Stability Analysis for Switched Discrete-Time Linear Singular Systems*

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Abstract

The stability of arbitrarily switched discrete-time linear singular (SDLS) systems is studied. Our analysis builds on the recently introduced one-step-map for SDLS systems of index-1. We first provide a sufficient stability conditions in terms of Lyapunov functions. Furthermore, we generalize the notion of joint spectral radius of a finite set of matrix pairs, which allows us to fully characterize exponential stability.

Keywords: switched singular systems, index-1, exponential stability, joint spectral radius, Lyapunov functions

1. Introduction

In this paper we investigate stability of switched discrete-time linear singular (SDLS) systems of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \tag{1}$$

where $\sigma : \mathbb{N} \to \{1, \ldots, \mathtt{n}\}$ denotes the switching signal that determines which of the $\mathtt{n} \in \mathbb{N}$ modes is active at time k.

Singular switched systems arise in many applications, such as power electronics and systems, flight control systems, network control systems, robot manipulators, economic systems, and so forth (see e.g. [13, 14, 21, 33, 38]). There are examples (e.g. the dynamic Leontief system in economic studies [20]) which are canonically given in discrete time; however, system models stemming from physical first principles usually are formulated in terms of differential equations in continuous time and also may contain external variables (inputs) which can be used to influence the dynamics of the system. In this paper we focus on the stability of the *closed loop* system, i.e. we assume that an adequate candidate feedback rule is already chosen and ask the question whether this controller leads to stability regardless of the switching signal. Furthermore, we assume that the continuous dynamics are discretized in time because many feedback controllers are nowadays implemented digitally. The consideration of a switching signal is often another modelling simplification where rapidly changing parameters are simplified as parameters which change their values instantaneously at some isolated points in time.

There are already quite a few works devoted to the stability of SDLS systems, some of them restrict themselves to the case of a constant E-matrix [33, 34, 35, 36] and some restrict the switching signal [1, 7]. The references [8, 13, 21] which actually study the general SDLS system (1) seem to have overlooked the fact that even if each mode is *causal* (i.e. regular and index-1, see Section 2.2) the corresponding switched system is not well-posed in general, see the recent publication [2]; in particular, these references lack a proper solution theory capable to handle arbitrary switching. In fact, to study the SDLS system (1) under arbitrary switching it seems reasonable to impose an additional *causality assumption with respect to the switching signal*, i.e. the future value of the switching signal should not influence the past values of the state-variable. In other words, we expect that the value $x(k)$ only depends on the previous state together with the system parameters of the modes active at time k or earlier. This intuition is formalized by an index-1 assumption for the overall SDLS which then results in a well defined *one-step-map*, see Section 3.

Based on this novel solution framework we are able to generalize the well known stability result in terms of a common Lyapunov function to SDLS systems (1), which can be seen as an discrete time version of the results presented in [15] for the continuous time and which seems not to have been reported so far.

Another main contribution is the generalization of the joint spectral radius to the family of matrix pairs $\{(E_i, A_i)\}$ associated to (1) and how it can be used to characterize exponential stability of the SDLS system (1). The continuous time counterpart of the the joint spectral radius is the Lyapunov exponent and was already studied for switched singular systems in [31]; however, the methods used in the discrete time case are very different to the ones used in the continuous time case. Furthermore, we are able to show that the joint spectral radius of (1) can also be obtained via a standard switched system of reduced size. This may have important consequences when calculating (or approximating) the joint spectral radius numerically, but this topic is outside the scope of the current paper.

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2. Preliminaries

2.1. Switched linear systems

We recall in the following basic properties of switched linear system of the form

$$x(k+1) = A_{\sigma(k)}x(k); \tag{2}$$

where $\sigma : \mathbb{N} \cup \{0\} \to \{1, 2, \dots, \mathtt{n}\}$ is the switching signal with $\mathtt{n} \in \mathbb{N}$ modes, $A_i \in \mathbb{R}^{n\times n}$ are given matrices, $i = 1, 2, .., \mathtt{n}$, $x(k) \in \mathbb{R}^n$ is the state at time $k \in \mathbb{N}$.
Define the state transition matrix $\Phi_\sigma(k, h)$ for system (2) as $\Phi_\sigma(k, h) = I$ for $k = h$ and, for $k > h$,

$$\Phi_\sigma(k, h) = A_{\sigma(k-1)}A_{\sigma(k-2)}...A_{\sigma(h)}.$$

The unique solution of system (2) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ is given by

$$x(k) = \Phi_\sigma(k, 0)x_0, \quad k \geq 0.$$

Definition 2.1. *System* (2) *is called exponentially stable if there exist $\gamma \geq 1$ and $0 \leq \lambda < 1$ such that for all switching signals and any solutions x of* (2) *with $x(0) = x_0 \in \mathbb{R}^n$ the following inequality holds:*

$$\|x(k)\| \leq \gamma\lambda^k \|x_0\| \quad \forall k \geq 0,$$

where $\|\cdot\|$ is some norm on $\mathbb{R}^n$.

Lemma 2.2. (see, [24, 26]) *System* (2) *is exponentially stable if and only if there exist $\gamma \geq 1$ and $0 < \lambda < 1$ such that for all switching signals the following inequality holds:*

$$\|\Phi_\sigma(k, j)\| \leq \gamma\lambda^{k-j}, \ \ \forall k \geq j,$$

where $\|\cdot\|$ is the induced matrix norm.

Lemma 2.3. (see, [17]) *System* (2) *is exponentially stable if, and only if, there exists a finite positive integer m such that*

$$\left\|A_{i_1}A_{i_1}...A_{i_m}\right\| < 1$$

for all m-tuples $(A_{i_j})_{j=1}^m$ with $A_{i_j} \in \{A_1, A_2, .., A_{\mathtt{n}}\}$.

The stability of (2) can be investigated by using the joint spectral radius of a set of matrices introduced in [23].

Definition 2.4. *The joint spectral radius of a family of matrices $\{A_i\}_{i=1}^{\mathtt{n}}$ is defined to be*

$$\rho(\{A_i\}_1^{\mathtt{n}}) = \lim_{k\to\infty} \max_{\sigma_k \in \{1,2,..,\mathtt{n}\}} \|A_{\sigma_1}A_{\sigma_2}...A_{\sigma_k}\|^{\frac{1}{k}}.$$

The existence of the limit in the definition of the joint spectral radius is based on following well-known Pólya – Szegö's result [22], which we will also need in Section 5.

Lemma 2.5. *Let $\{a_k\}_{k=1}^\infty$ be a sequence of positive numbers, such that $a_{k+\ell} \leq a_k a_\ell$ for all k, ℓ. Then the limit $\lim_{k\to\infty}(a_k)^{\frac{1}{k}}$ exists.*

Theorem 2.6. (see, [25]) *System* (2) *is exponentially stable if and only if $\rho(\{A_i\}_1^{\mathtt{n}}) < 1$.*

2.2. Singular systems

We recall some basic properties of discrete-time linear singular systems of the form

$$Ex(k+1) = Ax(k), \tag{3}$$

where $E, A \in \mathbb{R}^{n\times n}$ and $x(k) \in \mathbb{R}^n$. Usually it is assumed that E is singular, as otherwise, (3) could be multiplied with the inverse of E from the left to obtain a standard linear system.

Definition 2.7. *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called* regular *if, and only if, the polynomial* $\det(sE - A)$ *is not identically zero.*

Lemma 2.8 ([32, 11]). *A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \tag{4}$$

where $N \in \mathbb{R}^{n_N\times n_N}$ is nilpotent and $J \in \mathbb{R}^{n_J\times n_J}$ with $n_N + n_J = n$.

In view of [5] we call (4) a *quasi Weierstrass form (QWF)* of (E, A). The QWF is unique up to similarity of the matrices J and N; in particular, the nilpotency index of N (the smallest number $\nu \in \mathbb{N}$ such that $N^\nu = 0$) is independent of the choices for S and T and we will define the index of a regular matrix pair (E, A) as the nilpotency index of N in the QWF. In the *index-1 case* (also called sometimes *causal*) it is actually easy to see that $T = [T_1, T_2]$ and $S = [ET_1, AT_2]^{-1}$ with full column rank matrices T_1, T_2 matrices such that

$$\begin{aligned} \operatorname{im} T_1 &= \boxed{\mathcal{S} := A^{-1}(\operatorname{im} E)} := \{\xi \in \mathbb{R}^n \ : \ A\xi \in \operatorname{im} E\} \\ \operatorname{im} T_2 &= \ker E \end{aligned}$$

transform (E, A) into QWF, in particular, $\mathcal{S} \oplus \ker E = \mathbb{R}^n$. In fact, the following stronger result holds.

Lemma 2.9 ([12, Appendix A, Thm. 13], cf. [6, Prop. 9]). *The following three statements are equivalent for any $E, A \in \mathbb{R}^{n\times n}$ and $\mathcal{S} := A^{-1}(\operatorname{im} E)$.*

a. The matrix pair (E, A) is regular with index-1.

b. $\mathcal{S} \cap \ker E = \{0\}$.

c. $\mathcal{S} \oplus \ker E = \mathbb{R}^n$.

The main relevance of regularity and index-1 is the following statement about existence and uniqueness of solutions of (3).

Lemma 2.10 ([2, Lem. 2.5]). *Assume (E, A) is regular and of index-1, then the discrete-time singular system* (3) *with initial condition $x(0) = x_0 \in \mathbb{R}^n$ has a unique solution if, and only if $x_0 \in \mathcal{S}$ and the solution is then given by*

$$x(k) = \Phi_{(E,A)}^k x_0, \quad \textit{with } \Phi_{(E,A)} := T\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}T^{-1},$$

where T and J are given by the QWF (4) *and $\Phi_{(E,A)}$ is independent from the specific choice of S and T leading to* (4).

Remark 2.11. *The matrix $\Phi_{(E,A)}$ corresponds to the matrix A^{diff} in continuous time, see e.g. [28] and can be interpreted as the* one-step map *for* (3). *However, it is important to note that this interpretation is only valid if we assume that* (3) *holds for at least two time steps. In fact, from*

$$Ex(1) = Ax(0)$$

we can only conclude that

$$x(1) \in \{\Phi_{(E,A)}x(0)\} + \ker E.$$

In order to conclude that $x(1) = \Phi_{(E,A)}x(0)$ we additionally have to take into account

$$Ex(2) = Ax(1)$$

together with $\mathcal{S} \cap \ker E = \{0\}$.

Definition 2.12. *(see, [9]) Assume that there exists symmetric positive definite matrix $H \in \mathbb{R}^{n\times n}$ such that $A^\top HA - E^\top HE = -K$, where K is symmetric and positive in the sense $x^\top Kx > 0,\ \forall x \in \mathcal{S}\backslash\{0\}$. Then the function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined by $\mathcal{V}(x) = (Ex)^\top HEx$ is called a Lyapunov function for system* (3).

Lemma 2.13. *(see, [9]) Assume that system* (3) *is regular and of index-1. Then system* (3) *is exponentially stable if, and only if, there exists a Lyapunov function for system* (3).

3. Solution theory for switched singular systems

We now consider the solution properties of the switched SDLS (1)

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k),$$

where $\sigma : \mathbb{N} \cup \{0\} \to \{1, 2, .., \mathtt{n}\}$, $\mathtt{n} \in \mathbb{N}$, is a switching signal taking values in the finite set $\{1, \dots, \mathtt{n}\}$; $E_i, A_i \in \mathbb{R}^{n\times n}$ are given matrices, and $x(k) \in \mathbb{R}^n$ is state vector at time $k \in \mathbb{N}$. Suppose that the matrices E_i are singular for all $i = 1, 2, .., \mathtt{n}$. For notational convenience we put $\sigma(-1) = 1$.

Remark 3.1. *In* (1) *the E-matrix is also assumed to depend on the switching signal. Here we assume that the E and A matrix change synchronously, which leads to* (1). *However, one could also argue that the matrix in front of the future value $x(k+1)$ should also be the matrix valid in the future, leading to the following SDLS system*

$$E_{\sigma(k+1)}x(k+1) = A_{\sigma(k)}x(k) \tag{5}$$

which is the system class studied by one of the first papers on discrete time-varying singular systems by Luenberger [20]. One could even argue that the equation determining the future value $x(k+1)$ should be governed by the future coefficient matrices, i.e.

$$E_{\sigma(k+1)}x(k+1) = A_{\sigma(k+1)}x(k),$$

which, however, can be analyzed with the same methods used for (1) *by just considering a shifted switching signal. The "correct" modeling choice will in the end depend on the underlying application and here we decided to focus on the form* (1). *Some results concerning* (5) *have been developed in parallel to this work and have already appeared [18].*

For the continuous time case it suffices to assume that each matrix pair (E_i, A_i) is regular, to conclude existence and uniqueness of (distributional) solutions of the corresponding switched system. This nice solvability characterization is lost in the discrete time case (see e.g. [2, Example 1.1]); in particular, causality with respect to the switching signal may be lost. It is therefore *necessary* to impose more strict assumptions on the matrix pairs (E_i, A_i) to be able to conclude suitable solution properties. Inspired by our previous works on discrete-time singular systems [3, 4, 19, 2], we introduce the following index-1 notion for the overall switched system.

Definition 3.2. *System* (1) *is called an arbitrarily switched singular system of index-1 if*

$$\mathcal{S}_i \cap \ker E_j = \{0\}, \quad \forall i, j \in \{1, 2, .., \mathtt{n}\}, \tag{6}$$

where $\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$.

Since condition (6) also needs to hold for $i = j$ and in view of Lemma 2.9 it follows that each individual pair (E_i, A_i) needs to be regular and of index-1, i.e. the definition is indeed a generalization of the index-1 property for the non-switched case. Note furthermore, that we do *not assume* that the "mixed" matrix pairs (E_j, A_i) are regular and index-1 (because $\mathcal{S}_i$ is defined in terms of E_i and not E_j). We highlight the following consequences from the index-1 definition.

Lemma 3.3 ([2, Lem. 3.3]). *Suppose the SDLS system* (1) *is of index-1. Then the following statements hold:*

(a) $\operatorname{rank} E_i = r$, $\quad i = 1, \dots, \mathtt{n}$.

(b) $\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n$*, for all $i, j \in \{1, 2, .., \mathtt{n}\}$.*

Based on the index-1 assumption it is now possible to give an explicit solution formular for the SDLS system (1).

Lemma 3.4 ([2, 3.5]). *The SDSL* (1) *of index-1 in the sense of Definition 3.2 has a unique solution with $x(0) = x_0 \in \mathbb{R}$ if, and only if, $x_0 \in \mathcal{S}_{\sigma(0)}$. This solution satisfies*

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k) \quad \forall k \in \mathbb{N}, \tag{7}$$

where $\Phi_{i,j}$ is the one-step map *from mode j to mode i given by*

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j}\Phi_{(E_j,A_j)},$$

where $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is the unique projector onto $\mathcal{S}_i$ along $\ker E_j$ and $\Phi_{(E_j,A_j)}$ is the one-step map corresponding to mode j as in Lemma 2.10.

Note that a direct consequence of the solution formula (7) is that all solutions satisfy $x(k) \in \mathcal{S}_k$ for all k (as expected). Furthermore, the definition of the one-step-map consist of two parts: first the old state is mapped to an intermediate state with the one-step-map of the current mode, afterwards the new state is obtained by applying a projector, so that the new state is consistent with the new algebraic constraint, cf. also Remark 2.11.

Remark 3.5. *Similar as for classical switched systems* (2)*, it is now possible to define also a transition matrix* $\Phi_\sigma(k,h)$ *for SDLS system* (1) *as follows:*

$$\Phi_\sigma(k,h) = \Phi_{\sigma(k),\sigma(k-1)}\Phi_{\sigma(k-1),\sigma(k-2)}\cdots\Phi_{\sigma(h+1),\sigma(h)}$$

for $k > h$ *and*

$$\Phi_\sigma(h,h) = \Pi^{\ker E_{\sigma(h)}}_{\mathcal{S}_{\sigma(h)}}.$$

Then all solutions of the SDLS system (1) *are given by*

$$x(k) = \Phi_\sigma(k,0)x_0. \tag{8}$$

Note that also for $x_0 \notin \mathcal{S}_{\sigma(0)}$*, the formula* (8) *results in a valid solution, however, in general* $x(0) \neq x_0$ *and*

$$x(0) = \Pi^{\ker E_{\sigma(0)}}_{\mathcal{S}_{\sigma(0)}} x_0.$$

In what follows we give a constructive formula for the matrix $\Phi_{i,j}$ as well as for the unique solution to SDSL system (1). These formulas will be helpful in the forthcoming stability analysis.

Lemma 3.6 ([2, Lem. 4.1]). *Consider the SDLS system* (1) *and assume that it is index-1. For* $i = 1,\dots,\mathtt{n}$*, let* $T_i := [s_i^1,\dots,s_i^r,h_i^{r+1},\dots,h_i^n]$ *be such that it columns form bases of* $\mathcal{S}_i$ *and* $\ker E_i$*, respectively. Let* $P := \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0\end{smallmatrix}\right] \in \mathbb{R}^{n\times n}$*, where* I_r *is an* $r\times r$*, identity matrix,* $Q := I_n - P$*. Finally, let* $P_i := T_iPT_i^{-1} = \Pi^{\ker E_i}_{\mathcal{S}_i}$*,* $Q_i := I - P_i = \Pi^{\mathcal{S}_i}_{\ker E_i}$ *and* $Q_{i,j} := T_jQT_i^{-1}$ *for* $i,j = 1,\dots,\mathtt{n}$*. Then the following properties hold*

(i) $G_{i,j} := E_i + A_iQ_{i,j}$ *is nonsingular for all* $i,j \in \{1,2,\dots,\mathtt{n}\}$*,*

(ii) $\Pi^{\ker E_j}_{\mathcal{S}_i} = I - Q_{i,j}G_{i,j}^{-1}A_i$*,*

(iii) $\Phi_{(E_i,A_i)} = P_iG_{i,i}^{-1}A_i$*,*

(iv) $\Phi_{i,j} = (I - Q_{i,j}G_{i,j}^{-1}A_i)P_jG_{j,j}^{-1}A_j$*.*

4. Stability of switched system based on Lyapunov functions

We will establish a sufficient condition for the exponential stability for SDLS systems (1) in terms of the Laypunov functions of each mode. This approach is inspired by a similar result available for the continuous time case [15]; in particular, it generalizes the *common Lyapunov function* approach for standard switched system.

Theorem 4.1. *Consider the SDSL* (1) *of index-1 and assume that every subsystem* $E_ix(k+1) = A_ix(k)$ *is exponentially stable with a Lyapunov function* $T_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *in the sense of Definition 2.12. If*

$$\mathcal{V}_i(P_ix) \leq \mathcal{V}_j(P_ix), \quad \forall i,j = 1,2,..,\mathtt{n}, \tag{9}$$

where $P_i = \Pi^{\ker E_i}_{\mathcal{S}_i}$*, then the SDLS system* (1) *is exponentially stable for arbitrary switching signals*

Proof.
Step 1: We construct a Lyapunov function for the switched system which decreases along solutions.
If $x \in \mathcal{S}_i \cap \mathcal{S}_j$ then $x = P_ix = P_jx$. From condition (9) we have

$$\mathcal{V}_i(x) = \mathcal{V}_i(P_ix) \leq \mathcal{V}_j(P_ix) = \mathcal{V}_j(x),$$

$$\mathcal{V}_j(x) = \mathcal{V}_j(P_jx) \leq \mathcal{V}_i(P_jx) = \mathcal{V}_i(x).$$

Thus, $\mathcal{V}_i(x) = \mathcal{V}_j(x)$ for all $x \in \mathcal{S}_i \cap \mathcal{S}_j$, therefore we can define a common (piecewise-quadratic) Lyapunov function for system (1) as follows:

$$\mathcal{V} : \mathbb{R}^n \to \mathbb{R},\ x \to \begin{cases} \mathcal{V}_i(x) & \text{if } x \in \mathcal{S}_i, \\ 0 & \text{otherwise.}\end{cases}$$

Suppose $x(k)$, $k \in \mathbb{N}$, is the solution of system (1), then

$$\begin{aligned}
&\mathcal{V}(x(k+1)) - \mathcal{V}(x(k)) \\
&= \mathcal{V}_{\sigma(k+1)}(x(k+1)) - \mathcal{V}_{\sigma(k)}(x(k)) \\
&= \mathcal{V}_{\sigma(k+1)}(P_{\sigma(k+1)}x(k+1)) - \mathcal{V}_{\sigma(k)}(x(k)) \\
&\leq \mathcal{V}_{\sigma(k)}(x(k+1)) - \mathcal{V}_{\sigma(k)}(x(k)) \\
&= (E_{\sigma(k)}x(k+1))^\top H_{\sigma(k)}(E_{\sigma(k)}x(k+1)) \\
&\quad - (E_{\sigma(k)}x(k))^\top H_{\sigma(k)}(E_{\sigma(k)}x(k)) \\
&= (A_{\sigma(k)}x(k))^\top H_{\sigma(k)}(A_{\sigma(k)}x(k)) - (E_{\sigma(k)}x(k))^\top H_{\sigma(k)}(E_{\sigma(k)}x(k)) \\
&= x(k)^\top(A^\top_{\sigma(k)}H_{\sigma(k)}A_{\sigma(k)} - E^\top_{\sigma(k)}H_{\sigma(k)}E_{\sigma(k)})x(k) \\
&= -x(k)^\top K_{\sigma(k)}x(k).
\end{aligned}$$

Since $K_{\sigma(k)}$ is positive definite on $\mathcal{S}_{\sigma(k)}$ it is shown that indeed $\mathcal{V}$ decreases along solutions.

Step 2: We show existence of $\lambda_i \in (0,1]$ for all $i \in \{1,\dots,\mathtt{n}\}$ such that

$$x^\top K_ix \geq \lambda_i\mathcal{V}_i(x) \quad \forall x \in \mathcal{S}_i.$$

Let $\widetilde{\mathcal{S}_i} := \{y \in \mathcal{S}_i \mid \mathcal{V}_i(y) = 1\}$. Since $\widetilde{\mathcal{S}_i}$ is the preimage of a closed set under a continuous function it is closed. Seeking a contradiction suppose that $\widetilde{\mathcal{S}_i}$ is unbounded, then there exists a sequence $\{y_\ell\}$ within $\widetilde{\mathcal{S}_i}$, such that $\|y_\ell\| \to \infty$ as $\ell \to \infty$. Due to the positive definiteness of the matrix H_i, there exists a positive constant γ_i, such that $1 = \mathcal{V}_i(y_\ell) = (E_iy_\ell)^\top H_iE_iy_\ell \geq \gamma_i\|E_iy_\ell\|^2$. Let $\widetilde{E}_i$ be the restriction of E_i to $\mathcal{S}_i$. According to Lemma 3.3(b), $\mathcal{S}_i \oplus \ker E_i = \mathbb{R}^n$, hence, $\widetilde{E}_i : \mathcal{S}_i \to E_i(\mathcal{S}_i)$ is a bijective mapping, therefore, it has an inverse $\widetilde{E}_i^{-1}$. Since $y_\ell \in \mathcal{S}_i$, we find $\|y_\ell\| = \|\widetilde{E}_i^{-1}E_iy_\ell\| \leq \|\widetilde{E}_i^{-1}\|\|E_iy_\ell\| \leq \frac{1}{\sqrt{\gamma_i}}\|\widetilde{E}_i^{-1}\|$, which contradicts the assumption $\|y_\ell\| \to \infty$ as $\ell \to \infty$. We therefore have shown that $\widetilde{\mathcal{S}_i}$ is compact, hence

$$\bar{\lambda}_i := \min_{y\in\widetilde{\mathcal{S}_i}} y^\top K_iy > 0$$

is well defined. Furthermore, by definition of K_i and positive definiteness of H_i we have $y^\top K_iy = y^\top E_i^\top H_iE_iy - y^\top A_i^\top H_iA_iy = \mathcal{V}_i(y) - y^\top A_i^\top H_iA_iy \leq 1$ for all $y \in \widetilde{\mathcal{S}_i}$ and hence $\bar{\lambda}_i \leq 1$. Since for any $x \in \mathcal{S}_i \setminus \{0\}$ we have that $\frac{x}{\sqrt{\mathcal{V}_i(x)}} \in \widetilde{\mathcal{S}_i}$, we can therefore conclude that

$$x^\top K_ix = \left(\frac{x}{\sqrt{\mathcal{V}_i(x)}}\right)^\top K_i\frac{x}{\sqrt{\mathcal{V}_i(x)}}\mathcal{V}_i(x) \geq \bar{\lambda}_i\mathcal{V}_i(x).$$

Step 3: We show exponential stability.

Let $\bar{\lambda} := \min\{\bar{\lambda}_1, \dots, \bar{\lambda}_{\mathrm{n}}\} \in (0, 1]$ then from Step 2 we immediately obtain:

$$\mathcal{V}(x(k)) \le (1 - \bar{\lambda})^k \mathcal{V}(x(0)), \quad \forall k \ge 0.$$

Since K_i is positive definite on $\mathcal{S}_i$ there exists $\bar{\gamma}_i > 0$ such that $x^\top K_i x \ge \bar{\gamma}_i \|x\|^2$ for all $x \in \mathcal{S}_i$. Hence we have, with $\bar{\gamma} := \min\{\bar{\gamma}_1, \dots, \bar{\gamma}_{\mathrm{n}}\}$,

$$\begin{aligned}\|x(k)\|^2 \le \frac{x(k)^\top K_{\sigma(k)} x(k)}{\bar{\gamma}} \le \frac{\mathcal{V}(x(k))}{\bar{\gamma}} \le \frac{(1-\bar{\lambda})^k}{\bar{\gamma}} \mathcal{V}(x(0)) \\ \le \frac{\|E_{\sigma(0)} H_{\sigma(0)} E_{\sigma(0)}\|}{\bar{\gamma}} (1-\bar{\lambda})^k \|x(0)\|^2.\end{aligned}$$

We therefore have shown exponential stability

$$\|x(k)\| \le \gamma \lambda^k \|x_0\|$$

with $\lambda := \sqrt{1 - \bar{\lambda}} \in [0, 1)$ and (recalling that $x(0) = \Pi_{\mathcal{S}_{\sigma(0)}}^{\ker E_{\sigma(0)}} x_0$) $\gamma := \left\|\Pi_{\mathcal{S}_{\sigma(0)}}^{\ker E_{\sigma(0)}}\right\| \sqrt{\|E_{\sigma(0)} H_{\sigma(0)} E_{\sigma(0)}\| / \bar{\gamma}}$. ■

Remark 4.2. *Similar as in the continuous time case it is possible to relax the condition* (9) *to*

$$\mathcal{V}_i(P_i x) \le \mu \mathcal{V}_j(P_i x), \quad \forall i, j = 1, 2, .., \mathrm{n}.$$

where $\mu \ge 1$. Then exponential stability holds then for all switching signals whose dwell time is sufficiently large. To be precise, assume that every subsystem $E_i x(k+1) = A_i x(k)$ is exponentially stable with Lyapunov function $\mathcal{V}_i : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\mathcal{V}_i(x) = (E_i x)^\top H_i (E_i x)$, where H_i are the symmetric positive definite matrices, such that $A_i^\top H_i A_i - E_i^\top H_i E_i = -K_i$, $i = 1, 2, \dots, \mathrm{n}$, and K_i are symmetric and positive in the sense that $x^\top K_i x > 0$, $\forall x \in S_i \backslash \{0\}$ and let $\lambda = \min_i \inf_{x \in S_i \backslash \{0\}} \frac{x^\top K_i x}{\mathcal{V}_i(x)}$. As shown in the proof of Theorem 4.1 we have $\lambda \in (0, 1]$ and it is easily seen that the case $\lambda = 1$ is only possible if $x(k) = 0$ is the only solution of (1). *Otherwise, expontential stability holds if the dwell time d of the switching signal satisfies*

$$d > \log_{1-\lambda}(1/\mu).$$

5. The joint spectral radius and exponential stability

The previous section provided only a *sufficient* condition for exponential stability, we will now provide a *necessary and sufficient* condition for exponential stability in terms of the joint spectral radius. Furthermore, we will exploit the solution properties to calculate the joint spectral radius via a standard system of reduced size. Finally, we highlight that the conditions simplify significantly under a certain commutativity assumption.

5.1. Definitions

We first define notions of stability for SDLS systems.

Definition 5.1. *System* (1) *is called stable if there exists a positive constant γ such that for all switching signals, the corresponding solution $x(\cdot)$ with the initial condition $x(0) = \Pi_{\mathcal{S}_{\sigma(0)}}^{\ker E_{\sigma(0)}} x_0$ for $x_0 \in \mathbb{R}^n$, satisfies*

$$\|x(k)\| \le \gamma \|x_0\|.$$

Definition 5.2. *System* (1) *is called exponentially stable if there exist positive constants γ and $0 \le \lambda < 1$, such that for all switching signals, the corresponding solution $x(\cdot)$ with the initial condition $x(0) = \Pi_{\mathcal{S}_{\sigma(0)}}^{\ker E_{\sigma(0)}} x_0$ for $x_0 \in \mathbb{R}^n$ fulfills*

$$\|x(k)\| \le \gamma \lambda^k \|x_0\|.$$

Note that the (exponential) stability definitions are in terms of x_0 and not in terms of $x(0)$ (which is unequal from x_0 in general); this is important for precise characterizations in terms of the transition matrix (in particular for concluding an upper bound of the transition matrix, see e.g. the necessity part of the proof of the forthcoming Theorem 5.5).

Our goal is now to characterize exponential stability with a generalization of the joint spectral radius similar as in Theorem 2.6. Therefore, we first define the joint spectral radius of a family of matrix pairs $\{(E_1, A_1), (E_2, A_2), \dots, (E_{\mathrm{n}}, A_{\mathrm{n}})\}$ corresponding to the SDLS system (1) of index-1 as follows.

Definition 5.3. *The joint spectral radius of* (1) *of index-1 is*

$$\rho\left(\{(E_i, A_i)\}_1^{\mathrm{n}}\right) := \lim_{k \to \infty} \max_{\substack{i_0, i_1, \dots, i_k \\ \in \{1, \dots, \mathrm{n}\}}} \left\|\Phi_{i_k, i_{k-1}} \Phi_{i_{k-1}, i_{k-2}} \cdots \Phi_{i_1, i_0}\right\|^{1/k},$$

where $\Phi_{i,j}$ is the one-step-map as in Theorem 3.4 and $\|\cdot\|$ is the induced matrix norm.

We will show the existence of the limit in Definition 5.3. Letting $a_k = \max_{\substack{i_1, i_2, \dots, i_k \\ \in \{1, \dots, \mathrm{n}\}}} \left\|\Phi_{i_k, i_{k-1}} \Phi_{i_{k-1}, i_{k-2}} \cdots \Phi_{i_1, i_0}\right\|$, we have the following estimates:

$$\begin{aligned} a_{k+m} &= \max_{\substack{i_0, i_1, \dots, i_{k+m} \\ \in \{1, \dots, \mathrm{n}\}}} \left\|\Phi_{i_{k+m}, i_{k+m-1}} \Phi_{i_{k+m-1}, i_{k+m-2}} \cdots \Phi_{i_1, i_0}\right\| \\ &\le \max_{\substack{i_m, \dots, i_{k+m} \\ \in \{1, \dots, \mathrm{n}\}}} \left\|\Phi_{i_{k+m}, i_{k+m-1}} \Phi_{i_{k+m-1}, i_{k+m-2}} \cdots \Phi_{i_{m+1}, i_m}\right\| \\ &\quad \max_{\substack{i_0, i_1, \dots, i_m \\ \in \{1, \dots, \mathrm{n}\}}} \left\|\Phi_{i_m, i_{m-1}} \Phi_{i_{m-1}, i_{m-2}} \cdots \Phi_{i_1, i_0}\right\| \\ &= a_k a_m. \end{aligned}$$

Hence Lemma 2.5 ensures the existence of the required limit.

Remark 5.4. *In the index-0 case, i.e., when $E_i = I_n$, $i = 1, 2, .., \mathrm{n}$, we find $\Phi_{i_k, i_{k-1}} \Phi_{i_{k-1}, i_{k-2}} \cdots \Phi_{i_1, i_0} = A_{i_{k-1}} A_{i_{k-2}} \dots A_{i_0}$. Thus the joint spectral radius of a family of matrix pairs $\{(I_n, A_i)\}_{i=1}^{\mathrm{n}}$ equals the joint spectral radius of the family of matrices $\{A_i\}_{i=1}^{\mathrm{n}}$.*

5.2. Stability characterization via joint spectral radius

Theorem 5.5. *The switched system* (1) *of index-1 is exponentially stable if, and only if,*

$$\rho\left(\{(E_i, A_i)\}_1^{\mathrm{n}}\right) < 1.$$

Proof. *Sufficiency.* Let $K > 0$ and $\lambda \in (0,1)$ such that $\max_{i_0,i_1,\dots,i_k \in \{1,\dots,\mathtt{n}\}} \left\| \Phi_{i_k,i_{k-1}} \Phi_{i_{k-1},i_{k-2}} \cdots \Phi_{i_1,i_0} \right\|^{1/k} \le \lambda < 1$ for all $k \ge K$ and let $x(\cdot)$ be a solution of (1) for some switching signal σ, then due to Theorem 3.4 and in view of (8), for all $k \ge K$

$$\|x(k)\| \le \left\| \Phi_{\sigma(k),\sigma(k-1)} \cdots \Phi_{\sigma(1),\sigma(0)} \right\| \|x_0\| \le \lambda^k \|x_0\| \le \gamma \lambda^k \|x_0\|,$$

for any $\gamma \ge 1$. In fact, let $\gamma \ge 1$ be such that

$$\gamma \ge \max_{1 \le k < K} \max_{\substack{i_0,i_1,\dots,i_k \\ \in \{1,\dots,\mathtt{n}\}}} \frac{\left\| \Phi_{i_k,i_{k-1}} \Phi_{i_{k-1},i_{k-2}} \cdots \Phi_{i_1,i_0} \right\|}{\lambda^k},$$

then, clearly, for all $0 \le k < K$

$$\|x(k)\| \le \gamma \lambda^k \|x_0\|.$$

Necessity. In view of (8), we have that

$$\|x(k)\| = \|\Phi_\sigma(k,0) x_0\| \le \gamma \lambda^k \|x_0\|$$

for all $k \ge 0$, $x_0 \in \mathbb{R}^n$, all switching signals σ and all corresponding solutions $x(\cdot)$. Since $\|\cdot\|$ is an induced matrix norm we therefore have

$$\max_{\substack{i_0,i_1,\dots,i_k \\ \in \{1,\dots,\mathtt{n}\}}} \left\| \Phi_{i_k,i_{k-1}} \Phi_{i_{k-1},i_{k-2}} \cdots \Phi_{i_1,i_0} \right\| \le \gamma \lambda^k,$$

hence,

$$\rho\left(\{(E_i,A_i)\}_1^{\mathtt{n}}\right) \le \lim_{k \to \infty} \gamma^{1/k} \lambda = \lambda < 1. \qquad \blacksquare$$

Remark 5.6. *With similar arguments as in the proof of Theorem 5.5 it is actually possible to show that $\rho\left(\{(E_i,A_i)\}_1^{\mathtt{n}}\right) < 1$ if, and only if, there exists $K > 0$ such that*

$$\left\| \Phi_{i_K,i_{K-1}} \Phi_{i_{K-1},i_{K-2}} \cdots \Phi_{i_1,i_0} \right\| < 1 \quad \forall i_0, i_1, \dots, i_K \in \{1,\dots,\mathtt{n}\}.$$

In fact, if such a $K > 0$ exists, then exponential stability is guaranteed with

$$\lambda = \max_{\substack{i_0,i_1,\dots,i_K \\ \in \{1,\dots,\mathtt{n}\}}} \left\| \Phi_{i_K,i_{K-1}} \Phi_{i_{K-1},i_{K-2}} \cdots \Phi_{i_1,i_0} \right\|^{1/K}$$

$$\gamma = \max_{1 \le k < K} \max_{\substack{i_0,i_1,\dots,i_k \\ \in \{1,\dots,\mathtt{n}\}}} \frac{\left\| \Phi_{i_k,i_{k-1}} \Phi_{i_{k-1},i_{k-2}} \cdots \Phi_{i_1,i_0} \right\|}{\lambda^k}$$

5.3. Reduced order joint spectral radius

Although it is possible, to explicitly calculate the one-step-maps $\Phi_{i,j}$ via Lemma 3.6, we want to provide another approach to check for exponential stability. The advantage of the latter approach is that one can reduce the initial n-dimensional switched linear singular system (1) to a well-understood r-dimensional switched linear nonsingular system. This reduction is based on the following result.

Proposition 5.7. *Consider the switched singular system* (1) *of index-1 and let T_i and $G_{i,j}$ be given as in Lemma 3.6. Then*

$$T_i^{-1} G_{i,j}^{-1} A_i T_j =: \bar{A}_{i,j} = \begin{pmatrix} \bar{A}_{i,j}^1 & 0 \\ \bar{A}_{i,j}^2 & I_{n-r} \end{pmatrix}, \tag{10}$$

for some $\bar{A}_{i,j}^1 \in \mathbb{R}^{r \times r}$ and $\bar{A}_{i,j}^1 \in \mathbb{R}^{(n-r)\times r}$ where r as in Lemma 3.3(a) is assumed to be positive. Furthermore, for any switching signal we have that $x(\cdot)$ is a solution of (1) *if, and only if, $v(\cdot)$ is a solution of*

$$v(k+1) = \bar{A}_{\sigma(k)\sigma(k-1)}^1 v(k) \tag{11}$$

and

$$x(k) = T_{\sigma(k-1)} \begin{pmatrix} v(k) \\ -\bar{A}_{\sigma(k)\sigma(k-1)}^2 v(k) \end{pmatrix}.$$

In particular, (1) *is exponentially stable if, and only if,* (11) *is exponentially stable.*

Proof.

Observe that $G_{i,j} P_i = (E_i + A_i T_j Q T_i^{-1}) T_i P T_i^{-1} = E_i P_i + A_i T_j Q P T_i^{-1} = E_i P_i$. Further, since Q_i is the projection onto $\ker E_i$ along S_i, it follows $E_i Q_i = 0$, therefore, $E_i P_i = E_i(P_i + Q_i) = E_i$. Thus, $G_{i,j} P_i = E_i$, hence $P_i = G_{i,j}^{-1} E_i$.
Furthermore, according to the proof of item (ii) of Lemma 3.6 $G_{i,j} T_i Q = A_i T_j Q$, hence, $T_i^{-1} G_{i,j}^{-1} A_i T_j Q = Q$.
Therefore, we obtain

$$\bar{A}_{i,j} = T_i^{-1} G_{i,j}^{-1} A_i T_j = \begin{pmatrix} \bar{A}_{i,j}^1 & O \\ \bar{A}_{i,j}^2 & I_{n-r} \end{pmatrix},$$

$$\bar{E}_{i,j} := T_i^{-1} G_{i,j}^{-1} E_i T_i = \begin{pmatrix} I_r & O \\ O & O_{n-r} \end{pmatrix}.$$

Multiplying both sides of system (1) by $T_{\sigma(k)}^{-1} G_{\sigma(k)\sigma(k-1)}^{-1}$, and using the transformation $\bar{x}(k) = T_{\sigma(k-1)}^{-1} x(k)$, we get

$$\bar{E}_{\sigma(k)\sigma(k-1)} \bar{x}(k+1) = \bar{A}_{\sigma(k)\sigma(k-1)} \bar{x}(k). \tag{12}$$

Putting $\bar{x}(k) := (v(k)^\top, w(k)^\top)^\top$, where $v(k) \in \mathbb{R}^r$, $w(k) \in \mathbb{R}^{n-r}$, we can reduce system (12) to (11) in combination with

$$w(k) = -\bar{A}_{\sigma(k)\sigma(k-1)}^2 v(k).$$

Since (1) has by assumption only finitely many different modes, $\max_{i \in \{1,\dots,\mathtt{n}\}} \|T_i\| < \infty$, $\max_{i \in \{1,\dots,\mathtt{n}\}} \|T_i^{-1}\| < \infty$; as well as $\max_{i,j \in \{1,\dots,\mathtt{n}\}} \|\bar{A}_{i,j}^2\| < \infty$; consequently, $x(k)$ converges exponentially to zero if, and only if, $v(k)$ does. $\blacksquare$

Remark 5.8. *Comparing* (11) *with* (7) *it becomes apparent that $\bar{A}_{i,j}^1$ does not directly correspond to $\Phi_{i,j}$ for $i \neq j$, because in* (11) *we have that $\bar{A}_{\sigma(k),\sigma(k-1)}^1$ relates $v(k)$ with $v(k+1)$, while in* (7) *the matrix $\Phi_{\sigma(k+1),\sigma(k)}$ relates $x(k)$ with $x(k+1)$.*

Theorem 5.9. *Assume that system* (1) *is of index-1 with $r > 0$ as in Lemma 3.3(a) and consider the reduced system* (11)*. Then*

$$\rho\left(\{(E_i,A_i)\}_{i=1}^{\mathtt{n}}\right) = \rho\left(\{\bar{A}_{i,j}^1\}_{i,j=1}^{\mathtt{n}}\right) = \lim_{k \to \infty} \max_{\substack{i_{-1},i_0,\dots,i_{k-1} \\ \in \{1,\dots,\mathtt{n}\}}} \left\| \bar{A}_{i_{k-1},i_{k-2}}^1 \dots \bar{A}_{i_0,i_{-1}}^1 \right\|^{1/k}.$$

The proof of Theorem 5.9 is based on the following lemma.

Lemma 5.10. *Assume that system* (1) *is of index-1. Then the following statements hold for all $i, j, k \in \{1,\dots,\mathtt{n}\}$:*

(i) $\Phi_{i,j}P_j = \Phi_{i,j}$,

(ii) $\Phi_{i,j} = \widetilde{\Phi}_{i,j,k}P_j$ *where* $\widetilde{\Phi}_{i,j,k} := \Pi^{\ker E_j}_{\mathcal{S}_i} P_j G^{-1}_{j,k} A_j$.

(iii) $P_i G^{-1}_{i,j} A_i \Pi^{\ker E_j}_{S_i} = P_i G^{-1}_{i,j} A_i$,

Proof.

(i) From Lemma 2.10 it follows that

$$\Phi_{(E_j,A_j)}Q_j = T_j \begin{bmatrix} J_j & 0 \\ 0 & 0 \end{bmatrix} T_j^{-1} T_j \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T_j^{-1} = 0$$

and hence by Lemma 3.4 we have $\Phi_{i,j}Q_j = 0$ which concludes the proof because $P_j = I - Q_j$.

(ii) Due to (i) and Lemma 3.6 we have for any $\xi \in \mathbb{R}^n$

$$\Phi_{i,j}\xi = \Phi_{i,j}P_j\xi = \Pi^{\ker E_j}_{\mathcal{S}_i} P_j G^{-1}_{j,j} A_j P_j \xi.$$

Since $P_j\xi \in \mathcal{S}_j = A_j^{-1}(\operatorname{im} E_j)$ there exists $\eta \in \mathbb{R}^n$ such that $A_jP_j\xi = E_j\eta$ and according to the proof of Proposition 4.7 we have $P_j = G^{-1}_{j,k}E_j$ for all j,k, in particular, $G^{-1}_{j,j}E_j = P_j = G^{-1}_{j,k}E_j$, so that

$$\begin{aligned}\Phi_{i,j}\xi &= \Pi^{\ker E_j}_{\mathcal{S}_i} P_j G^{-1}_{j,j} E_j \eta = \Pi^{\ker E_j}_{\mathcal{S}_i} P_j G^{-1}_{j,k} E_j \eta \\ &= \Pi^{\ker E_j}_{\mathcal{S}_i} P_j G^{-1}_{j,k} A_j P_j \xi = \widetilde{\Phi}_{i,j,k} P_j \xi.\end{aligned}$$

Since $\xi \in \mathbb{R}^n$ was arbitrary, the claim is shown.

(iii) We have

$$\begin{aligned} P_i G^{-1}_{i,j} A_i \Pi^{\ker E_j}_{S_i} &= P_i G^{-1}_{i,j} A_i (I - Q_{i,j} G^{-1}_{i,j} A_i) \\ &= P_i G^{-1}_{i,j} A_i - P_i G^{-1}_{i,j} A_i Q_{i,j} G^{-1}_{i,j} A_i \\ &= P_i G^{-1}_{i,j} A_i - T_i P \underline{T_i^{-1} G^{-1}_{i,j} A_i T_j Q} T_i^{-1} G^{-1}_{i,j} A_i \\ &= P_i G^{-1}_{i,j} A_i P_j - T_i P \underline{Q} T_i^{-1} G^{-1}_{i,j} A_i P_j \\ &\overset{PQ=0}{=} P_i G^{-1}_{i,j} A_i P_j = P_i G^{-1}_{i,j} A_i, \end{aligned}$$

where we used $T_i^{-1}G^{-1}_{i,j}A_iT_jQ = Q$, which was already shown in the proof of Proposition 5.7. ■

Proof of Theorem 5.9

From Lemma 5.10 it follows that

$$\begin{aligned} &\Phi_{i_k,i_{k-1}}\Phi_{i_{k-1},i_{k-2}} \cdots \Phi_{i_2,i_1}\Phi_{i_1,i_0} \\ &= \widetilde{\Phi}_{i_k,i_{k-1},i_{k-2}} P_{i_{k-1}} \widetilde{\Phi}_{i_{k-1},i_{k-2},i_{k-3}} P_{i_{k-2}} \cdots \widetilde{\Phi}_{i_2,i_1,i_0} P_{i_1} \widetilde{\Phi}_{i_1,i_0,i_{-1}} P_{i_0} \\ &= \Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}}\, P\bar{A}_{i_{k-1}i_{k-2}}\, P\bar{A}_{i_{k-2}i_{k-3}} \cdots P\bar{A}_{i_1,i_0}\, P\bar{A}_{i_0,i_{-1}}\, T^{-1}_{i_{-1}} P_{i_0}. \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} &\rho\left(\{(E_i,A_i)\}_{i=1}^{\mathrm{n}}\right) \\ &= \lim_{k\to\infty} \max_{\substack{i_0,i_1,\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Phi_{i_k,i_{k-1}}\Phi_{i_{k-1},i_{k-2}} \cdots \Phi_{i_1,i_0}\right\|^{1/k} \\ &\overset{(13)}{=} \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}} P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_0,i_{-1}} T^{-1}_{i_{-1}} P_{i_0}\right\|^{1/k} \\ &\le \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_{k-1} \\ \in\{1,\dots,\mathrm{n}\}}} \left\|P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_0,i_{-1}}\right\|^{1/k} \cdot \\ &\quad \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left(\left\|\Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}}\right\| \left\|T^{-1}_{i_{-1}} P_{i_0}\right\|\right)^{1/k} \\ &= \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_{k-1} \\ \in\{1,\dots,\mathrm{n}\}}} \left(\left\|P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_0,i_{-1}}\right\|\right)^{1/k} \\ &= \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_{k-1} \\ \in\{1,\dots,\mathrm{n}\}}} \left(\left\|\bar{A}^1_{i_{k-1},i_{k-2}} \dots \bar{A}^1_{i_0,i_{-1}}\right\|\right)^{1/k} = \rho\left(\{\bar{A}^1_{i,j}\}_{i,j=1}^{\mathrm{n}}\right). \end{aligned}$$

On the other hands, we clearly have $P_i\Pi^{\ker E_j}_{\mathcal{S}_i} = \Pi^{\ker E_j}_{\mathcal{S}_i}$ which together with Lemma 5.10(iii) and (13) yields

$$\begin{aligned} &\Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}} P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_1,i_0} P\bar{A}_{i_0,i_{-1}} T^{-1}_{i_{-1}} \\ &= \Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}} P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_1,i_0} P\bar{A}_{i_0,i_{-1}} T^{-1}_{i_{-1}} \Pi^{\ker E_{i_{-1}}}_{\mathcal{S}_{i_0}} \\ &= \Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}} P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_1,i_0} P\bar{A}_{i_0,i_{-1}} T^{-1}_{i_{-1}} P_{i_0} \Pi^{\ker E_{i_{-1}}}_{\mathcal{S}_{i_0}} \\ &= \Phi_{i_k,i_{k-1}} \dots \Phi_{i_1,i_0} \Pi^{\ker E_{i_{-1}}}_{\mathcal{S}_{i_0}}. \end{aligned} \tag{14}$$

Note that $P_i\Pi^{\ker E_j}_{\mathcal{S}_i}P_i = P_i$, which is equivalent to

$$PT_i^{-1}\Pi^{\ker E_j}_{\mathcal{S}_i} T_i P = P, \tag{15}$$

therefore,

$$\begin{aligned} &\rho\left(\{\bar{A}^1_{i,j}\}_{i,j=1}^{\mathrm{n}}\right) \\ &= \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_{k-1} \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\bar{A}^1_{i_{k-1},i_{k-2}} \dots \bar{A}^1_{i_0,i_{-1}}\right\|^{1/k} \\ &= \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_{k-1} \\ \in\{1,\dots,\mathrm{n}\}}} \left\|P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_0,i_{-1}}\right\|^{1/k} \\ &\overset{(15)}{=} \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|PT^{-1}_{i_{k-1}} \Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}} P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_0,i_{-1}}\right\|^{1/k} \\ &\le \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Pi^{\ker E_{i_{k-1}}}_{\mathcal{S}_{i_k}} T_{i_{k-1}} P\bar{A}_{i_{k-1},i_{k-2}} \dots P\bar{A}_{i_0,i_{-1}} T^{-1}_{i_{-1}}\right\|^{1/k} \cdot \\ &\quad \lim_{k\to\infty} \max_{\substack{i_{-1},i_{k-1} \\ \in\{1,\dots,\mathrm{n}\}}} \left(\left\|PT^{-1}_{i_{k-1}}\right\| \left\|T_{i_{-1}}\right\|\right)^{1/k} \\ &\overset{(14)}{=} \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Phi_{i_k,i_{k-1}} \dots \Phi_{i_1,i_0} \Pi^{\ker E_{i_{-1}}}_{S_{i_0}}\right\|^{1/k} \\ &\le \lim_{k\to\infty} \max_{\substack{i_{-1},\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Phi_{i_k,i_{k-1}} \dots \Phi_{i_1,i_0}\right\|^{1/k} \cdot \lim_{k\to\infty} \max_{\substack{i_{-1},i_0 \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Pi^{\ker E_{i_{-1}}}_{S_{i_0}}\right\|^{1/k} \\ &= \lim_{k\to\infty} \max_{\substack{i_0,\dots,i_k \\ \in\{1,\dots,\mathrm{n}\}}} \left\|\Phi_{i_k,i_{k-1}} \dots \Phi_{i_1,i_0}\right\|^{1/k} = \rho\left(\{(E_i,A_i)\}_{i=1}^{\mathrm{n}}\right). \end{aligned}$$

Thus, we get $\rho\left(\{(E_i,A_i)\}_{i=1}^{\mathrm{n}}\right) = \rho\left(\{\bar{A}^1_{i,j}\}_{i,j=1}^{\mathrm{n}}\right)$. ■

Remark 5.11. *The joint spectral radius in Definition 5.3 may also called Bohl exponent for the index-1 SDSL* (1)*, cf. [10] and the references therein. Concerning the computation of the joint-spectral radius (or Bohl exponent) of the SDLS system* (1) *we first remark, that already in the nonsingular case the calculation (of an approximation) is an NP-hard problem [27] and the situation gets even worse for the SDLS system* (1) *because according to Definition 5.3 the joint spectral radius needs to be calculated for* n^2 *matrices (instead of* n *matrices in the nonsingular case). This complexity issues is to a little extend resolved by Theorem 5.9, because it shows that the calculation can be carried out with smaller* $r \times r$ *matrices instead of* $n \times n$ *matrices.*

We illustrate Theorem 5.9 by applying it to the following Example.

Example 5.12. *Let*

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 3 & -1 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \right),$$

$$(E_2, A_2) = \left(\begin{bmatrix} 3 & -6 & -3 \\ 0 & -3 & 0 \\ -6 & -9 & 6 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & 3 \end{bmatrix} \right)$$

A simple computation shows that $\ker E_1 = \ker E_2 = \operatorname{span}\{(1,0,1)^\top\}$, $S_1 = S_2 = \operatorname{span}\{(-1,1,0)^\top, (0,-1,1)^\top\}$, *hence* $S_i \cap \ker E_j = \{0\}$, $i, j = 1, 2$. *Hence system* (1) *with the above data is of index-1. An easy computation shows that with the choice of* $T_1 = T_2 = \left[\begin{smallmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right]$, *we have*

$$\bar{A}^1_{1,1} = \bar{A}^1_{1,2} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad \bar{A}^1_{2,2} = \bar{A}^1_{2,1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix},$$

$$\Phi_{11} = \Phi_{21} = \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}, \qquad \Phi_{22} = \Phi_{12} = \begin{bmatrix} \frac{1}{6} & \frac{-1}{6} & \frac{-1}{6} \\ 0 & \frac{1}{3} & 0 \\ \frac{-1}{6} & \frac{-1}{6} & \frac{1}{6} \end{bmatrix},$$

Using Theorem 5.9, we find $\rho(\{(E_i, A_i)\}_{i=1}^2) = \frac{1}{2} < 1$. *According to Theorem 5.5, the SDLS system with the above data* $\{(E_i, A_i)\}_{i=1}^2$ *is therefore exponentially stable.*
We see that the spectral radius cannot be read off directly from $\Phi_{i,j}$ *but can be easily determined from* $\bar{A}^1_{i,j}$.

5.4. Commutativity and exponential stability

It is well known (see e.g. [17, 37]), that a certain commutativity conditions ensure exponential stability under arbitrary switching for switched systems, hence Lemma 3.4 immediately leads to the following result on exponential stability under arbitrary switching.

Corollary 5.13. *Consider the SDLS system* (1) *of index-1 and assume that all one-step maps* $\Phi_{i,j}$ *as in Theorem 3.4 have eigenvalues in the (open) unit circle and commute with each other. Then the SDLS system* (1) *is exponentially stable. In particular, the joint spectral radius is the the largest absolut value of all eigenvalues of all one-step-maps* $\Phi_{i,j}$.

Due to Proposition 5.7 it is also possible to test the commutativity condition on the smaller matrices $\bar{A}^1_{i,j}$.

Corollary 5.14. *Consider the SDLS system* (1) *of index-1 and let* $\bar{A}^1_{i,j}$ *be given according to Proposition 5.7. If all eigenvalues of* $\bar{A}^1_{i,j}$ *are in the (open) unit circle and all* $\bar{A}^1_{i,j}$ *commute with each other then the SDLS system* (1) *is exponentially stable.*

Remark 5.15. *(i) Since*

$$\begin{aligned} \det(\lambda I_r - \bar{A}_{i,j}) &= \det(\lambda T_i^{-1} G_{i,j}^{-1} E_i T_i - T_i^{-1} G_{i,j}^{-1} A_i T_j) \\ &= \det(T_i^{-1} G_{i,j}^{-1}) \det(\lambda E_i T_i - A_i T_j), \end{aligned}$$

and due to the block upper diagonal structure of $\bar{A}_{i,j}$, *all eigenvalues of* $\bar{A}^1_{i,j}$ *are also finite (generalized) eigenvalues of the pair* $(E_i T_i, A_i T_j)$ *(i.e. those* $\lambda \in \mathbb{C}$ *such that* $\det(\lambda E_i T_i - A_i T_j) = 0$*). Hence a sufficient condition for* $\bar{A}^1_{i,j}$ *having only eigenvalues in the unit circle is that the pair* $(E_i T_i, A_i T_j)$ *has this property.*

(ii) There seems to be no obvious connection between the eigenvalues of $\Phi_{i,j}$ *and* $\bar{A}_{i,j}$ *and also not between the commutativity of each. Hence as of now it is not clear, which stability condition is more conservative.*

Example 5.16. *Let us consider Example 5.12 again. Clearly, the diagonal matrices* $\bar{A}^1_{i,j}$ *commute and the eigenvalues have magnitude smaller than one, hence (without explicitely calculating the joint spectral radius we can conclude via Corollary 5.14 that* (1) *is exponentially stable under arbitrary switching. Note that the matrices* $\Phi_{i,j}$ *also commute and their spectrum is given by* $\{0, 1/2, 1/2\}$ *or* $\{0, 1/3, 1/3\}$, *so Corollary 5.13 can also be used to establish exponential stability of* (1).

Remark 5.17. *There are many similarities between the continuous and discrete time case. In fact, the single-mode one-step map* $\Phi_{(E,A)}$ *is* identical *to the matrix* A^{diff} *which describes the dynamics of* $E\dot{x} = Ax$ *via the equation* $\dot{x} = A^{diff} x$ *(see [28, 29]). There is also a similarity when considering switching, in both cases the dynamics are described by the single-mode one-step map in combination with a certain projector. However, the major difference between continuous and discrete time case is that for the discrete time case, the resulting dynamics can still be described in a usual way (i.e.* $x(k+1) = Mx(k)$ *for some matrix* M*), while in the continuous time case the resulting dynamics are a combination of a continuous flow with a discrete jump. In particular, an adequate notion of commutativity for the continuous time case and the corresponding proof of exponential stability [16] is much more complicated than for the discrete time case.*

6. Conclusion

In this paper a class of SDLS systems of index-1 is introduced and a corresponding one-step-map is established. The stability of such systems is studied by using Lyapunov functions as well as the joint spectral radius of a set of matrix pairs. Furthermore, certain commutativity conditions are shown to preserve exponential stability.

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