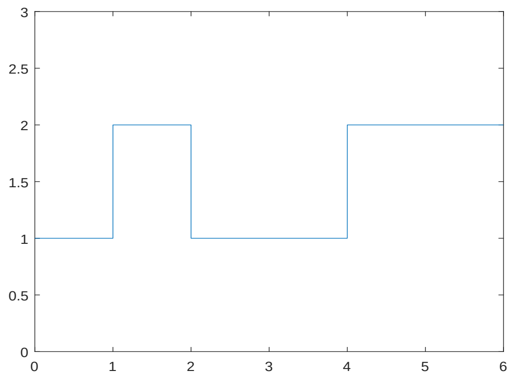

Fig. 6. Considered switching signal for Example 16.

In figure 4, we show that the first singular value is significantly larger than the others. So we can truncate the small four singular values to obtain a 1st order reduced model. Figure 5 displays the output of original switched system (S_σ), proposed time-varying approximate system (S_ε) and 1st order reduced system ($\hat{S}_\varepsilon(t)$) which shows that they are nicely matching. The switching signal σ is depicted in figure 6.

6. CONCLUSION

In this paper, we have presented a time-varying approach which approximate switched linear system. We defined controllability and observability Gramians for a finite time interval. Based on the Gramians, we have discussed a balanced coordinate system of the time-varying approximate system. Then we have applied time-varying balanced truncation method to compute the reduced order model. Two numerical examples illustrate the applicability and good performance.

In this paper, we do not present error bounds yet; this ongoing research. To compare the original switched system, our proposed model forces the value of ε tends to zero which gives impact on the error bounds. Furthermore, computational time should be taken care for higher-order system to obtain the Gramians.

Finally, we want to stress that the reduced model is not a simple switched system anymore. However, our long term goal is to study the effect of ε in the first approximation tending to zero and conclude jump rules between reduced modes in order to obtain a simpler reduced model.

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