

# Asymptotic tracking with funnel control

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Funnel control is a strikingly simple control technique to ensure model free practical tracking for quite general nonlinear systems. It has its origin in the adaptive control theory, in particular, it is based on the principle of high gain feedback control. The key idea of funnel control is to chose the feedback gain large when the tracking error approaches the prespecified error tolerance (the funnel boundary). It was long believed that it is a theoretical limitation of funnel control not being able to achieve asymptotic tracking, however, in this contribution it will be shown that this is not the case.

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## 1 Introduction

The general control objective of funnel control is tracking of a given reference signal with a prespecified (time-varying) error accuracy—the funnel—without knowing a detailed model of the system, see Figure 1.

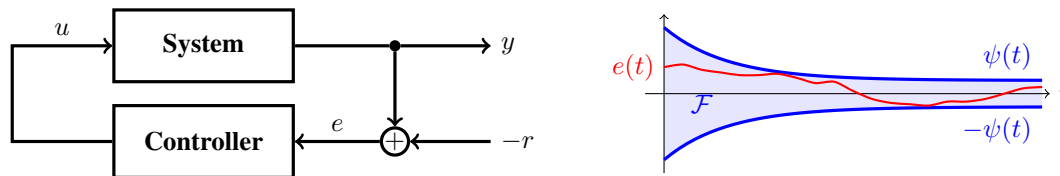


Fig. 1: The overall tracking control feedback structure (left) and the funnel as a prespecified error bound (right).

The solution to this control objective was proposed in [1], see also the survey [2]. The original idea of funnel control was extended in many directions, e.g. more general feedback gain [3], input constraints [4–8], higher relative degree [9–13], for differential-algebraic equations [14, 15] and is also included in the engineering textbook [16]. However, all of these references exclude a funnel boundary which converges to zero. The only remarkable exception is the reference [17] which try to achieve asymptotic tracking with funnel control, but the authors have to rely on an *internal model principle* to prove their result. An approach similar to funnel control was proposed by Bechlioulis & Rovithakis [18] which also aims at ensuring that the error evolves within a prespecified time-varying error bound. But also in this approach it is assumed that the error-bound is not approaching zero.

Altogether it seems to be a common assumption in the community, that asymptotic tracking of an arbitrary reference signal (not produced by a known exo-system) with prescribed performance is not possible. We will show here that this is a misconception and with a simple trick of rewriting the funnel control law it is indeed possible to show that asymptotic tracking is possible!

## 2 Problem setting

We consider nonlinear SISO system of the following input-affine form:

$$\dot{y} = f(p_f, y, z) + g(p_g, y, z) \cdot u, \tag{1a}$$

$$\dot{z} = h(p_h, y, z), \tag{1b}$$

where we assume that (A1)  $g(p_g, y, z) > 0$  for all  $p_g, y, z$ , (A2) the zero dynamics (1b) are BIBO stable with respect to the “inputs”  $y$  and  $p_h$  and (A3) the perturbations  $p_f, p_g, p_h$  are bounded.

The control objective is to find an output feedback rule such that the output  $y$  of the system tracks a given reference signal  $r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  with prespecified error performance. The latter is given via a time varying strict error bound  $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ , i.e. it is required that  $e(t) := y(t) - r(t) \in (-\psi(t), \psi(t)) \quad \forall t \geq 0$ .

The time-varying region where the error is allowed to evolve in is given by  $\mathcal{F} := \{(t, e) \mid |e| < \psi(t)\}$  and is called *funnel* and  $\psi$  is called *funnel boundary*. The following assumptions are made on the tracking signal and the funnel boundary: (A4)  $r$  is continuously differentiable, bounded and with bounded derivative, (A5)  $\psi$  is continuously differentiable, bounded and with bounded derivative and (A6)  $|e(0)| < \psi(0)$ .

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### 3 Asymptotic funnel control

The classical funnel control takes the form

$$u(t) = -K_{\mathcal{F}}(t, e(t))e(t) \quad \text{where} \quad K_{\mathcal{F}}(t, e) = \frac{1}{\psi(t) - |e|} \quad (2)$$

is a positive gain function which approaches infinity when the error variable  $e(t)$  approaches the funnel boundary (more general gain functions are possible, see [3]). Since by design the gain grows unbounded when the distance  $\psi(t) - |e|$  tends to zero, it follows that asymptotic tracking is impossible with bounded internal variables (including the gain), because enforcing asymptotic tracking means that  $\psi(t) \rightarrow 0$  as  $t \rightarrow \infty$  and this implies  $\psi(t) - |e(t)|$  also tends to zero for  $t \rightarrow \infty$  (or even earlier, when the error leaves the funnel). Inspired by the proof technique used in [19] we overcome this problem by introducing the *ratio* between the error and the funnel boundary  $\eta(t) := \frac{e(t)}{\psi(t)}$  and then by *rewriting* the classical funnel control law (2) as

$$u(t) = -\frac{1}{\psi(t) - |e(t)|} e(t) = -\frac{1}{1 - |\eta(t)|} \eta(t) \quad \text{or, in general,} \quad \boxed{u(t) = -\alpha(\eta(t))\beta(\eta(t))}, \quad (3)$$

where  $\alpha$  and  $\beta$  in general satisfy the following conditions (A7)  $\alpha : (-1, 1) \rightarrow [0, \infty)$  is continuous and satisfies  $\alpha(\eta) \rightarrow \infty$  as  $|\eta| \rightarrow 1$  and (A8)  $\beta : (-1, 1) \rightarrow \mathbb{R}$  is continuous,  $\beta(\eta) \neq 0$  as  $|\eta| \rightarrow 1$  and  $\text{sgn}(\beta(\eta)) = \text{sgn}(\eta)$  for all  $\eta \neq 0$ .

In contrast to the classical funnel controller where  $\psi(t) \rightarrow 0$  as  $t \rightarrow \infty$  leads to an unbounded gain, this is not the case any more with the alternative formulation because the gain  $\alpha(e(t)/\psi(t))$  can remain bounded even if  $1/\psi(t)$  tends to infinity.

**Theorem 3.1** Consider the nonlinear system (1) satisfying assumptions (A1)–(A3). Then for any reference signal  $r$  and any funnel boundary  $\psi$  satisfying assumptions (A4)–(A6), the funnel controller (3) with arbitrary  $\alpha$  and  $\beta$  satisfying (A7) and (A8) results in a closed loop where all solutions exist on  $[0, \infty)$  and remain bounded. In particular, there exists  $\varepsilon > 0$  such that

$$|e(t)| \leq (1 - \varepsilon)\psi(t) \quad \text{for all } t \geq 0,$$

*i.e.* the error between the output and the reference signal remains within the funnel for all times and the distance between the fraction  $e(t)/\psi(t)$  and the boundary  $\pm 1$  is bounded away from zero. Consequently, asymptotic tracking via funnel control can be achieved with arbitrary convergence rates, for example by using  $\psi(t) = ce^{-\lambda t}$  for some  $c > 0$  and  $\lambda > 0$ .

Due to space limitation the proof is omitted, however it is contained in the recently submitted paper [20], where also the extension to the MIMO case is contained. An interesting application of this result is used in [21] to solve a decentralized optimization problem with global asymptotic convergence towards the optimal solution and with prescribed transient behavior.

**Remark 3.2** A careful analysis of the proof used in [20] shows that it is even possible to achieve *finite time convergence* of the error by simply choosing a funnel boundary which approach zero in finite time. However, in that case the solution stops to exist once it reaches the end-point.

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