Synchronization of heterogeneous agents via funnel coupling

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Problem statement

Given

- N agents with individual n-dimensional dynamics:

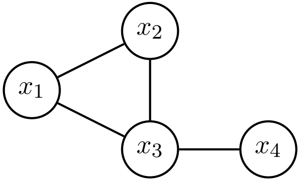
$$\dot{x}_i = f_i(t, x_i) + u_i$$

- undirected connected coupling-graph $G = (V, E)$
- local feedback $u_i = \gamma_i(x_i, x_{\mathcal{N}_i})$

Desired

Control design for practical synchronization

$$x_1 \approx x_2 \approx \ldots \approx x_n$$



$$u_1 = \gamma_1(x_1, x_2, x_3)$$
$$u_2 = \gamma_2(x_2, x_1, x_3)$$
$$u_3 = \gamma_3(x_3, x_1, x_2, x_4)$$
$$u_4 = \gamma_4(x_4, x_3)$$

A „high-gain“ result

Let $\mathcal{N}_i := \{j \in V \mid (j,i) \in E\}$ and $d_i := |\mathcal{N}_i|$ and $\mathcal{L}$ be the Laplacian of G.

Diffusive coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j) \quad \text{or, equivalently,} \quad u = -k\ \mathcal{L}\ x$$

Theorem (Practical synchronization, Kim et al. 2013)

Assumptions: G connected, all solutions of average dynamics

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))$$

remain bounded. Then $\forall \varepsilon > 0\ \exists K > 0\ \forall k \geq K$: Diffusive coupling results in

$$\boxed{\limsup_{t \to \infty} \|x_i(t) - x_j(t)\| < \varepsilon \quad \forall i,j \in V}$$

Remarks on high-gain result

Common trajectory

It even holds that

$$\limsup_{t\to\infty} |x_i(t) - s(t)| < \varepsilon/2,$$

where $s(\cdot)$ solves $\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \quad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i.$

Independent of coupling structure and amplification k.

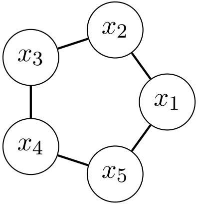
Error feedback

With $e_i := x_i - \overline{x}_i$ and $\overline{x}_i := \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j$ diffusive coupling has the form

$$u_i(t) = -ke_i(t)$$

Attention: $e_i \neq x_i - s$, in particular, agents do not know „limit trajectory“ $s(\cdot)$

Example (taken from Kim et al. 2015)



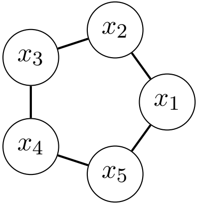
Simulations in the following for $N = 5$ agents with dynamics

$$f_i(t, x_i) = (-1+\delta_i)x_i + 10\sin t + 10m_i^1 \sin(0.1t+\theta_i^1) + 10m_i^2 \sin(10t+\theta_i^2),$$

with randomly chosen parameters $\delta_i, m_i^1, m_i^2 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$.

Parameters identical in all following simulations, in particular $\delta_2 > 1$, hence agent 2 has unstable dynamics (without coupling).

Example (taken from KIM et al. 2015)

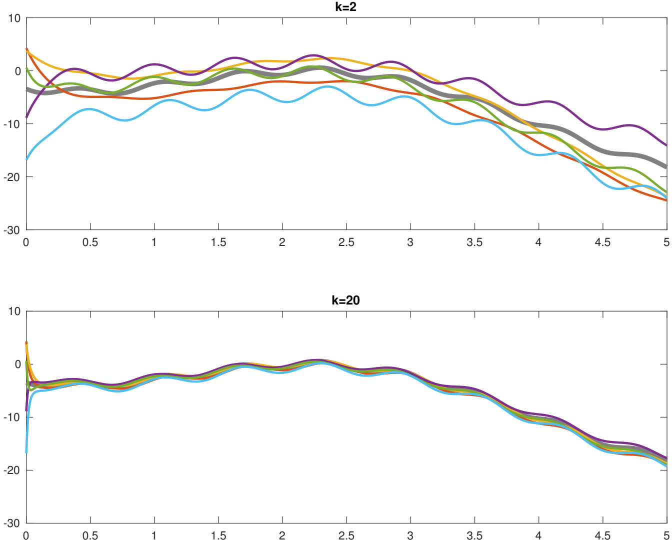


$$u = -k\ \mathcal{L}\ x$$

gray curve:

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t))$$

$$s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$$



Remaining problems

Problems

- necessary gain depends on global network structure
- all agents must use the same gain
- no direct control over convergence rate
- only practical synchronization, no asymptotic convergence

High gain adaptive control

Problems very similar to high gain control problems!

$\hookrightarrow$ Funnel Control

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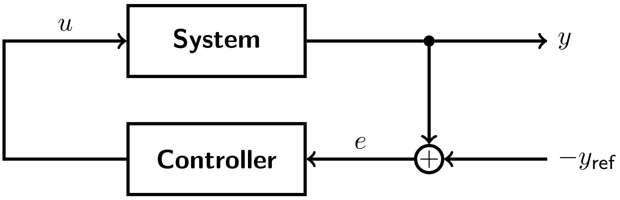
Control Task

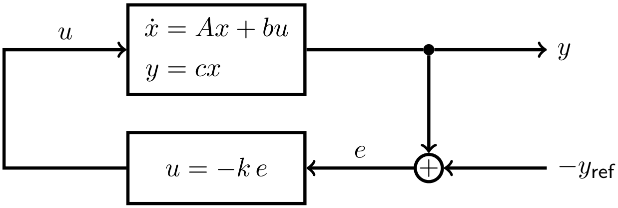


Goal: Output tracking

without

- exact knowledge of system model
- knowledge of reference signal (only error e is available for controller)

High-gain-feedback: linear case



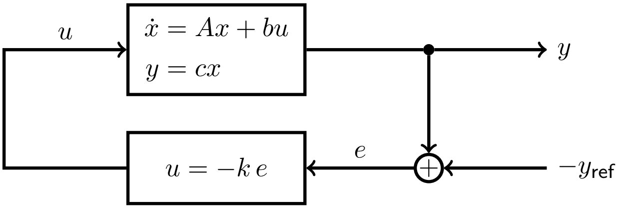
Assumptions:

- Relative degree $r = 1$ $\Leftrightarrow$ $\gamma := cb \neq 0$, in particular:

$$\text{System} \quad \Leftrightarrow \quad \begin{aligned} \dot{y} &= a_{11}y + a_{12}z + \gamma u \\ \dot{z} &= a_{21}y + A_{22}z \end{aligned}$$

- positive high frequency gain $\Leftrightarrow$ $\gamma > 0$
- stable zero-dynamics (minimum phase) $\Leftrightarrow$ A_{22} Hurwitz

High-gain-feedback: linear case



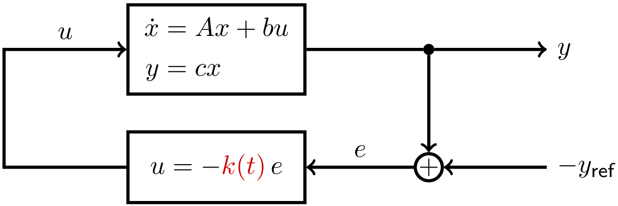
Theorem (High-Gain Feedback)

$cb > 0$, *stable zero-dynamics and* $y_{\text{ref}} = 0$

$$\Rightarrow \quad \exists K > 0\ \forall k \geq K: \quad |e(t)| \to 0$$

Problem: How to find K?

High-gain-feedback: linear case



Idea: Make gain k time varying

Theorem (Adaptive High-Gain Feedback, Byrnes & Willems 1984)

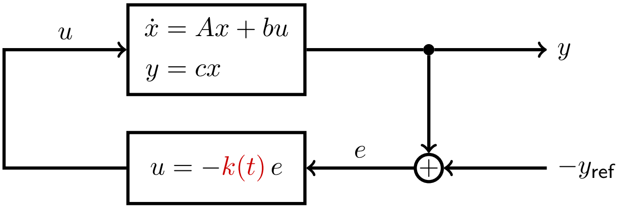
$cb > 0$, stable zero-dynamics and $y_{\text{ref}} = 0$ $\Rightarrow$

$$\dot{k}(t) = e(t)^2 \text{ makes closed loop asymptotically stable}$$

and $k(\cdot)$ remains bounded

Problem: Disturbances or $y_{\text{ref}} \neq 0$ lead to unbounded $k(\cdot)$!

High-gain-feedback: linear case



Solution: Aim for practical stability, i.e. $|e(t)| \leq \lambda$ for $t >> 0$ and some small $\lambda > 0$

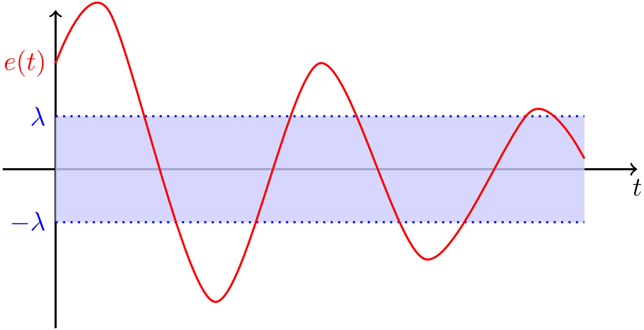
Theorem (λ-tracking, Ilchmann & Ryan 1994)

Assume $cb > 0$, stable zero-dynamics and $y_{\text{ref}}, \dot{y}_{\text{ref}}$ bounded. For $\lambda > 0$ consider

$$\dot{k}(t) = \begin{cases} |e(t)|\big(|e(t)| - \lambda\big), & |e(t)| > \lambda, \\ 0, & |e(t)| \leq \lambda. \end{cases}$$

Then the closed loop is practically stable.

Remaining problems of λ-tracker



Problems:

- No guarantees when $|e(t)| \leq \lambda$
- No bounds on transient behaviour
- Monotonically growing $k(\cdot)$ $\Rightarrow$ Measurement noise unnecessarily amplified

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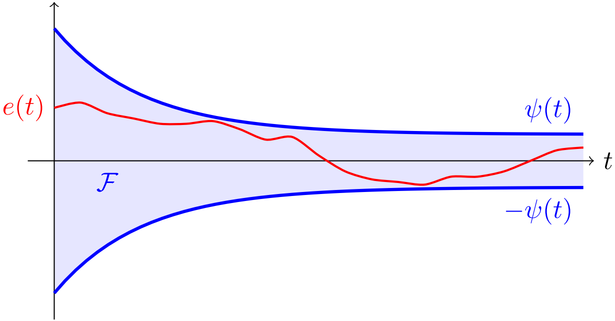
The funnel as time-varying error bound

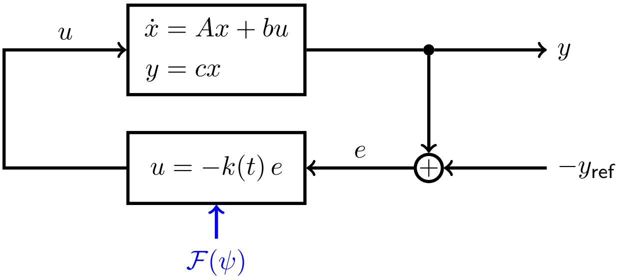
$$\mathcal{F} = \mathcal{F}(\psi) := \{(t,e) \mid |e| < \psi(t)\}$$



Idea: $k(t)$ large $\Leftrightarrow$ Distance of $e(t)$ to funnel boundary small

$$u(t) = -k(t)e(t) \quad \text{with} \quad \boxed{k(t) = \frac{1}{\psi(t) - |e(t)|}}$$

The linear SISO case



Theorem (Funnel Control, Ilchmann, Ryan, Sangwin 2002)

Let $cb > 0$, A_{22} *Hurwitz,* $y_{\text{ref}}, \dot{y}_{\text{ref}}, \psi, \dot{\psi}$ *bounded,* $\liminf_{t\to\infty} \psi(t) > 0$ *and* $|e(0)| < \psi(0)$. *Then*

$$k(t) = \frac{1}{\psi(t) - |e(t)|}$$

remains bounded in the closed loop, i.e. $e(t)$ *remains within funnel.*

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Asymptotic tracking impossible by design?

$$u(t) = -k(t)e(t) \quad \text{with} \quad \boxed{k(t) = \frac{1}{\psi(t) - |e(t)|}}$$

Asymptotic tracking only with unbounded gain

Asymptotic tracking $\Leftrightarrow \quad \psi(t) \underset{t\to\infty}{\to} 0 \quad \Rightarrow \quad \psi(t) - |e(t)| \underset{t\to\infty}{\to} 0$

$\Leftrightarrow \quad k(t) \underset{t\to\infty}{\to} \infty$

Conclusion: Funnel control and asymptotic tracking not compatible **?**

Important observation

$\psi(t) \underset{t\to\infty}{\to} 0 \quad \Longrightarrow \quad |e(t)| \underset{t\to\infty}{\to} 0$ hence

$$u(t) = -k(t)e(t) \to \infty \cdot 0$$

not necessarily unbounded!

Rewrite rule for funnel control

$$u(t) = -\frac{1}{\psi(t) - |e(t)|} e(t) = -\frac{1}{1 - \left|\frac{e(t)}{\psi(t)}\right|} \frac{e(t)}{\psi(t)} =: -\alpha(\eta(t)) \cdot \beta(\eta(t))$$

where $\eta(t) := \frac{e(t)}{\psi(t)}$ and

- $1 - \eta(t)$ is the relative difference between error and funnel boundary
- $\alpha : (-1, 1) \to \mathbb{R}$ with $\alpha(\eta) \to \infty$ as $|\eta| \to 1$
- $\beta : (-1, 1) \to \mathbb{R}$ bounded with $\beta(\eta) \not\to 0$ as $|\eta| \to 1$ and $\operatorname{sgn} \beta(\eta) = \operatorname{sgn} \eta$

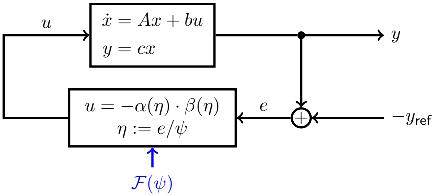
Remark

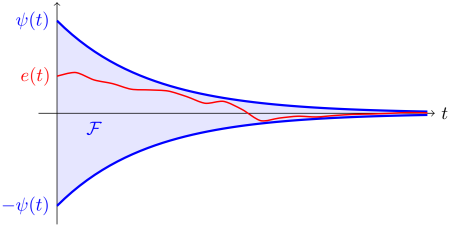
The original funnel controller already had a very similar structure:

$$u(t) = -\alpha(\varphi(t)e(t)) \cdot e(t)$$

with $\varphi := 1/\psi$, but $e(t) \to 0$ still requires unbounded gain.

Asymptotic tracking via funnel control





Theorem (Lee & T. 2019 (CDC))

Let $cb > 0$, A_{22} *Hurwitz,* $y_{\text{ref}}, \dot{y}_{\text{ref}}, \psi, \dot{\psi}$ *bounded,* $\psi > 0$, $|e(0)| < \psi(0)$, $\alpha : (-1,1) \to \mathbb{R}$ *with* $\alpha(\eta) \to \infty$ *as* $|\eta| \to 1$, $\beta : (-1,1) \to \mathbb{R}$ *bounded with* $\beta(\eta) \not\to 0$ *as* $|\eta| \to 1$ *and* $\operatorname{sgn} \beta(\eta) = \operatorname{sgn} \eta$ *then* $\exists > 0$

$$|e(t)| < (1 - \varepsilon)\psi(t) \quad \forall t \geq 0$$

In particular, for $\psi(t) \to 0$ *asymptotic tracking is achieved.*

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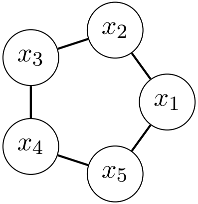
Back to synchronization problem

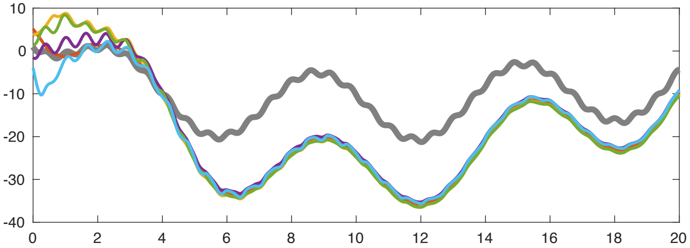
Reminder diffusive coupling: $u_i = -k_i e_i$ with $e_i = x_i - \overline{x}_i$.

Combine diffusive coupling with Funnel Controller

$$u_i(t) = -k_i(t)\, e_i(t) \quad \text{with} \quad k_i(t) = \frac{1}{\psi(t) - |e_i(t)|}$$

First simulations



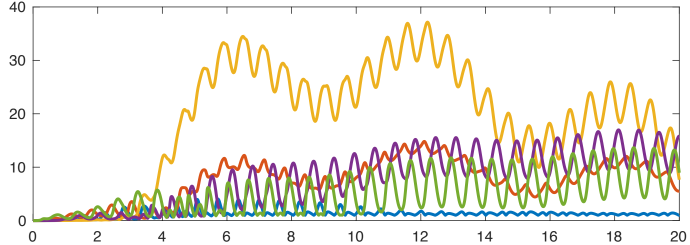


$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\psi(t) - |e_i(t)|}$$

$$\psi(t) = \underline{\psi} + (\overline{\psi} - \underline{\psi})e^{-\lambda t}$$

$\overline{\psi} = 20$, $\underline{\psi} = 1$,
$\lambda = 1$



Observations from simulations

Funnel synchronization seems to work

- errors remain within funnel
- practical synchronizations is achieved
- limit trajectory does not coincide with solution $s(\cdot)$ of

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \qquad s(0) = \tfrac{1}{N}\textstyle\sum_{i=1}^{N} x_i(0).$$

What determines the new limiting trajectory?

- Coupling graph?
- Funnel shape?
- Gain function?

Weakly centralized funnel synchronization

Weakly centralized Funnel synchronization

Analogously as for diffusive coupling, all agents use the same gain:

$$u_i(t) = -k_{\max}(t)\, d_i\, e_i(t) \quad \text{with} \quad k_{\max}(t) := \max_{i \in V} \frac{1}{\psi(t) - |e_i(t)|}$$

Theorem (Shim & T. 2015 (CDC))

Assumption:

- *No „finite escape time“ of x_i*
- *The graph is connected, undirected and d-regular with $d > \frac{N}{2} - 1$*
- *Funnel boundary $\psi : [0, \infty) \to [\underline{\psi}, \overline{\psi}]$ is differentiable, non-increasing and*

$$|e_i(0)| < \psi(0), \quad \forall i = 1, 2, \ldots, N.$$

Then weakly centralized funnel synchronization works.

Node wise funnel synchronization: general case

Theorem (Lee, T. & Shim 2019, submitted)

Multiagent system with symmetric, connected coupling graph under funnel coupling:

$$\boxed{\dot{x}_i = f_i(t, x_i) - \mu\left(\frac{e_i(t)}{\psi(t)}\right)} \qquad e_i(t) := x_i - \tfrac{1}{d_i}\textstyle\sum_{j\in\mathcal{N}_i} x_j, \quad \mu(\eta) := \frac{\eta}{1-|\eta|}$$

Assume that $\dot{\overline{\chi}}(t) = \max_{i\in\mathcal{N}} f_i(t, \overline{\chi}(t))$ *and* $\dot{\underline{\chi}}(t) = \min_{i\in\mathcal{N}} f_i(t, \underline{\chi}(t))$ *do not exhibit finite escape time, then*

$$|e_i(t)| < \psi(t) \quad \forall t \in [0, \infty).$$

Furthermore, the emergent behavior is given by

$$\dot{\xi} = h_\mu(f_1(t,\xi), f_2(t,\xi), \ldots, f_N(t,\xi)),$$

where h_μ *is unique function implicitely given by* $\sum_{i=1}^N \mu^{-1}\left(h_\mu(f_1, \ldots, f_N) - f_i\right) = 0$.

Why not average dynamics?

Laplacian feedback

Diffusive coupling

$$u = -k\,\mathcal{L}\,x$$

has Laplacian feedback matrix $k\mathcal{L}$

Non-Laplacian feedback

Funnel synchronization

$$u = -K(t)\,\mathcal{L}\,x = -\begin{bmatrix} k_1(t) & & & \\ & k_2(t) & & \\ & & \ddots & \\ & & & k_N(t) \end{bmatrix} \mathcal{L}\,x$$

has non-Laplacian feedback matrix $K(t)\mathcal{L}$, in particular $[1, 1, \ldots, 1]^\top$ is not a left-eigenvector of $K(t)\mathcal{L}$.

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Diffusive coupling revisited

Diffusive coupling for weighted graph

$$u_i = -k \sum_{i}^{N} \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = - \sum_{i}^{N} k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j)$$

where $\alpha_{ij} = \alpha_{ji} \in \{0, 1\}$ is the weight of edge (i, j)

Conjecture

If $k_{ij} = k_{ji}$ are all sufficiently large, then practical synchronization occurs with desired limit trajectory s of average dynamics.

Proof technique from Kim et al. 2013 should still work in this setup.

Edgewise Funnel synchronization

Diffusive coupling → edgewise Funnel synchronization

$$u_i = -\sum_i^N k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_i^N k_{ij}(t) \cdot \alpha_{ij} \cdot (x_i - x_j)$$

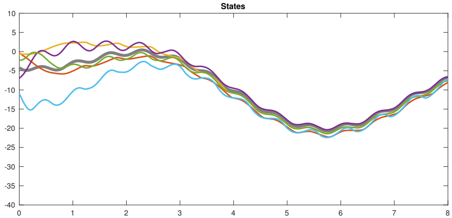
Edgewise error feedback

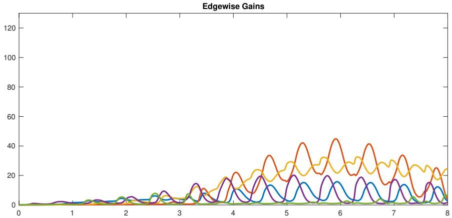
$$k_{ij}(t) = \frac{1}{\psi(t) - |e_{ij}|}, \quad \text{with} \quad e_{ij} := x_i - x_j$$

Properties:

- Decentralized, i.e. u_i only depends on state of neighbors
- Symmetry, $k_{ij} = k_{ji}$
- Laplacian feedback, $u = -\mathcal{L}_K(t, x)x$

Simulation (from T. 2017)





Properties

+ Synchronization occurs
+ Predictable limit trajectory (given by average dynamics)
+ Local feedback law

Edgwise-wise funnel control

Theorem (Lee, Berger, Shim, T. 2019, in preperation)

Multiagent system with symmetric, connected coupling graph under funnel coupling:

$$\dot{x}_i = f_i(t, x_i) - \sum_{j \in \mathcal{N}_i} \frac{x_i - x_j}{\psi(t) - |x_i - x_j|}$$

If f_i is globally Lischitz in x_i then

$$|x_i(t) - x_j(t)| < \psi(t) \quad \forall t \in [0, \infty)$$

Furthermore, the emergent behaviour is given by $s(\cdot)$ where

$$\dot{s} = \sum_{i=1}^{N} f_i(t, s), \quad s(0) = \tfrac{1}{N} \textstyle\sum_{i=1}^{N} x_i(0).$$

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The problem

Minimization problem

For some differentiable and convex $F : \mathbb{R}^n \to \mathbb{R}$ find $x^* \in \mathbb{R}^n$ which solves

$$\min_{x \in \mathbb{R}^n} F(x)$$

Challenge

$$F(x) = \sum_{i=1}^{N} F_i(x)$$

Find x^* in a decentralized way with N agents who only know their $F_i : \mathbb{R}^n \to \mathbb{R}$, which are also not assumed to be convex.

Average dynamics and gradient descent

Local gradient descent + coupling

$$\dot{x}_i = -\nabla F_i(x_i) + u_i \qquad x_i(0) = x_{0,i} \in \mathbb{R}^n$$

Average dynamics for local gradient descent

For $f_i(t,s) = -\nabla F_i(s)$ we have

$$\dot{s}(t) = -\frac{1}{N}\sum_{i=1}^{N} \nabla F_i(s(t)) = -\frac{1}{N}\nabla F(s(t))$$

$\hookrightarrow$ Average dynamics = global gradient descent

Corollary (Lee, Berger, T. & Shim 2019 submitted)

Edgewise funnel coupling solves decentralized optimization asymptotically.

Convergence rate

Average dynamics = gradient descent

$$\dot{s} = -\frac{1}{N}\sum_{i=1}^{N} \nabla F_i(s)$$

Convergence towards x^* not influenced by choice of coupling rule

Edge-wise funnel coupling

$$x_i(t) - s(t) \to 0 \quad \text{as } t \to \infty$$

Convergence rate (and transient behavior) directly influenceable by choice of ψ

Example: Distributed Least-square solver

Problem

Find least-square solution of $Ax = b$, where $A \in \mathbb{R}^{M\times n}$ with $M = \sum_{i=1}^{N} m_i$. Then

$$\frac{1}{2}\|Ax - b\|_2^2 = \sum_{i=1}^{N} \frac{1}{2}\|A_i x - b_i\|_2^2$$

where $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$ are the corresponding block rows of A and b.

Local gradient descent with funnel coupling

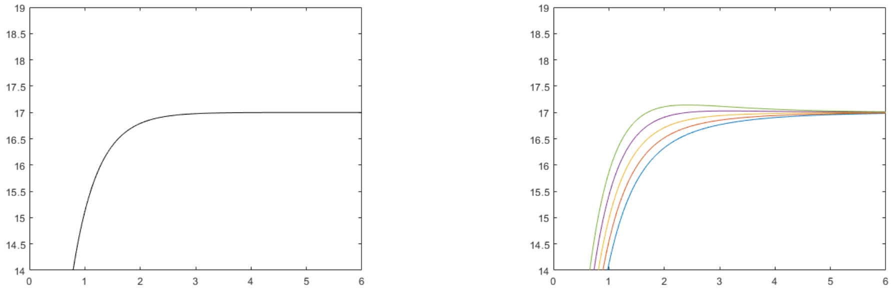
$$\dot{x}_i = -A_i^\top (A_i x_i - b_i) - \sum_{j\in\mathcal{N}_i} \left(\frac{e_{ij}^1}{\psi - |e_{ij}^1|}, \frac{e_{ij}^2}{\psi - |e_{ij}^2|}, \ldots, \frac{e_{ij}^n}{\psi - |e_{ij}^n|} \right)^\top$$

Simulation

For $A = [1, 1, 2, 2, 1]^\top$ and $b = [1, 10, 20, 18, 100]^\top$,
$x_1(0) = 0$, $x_2(0) = -x_3(0) = 0.1$, $x_4(0) = -x_5(0) = 0.2$,
$\psi(t) = \exp(-0.8t)$ and a line coupling we obtain

Average dynamics
$=$ global gradient descent

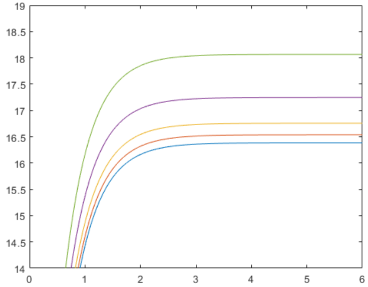
Funnel synchronization



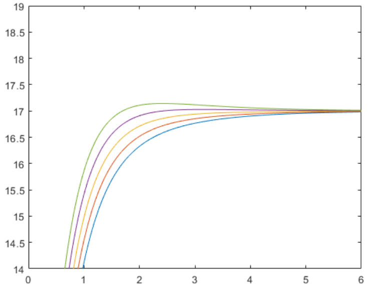
Simulation

For $A = [1, 1, 2, 2, 1]^\top$ and $b = [1, 10, 20, 18, 100]^\top$,
$x_1(0) = 0$, $x_2(0) = -x_3(0) = 0.1$, $x_4(0) = -x_5(0) = 0.2$,
$\psi(t) = \exp(-0.8t)$ and a line coupling we obtain

Constant gain
$k = 100$



Funnel synchronization



Summary

- **Synchronization of heterogeneous agents**
  - constant high-gain $\rightarrow$ practical synchronization
  - disadvantage: necessary gain unknown + must be same for all agents
- **Funnel control**
  - based on ideas from high gain adaptive control
  - prescribed transient behavior: convergence rate and no overshoot
  - asymptotic tracking possible
- **Node-wise funnel coupling**
  - simple initial idea
  - synchronization occurs
  - emergent behavior not equal to average dynamics
- **Edge-wise funnel coupling**
  - Laplacian feedback
  - emergent behavior = average dynamics
  - application to decentralized optimization

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