Synchronization with prescribed transient behavior: Heterogeneous multi-agent systems under funnel coupling ⋆

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Abstract

In this paper, we introduce a nonlinear time-varying coupling law, which can be designed in a fully decentralized manner and achieves approximate synchronization with arbitrary precision, under only mild assumptions on the individual vector fields and the underlying graph structure. The proposed coupling law is motivated by the funnel control studied in adaptive controls under the observation that arbitrary precision synchronization can be achieved for heterogeneous multi-agent systems by the high-gain coupling, and thus, we follow to call our coupling law as '(node-wise) funnel coupling.' By getting out of the conventional proof technique in the funnel control study, we now can obtain even asymptotic or finite-time synchronization with the same funnel coupling law. More interestingly, the emergent collective behavior that arises for a heterogeneous multi-agent system when enforcing arbitrary precision synchronization by the proposed funnel coupling law, has been analyzed in this paper. In particular, we introduce a single scalar dynamics called 'emergent dynamics' that is capable of illustrating the emergent synchronized behavior by its solution trajectory. Characterization of the emergent dynamics is important because, for instance, one can design the emergent dynamics first such that the solution trajectory behaves as desired, and then, provide a design guideline to each agent so that the constructed vector fields yield the desired emergent dynamics. A particular example illustrating the utility of the emergent dynamics is given also in the paper as a distributed median solver.

Key words: synchronization, heterogeneous multi-agents, emergent dynamics, funnel control

1 Introduction

During the last decade, synchronization and collective behavior of a multi-agent system have attracted increasing attention because of numerous applications in diverse areas, e.g., biology, physics, and engineering. An initial study was about identical multi-agents (Olfati-Saber & Murray, 2004; Moreau, 2004; Ren & Beard, 2005; Seo, Shim, & Back, 2009), but the interest soon transferred to the heterogeneous case because, uncertainty, disturbance, and noise are prevalent in practice. Therefore, it may be a natural follow-up to study synchronization of a heterogeneous multi-agent system. Earlier results in this direction such as (Wieland, Wu, & Allgöwer, 2013) have found that for synchronization to happen in a heterogeneous network, each agent must contain a common internal model that is the same for all the agents. However, recalling that heterogeneity can be given, for instance, by noise, the assumption that a common internal model exists may be too ideal, and approximate (practical) synchronization has been studied as an alternative (Montenbruck, Bürger, & Allgöwer, 2015; Ha, Noh, & Park, 2015). We want to note that it is only recent that some attempts are made to analyze the emergent collective behavior of a heterogeneous multi-agent system that achieves approximate synchronization (Kim, Yang, Shim, Kim, & Seo, 2016; Panteley & Loría, 2017; Lee & Shim, 2019).

In this paper, we introduce a nonlinear time-varying coupling law, which can be designed in a fully decentralized manner and achieves approximate synchronization with arbitrary precision, for heterogeneous agents given by

$$\dot{x}_i(t) = f_i(t, x_i(t)) + u_i(t, \nu_i(t)), \quad i \in \mathcal{N},$$
$$\nu_i(t) = \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_j(t) - x_i(t)), \tag{1}$$

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where $\mathcal{N} := \{1, \ldots, N\}$ is the set of agent indices with the number of agents N, and $\mathcal{N}_i$ is a subset of $\mathcal{N}$ whose elements are the indices of the agents that send the information to agent i. The coefficient α_{ij} is the ij-th element of the adjacency matrix that represents the interconnection graph. In the description, the internal state at time $t \in \mathbb{R}$ is represented by $x_i(t) \in \mathbb{R}$, and $u_i : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is the nonlinear time-varying coupling law to be presented later on, which is a continuous mapping from the diffusive coupling term ν_i to the control input and is possibly time-varying. It is a *heterogeneous* multi-agent system in the sense that the vector field $f_i : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is possibly different from each other as long as they are measurable in t, locally Lipschitz with respect to x_i uniformly in t, and $f_i(\cdot, x)$ is bounded for each $x \in \mathbb{R}$. Note that the time-varying f_i can include an external input, a disturbance, and/or noise as well.

In particular, we propose our coupling law as

$$u_i(t, \nu_i) := \mu_i \left(\frac{\nu_i}{\psi(t)} \right) := \gamma_i \left(\frac{|\nu_i|}{\psi(t)} \right) \frac{\nu_i}{\psi(t)} \in \mathbb{R} \qquad (2)$$

where $\psi : [0, \infty) \to [0, \overline{\psi}]$ and $\gamma_i : [0, r_i) \to [0, \infty)$ are continuously differentiable functions with positive coefficients $\overline{\psi}$ and r_i such that γ_i is strictly increasing and satisfies $\lim_{s \to r_i} \gamma_i(s) = \infty$, and ψ has a bounded derivative, i.e., there exists $\theta_\psi < \infty$ such that $|d\psi(t)/dt| \le \theta_\psi$ for all $t \ge 0$. The proposed coupling law allows fully decentralized design, since the function μ_i can be selected individually, and the performance function ψ can be prespecified. The coupling law is motivated by the funnel controller proposed in an adaptive control framework (Ilchmann, Ryan, & Sangwin, 2002) under the observation that approximate synchronization with arbitrary precision can be obtained by the high-gain linear coupling law $u_i(t, \nu_i) = k\nu_i$ (Lee & Shim, 2019) (which corresponds to the high-gain property in the funnel control study), and thus, we call our coupling law '(nodewise) funnel coupling.' In fact, as for the funnel controller, it is proven that the funnel coupling law achieves synchronization with respect to the given performance function ψ under only mild assumptions, i.e., we have $|\nu_i(t)| < r_i\psi(t)$ for all $t \ge 0$ and $i \in \mathcal{N}$. We emphasize that now transient performance can also be guaranteed as done by the funnel control.

This idea has been first proposed in (Shim & Trenn, 2015), however, due to some technical reasons, the analysis was conducted only for the weakly centralized funnel coupling, i.e., $u_i(t) = \max_j \gamma_j(|\nu_j|/\psi(t))\nu_i/\psi(t)$, and only when the underlying graph is d-regular with $d > N/2 - 1$. By getting out of the conventional proof technique in the funnel control study, these technical limitations have been resolved in this paper, and we can now consider fully decentralized coupling law (2) with an arbitrarily given graph which is undirected and connected. This new approach also allows the performance function ψ to converge asymptotically or in a finite time to zero, i.e., $\lim_{t \to \omega} \psi(t) = 0$ with some $\omega \in (0, \infty]$, by which we obtain *asymptotic* or *finite-time* synchronization for heterogeneous multi-agent systems with the same funnel coupling law, but possibly with a different performance function ψ. This, in fact, seems to violate the common presumption, in the synchronization community, that heterogeneous multi-agent systems can not asymptotically synchronize without a common internal model. This violation is resolved by observing that we use a time-varying coupling law, which is not considered in the framework of the internal model principle for multi-agent systems (Wieland et al., 2013). In fact, unlike the internal model principle results, it is observed in this paper, that as the performance function approaches zero, the coupling term approaches to possibly non-zero time-varying signal, which compensates the heterogeneity of the individual agents. Specific use of this idea to solve distributed consensus optimization can be found in (Lee, Berger, Trenn, & Shim, 2019a). We want to emphasize that even when asymptotic or finite-time synchronization is achieved, the input $u_i(t, \nu_i(t))$ can still be bounded. Some sufficient conditions which guarantee boundedness of the input are also provided.

More interestingly, as in (Kim et al., 2016; Panteley & Loría, 2017; Lee & Shim, 2019), the emergent collective behavior that arises for a heterogeneous multi-agent system (1), when enforcing arbitrary precision synchronization by the proposed funnel coupling law (2), has been analyzed in this paper. In particular, we introduce a single scalar dynamics which we call 'emergent dynamics' (which depends on the individual vector field f_i and the function μ_i for all $i \in \mathcal{N}$) that is capable of illustrating the emergent synchronized behavior of the whole network by its solution trajectory. Characterization of the emergent collective behavior or the emergent dynamics is important, for instance, when synthesizing a heterogeneous network for some specific purposes. In particular, one can design the emergent dynamics first such that the solution trajectory behaves as desired, and then, provide a design guideline to each agent (which allows fully decentralized design) so that the constructed vector field and μ_i function yields the desired emergent dynamics. This scheme of constructing a heterogeneous network with the desired collective behavior is first introduced in (Lee & Shim, 2019) and has many interesting applications, e.g., distributed state estimation, estimation of the number of agents, and economic dispatch problem. It is interesting to note that in (Lee & Shim, 2019), the emergent behavior of a heterogeneous network under the high-gain coupling $u_i(t, \nu_i) = k\nu_i$, has been approximated by the solution trajectory of the 'blended dynamics' given by

$$\dot{s} = \tfrac{1}{N} \textstyle\sum_{i=1}^{N} f_i(t, s). \qquad (3)$$

The emergent dynamics takes clearly different form compared to the blended dynamics, in fact, the vector field of the emergent dynamics cannot be represented as a linear combination of the individual vector fields. By this difference, a new application might occur, which needs further inspection. A particular example illustrating the utility of the emergent dynamics is given in Section 5 as a distributed median solver.

Relying also on the observation that arbitrary precision synchronization can be achieved by the high-gain linear coupling law, a dynamic coupling law motivated by the λ-tracking studied in adaptive controls (Ilchmann & Ryan, 1994) given, for instance, as

$$u_i(t, \nu_i(t)) = k_i(t)\nu_i(t),$$
$$\dot{k}_i(t) = \begin{cases} |\nu_i(t)|(|\nu_i(t)| - r_i) & \text{if } |\nu_i(t)| > r_i, \\ 0 & \text{otherwise,} \end{cases}$$

has been introduced in (Shafi & Arcak, 2014; Li, Ren, Liu, & Fu, 2013; Lv, Li, Duan, & Feng, 2017; Kim & De Persis, 2017; Lee, Yun, & Shim, 2018). But, most of them considered homogeneous network, and for a heterogeneous network, additional communication between the coupling gains k_i has been introduced to ensure that the collective behavior of the network is as desired. In fact, funnel control is the study that has resolved two main drawbacks of λ-tracker, where one is that the transient behavior has no direct control, i.e., we don't know how fast the desired accuracy is reached, and the other is that the gain is monotonically increasing, so that even when the error is small, the gain remains large, and thus, unnecessarily amplifying the measurement noise.

The paper is organized as follows. In Section 2, it is proven that the proposed node-wise funnel coupling law achieves synchronization with respect to the given performance function. Some sufficient conditions that ensure boundedness of the inputs are also given at the end of that section. Section 3 analyzes the emergent collective behavior that arises when enforcing synchronization by the proposed funnel coupling law. Then, in Section 4, we discover the properties of the emergent dynamics, and an application related to these properties is provided in Section 5.

Notation: Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of a graph is defined as $\mathcal{L} := \mathcal{D} - \mathcal{A}$, where $\mathcal{A} = [\alpha_{ij}]$ is the adjacency matrix of the graph and $\mathcal{D}$ is the diagonal matrix whose diagonal entries are determined such that each row sum of $\mathcal{L}$ is zero. By its construction, it contains at least one eigenvalue of zero, whose corresponding eigenvector is $1_N := [1, \dots, 1]^T \in \mathbb{R}^N$, and all the other eigenvalues have non-negative real parts. For undirected graphs, the zero eigenvalue is simple if and only if the corresponding graph is connected. For vectors or matrices a and b, $\mathrm{col}(a, b) := [a^T, b^T]^T$. For matrices $A_1, \dots, A_k$, we denote by $\mathrm{diag}(A_1, \dots, A_k)$ the corresponding block diagonal matrix. For a set $\Xi \subseteq \mathbb{R}$, $|x|_\Xi$ denotes the distance between the value $x \in \mathbb{R}$ and Ξ, i.e., $|x|_\Xi := \inf_{y\in\Xi} |x-y|$.

2 Heterogeneous multi-agent systems under node-wise funnel coupling

The intuition of the funnel coupling law (2) is simple, following that of funnel control, which is to enlarge the gain infinitely large as the error (diffusive term) approaches the funnel boundary. Then, the high-gain coupling (or the high-gain property in the funnel control study) precludes boundary contact. In particular, if say agent i has only one neighbor j, and that the difference between two agents, $\nu_i(t) = \alpha_{ij}(x_j(t) - x_i(t))$, approaches the funnel boundary $\pm r_i\psi(t)$ so that $r_i\psi(t) - |\nu_i(t)|$ becomes closer to zero, then the gain $\gamma_i(|\nu_i(t)|/\psi(t))$ gets larger towards infinity, and the state x_i will tend to its neighbor x_j since the large coupling term dominates the vector field $f_i(t, x_i)$, and the error $\nu_i(t)$ will remain inside the funnel. However, with more than one neighbor, this intuition becomes no longer straightforward because two neighbors may attract x_i in the opposite direction with almost infinite power. In the following, we will prove that all the errors $\nu_i(t)$ remain inside the funnel, which is however far more complicated and also requires the following technical assumption, which guarantees that if the diffusive term is contained in the funnel, i.e., $|\nu_i(t)| < r_i\psi(t)$, then finite time escape cannot occur.

Assumption 1 *The dynamical systems defined by*

$$\dot{\overline{\chi}}(t) = \max_{i\in\mathcal{N}} f_i(t, \overline{\chi}(t)), \quad \dot{\underline{\chi}}(t) = \min_{i\in\mathcal{N}} f_i(t, \underline{\chi}(t)),$$

have global solutions $\overline{\chi}, \underline{\chi} : [0, \infty) \to \mathbb{R}$ *for any initial condition.* □

We stress that if the functions f_i are globally Lipschitz in x_i, then Assumption 1 is satisfied. An alternative sufficient condition is that f_i is continuously differentiable in x_i and there exists $c_i \in \mathbb{R}$ such that $(\partial f_i/\partial x)(t, x) \le c_i$ for all $t \ge 0$ and $x \in \mathbb{R}$. [1]

Lemma 1 *In addition to Assumption 1, let us assume that the solution of the system* (1) *coupled via* (2) *exists on* $[0, \omega)$ *for some* $\omega < \infty$ *and satisfies* $|\nu_i(t)| < r_i\psi(t)$ *for all* $t \in [0, \omega)$ *and* $i \in \mathcal{N}$ *(*$\psi(t) > 0$ *for all* $t \in [0, \omega)$*). Then, there exists* $M > 0$ *such that* $|x_i(t)| < M$ *for all* $i \in \mathcal{N}$ *and* $t \in [0, \omega)$*.* □

Lemma 1 tells us that Assumption 1 guarantee that there is no finite escape time, even when the vector field f_i combined with the funnel coupling law u_i, which can have arbitrarily large size, yields arbitrarily fast switching in the index of the maximum (or minimum) agent.

[1] This is because, we then have

$$\dot{\overline{\chi}}(t) \le \bar{c}\,\overline{\chi}(t) + \max_i f_i(t, 0), \quad \dot{\underline{\chi}}(t) \ge -\bar{c}\,\underline{\chi}(t) + \min_i f_i(t, 0),$$

where $\bar{c} := \max_i c_i$, which can have a positive value.

PROOF. Choose a time-varying index $J(t) \in \mathcal{N}$ such that $x_{J(t)}(t) = \max_i x_i(t)$ and $\dot{x}_{J(t)}(t) \geq \dot{x}_k(t)$ for all those $k \in \mathcal{N}$ with $x_k(t) = \max_i x_i(t)$. Then, the right-hand side Dini derivative of $V(t) := \max_i x_i(t)$ satisfies

$$\begin{aligned}\dot{V}(t) &\leq \dot{x}_{J(t)}(t)\\ &= f_{J(t)}(t, x_{J(t)}(t)) + \gamma_{J(t)}\left(\frac{|\nu_{J(t)}(t)|}{\psi(t)}\right)\frac{\nu_{J(t)}(t)}{\psi(t)}\\ &\leq f_{J(t)}(t, x_{J(t)}(t)) \leq \max_i f_i(t, V(t))\end{aligned}$$

where the second inequality follows from the fact that $\gamma_{J(t)}$ and ψ are non-negative and $\nu_{J(t)}(t)$ is non-positive, because $x_{J(t)}(t)$ is a maximum. Hence, by Assumption 1, there exists $M_+ > 0$ such that $V(t)$ is upper bounded by M_+ for $t \in [0, \omega)$. Similarly, we can find $M_- > 0$ such that $\min_i x_i(t) \geq -M_-$ for all $t \in [0, \omega)$, which concludes our claim. □

In the following, we state our main result assuming connectivity of the network, which tells us that under these mild assumptions the diffusive term $\nu_i(t)$ stays inside the prescribed funnel boundary $(-r_i\psi(t), r_i\psi(t))$, and thus, (approximate, or asymptotic, or finite-time) synchronization follows. In particular, if our goal is to obtain approximate synchronization, e.g., $|\nu_i(t)| < \eta$ for all $t \geq T$ and $i \in \mathcal{N}$, then this is achieved by the proposed coupling law (2) with the performance function ψ satisfying $\psi(t) < \eta$ for all $t \geq T$ and with $r_i < 1$ for all $i \in \mathcal{N}$. More interestingly, if we let the performance function ψ to converge asymptotically to zero, i.e., $\psi(t) > 0$ for all $t \geq 0$ and $\lim_{t\to\infty} \psi(t) = 0$, then we obtain asymptotic synchronization. Finally, by taking the performance function ψ to satisfy $\omega = \sup\{t : \psi(\tau) > 0 \ \forall \tau \in [0, t)\}$ with some finite $\omega < \infty$, we cover the problem of finite-time synchronization for heterogeneous multi-agent systems. We want to emphasize that in all of the cases, each agent i is required to know only the individual vector field f_i, the pre-specified performance function ψ, and the diffusive coupling term ν_i, hence no global information (e.g., vector field of other agents or the number of agents in the network) is needed.

Assumption 2 *The communication graph induced by the adjacency element α_{ij} is undirected and connected, and thus, the Laplacian matrix $\mathcal{L}$ is symmetric, having one simple eigenvalue of zero.* □

Theorem 2 *Consider the system* (1) *coupled via node-wise funnel coupling* (2)*. Under Assumptions 1, 2 and the assumption that $|\nu_i(0)| < r_i\psi(0)$ for all $i \in \mathcal{N}$, funnel coupling leads to a solution defined on the whole time interval $[0, \omega)$, where $\omega := \sup\{t : \psi(\tau) > 0 \ \forall \tau \in [0, t)\} \in (0, \infty]$. In particular, we have $|\nu_i(t)| < r_i\psi(t)$ for all $t \in [0, \omega)$ and $i \in \mathcal{N}$.* □

The proof of the main theorem relies on the following technical result, which will be proven afterward.

Lemma 3 *In addition to the assumptions of Theorem 2, let us assume that the solution of the system* (1) *coupled via* (2) *exists on $[0, \omega')$ for some $\omega' \in (0, \infty]$ and satisfies $|\nu_i(t)| < r_i\psi(t)$ for all $t \in [0, \omega')$ and $i \in \mathcal{N}$ ($\psi(t) > 0$ for all $t \in [0, \omega')$). If there exists $\overline{M}_f$ such that*

$$|f_j(t, x_j(t)) - f_i(t, x_i(t))| \leq \overline{M}_f, \quad \forall i \in \mathcal{N},\ j \in \mathcal{N}_i, \quad (4)$$

for all $t \in [0, \omega')$, then the index sets defined as

$$\mathcal{I}_+ := \left\{ i \in \mathcal{N} : \exists \{t_k\} \to \omega',\ s.t.\ \lim_{k\to\infty} \frac{\nu_i(t_k)}{\psi(t_k)} = r_i \right\}$$
$$\mathcal{I}_- := \left\{ i \in \mathcal{N} : \exists \{t_k\} \to \omega',\ s.t.\ \lim_{k\to\infty} \frac{\nu_i(t_k)}{\psi(t_k)} = -r_i \right\}$$

are empty. □

PROOF OF THEOREM 2. Let

$$\Omega_\psi := \{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^N : |\mathcal{L}_r x|_\infty < \psi(t)\}$$

where $\mathcal{L}_r := \mathrm{diag}(1/r_1, \ldots, 1/r_N)\mathcal{L}$ and $|\cdot|_\infty$ denotes the maximum norm on $\mathbb{R}^N$. Since the right-hand side of (1) with (2) is continuous on $\Omega_\psi \subseteq \mathbb{R}^{N+1}$, the standard theory of ODEs yields existence and uniqueness of a maximally extended solution $x : [0, \omega') \to \mathbb{R}^N$ such that $(t, x(t)) \in \Omega_\psi$ for all $t \in [0, \omega')$ (Hartman, 1964). If $\omega' = \omega$, then nothing is to show anymore.

Thus, suppose that $\omega' < \omega$ for a given maximal solution. Then, again from (Hartman, 1964), there exists a time sequence $\{t_k\} \to \omega'$ such that $\lim_{k\to\infty} |\mathcal{L}_r x(t_k)|_\infty = \psi(\omega')$. In fact, there exist a subsequence $\{k_p\} \to \infty$ and an index i, such that

$$\lim_{p\to\infty} \frac{\nu_i(t_{k_p})}{\psi(t_{k_p})} = r_i \quad \text{or} \quad \lim_{p\to\infty} \frac{\nu_i(t_{k_p})}{\psi(t_{k_p})} = -r_i.$$

This is because, otherwise we can continue the solution after ω', which violates the fact that x is maximal.

However, since we have $\omega' < \infty$ even when $\omega = \infty$, there exists $M > 0$ such that $|x_i(t)| < M$ for all $i \in \mathcal{N}$ and $t \in [0, \omega')$, by Lemma 1. Therefore, there exists $\overline{M}_f$ such that (4) is satisfied for all $t \in [0, \omega')$, and thus, by Lemma 3, $\omega' < \omega$ is impossible. This completes the proof. □

PROOF OF LEMMA 3. Assume first that the index set $\mathcal{I}_+$ is nonempty. In order to arrive at a contradiction, we will construct a sequence of strictly increasing index sets $\mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \mathcal{J}_3 \subsetneq \ldots$ that are all contained in $\mathcal{I}_+$; which of course is impossible due to the finiteness of $\mathcal{I}_+$. Hence $\mathcal{I}_+$ must be empty, and analogous argument yield that $\mathcal{I}_-$ must also be empty.

So, first, take any element of $\mathcal{I}_+$, say j_1. Then, by the definition of $\mathcal{I}_+$ there exists a strictly increasing time sequence $\{t_k^1\}$ such that $\lim_{k\to\infty} t_k^1 = \omega'$ and satisfies $\lim_{k\to\infty} \nu_{j_1}(t_k^1)/\psi(t_k^1) = r_{j_1}$. Then

$$\mathcal{J}_1 := \left\{ i \in \mathcal{I}_+ : \lim_{k\to\infty} \frac{\nu_i(t_k^1)}{\psi(t_k^1)} = r_i \right\},$$

is nonempty (because it contains j_1) and is a subset of $\mathcal{I}_+$. Inductively, assume now that for $n \geq 1$ a nonempty index set is given by

$$\mathcal{J}_n := \left\{ i \in \mathcal{I}_+ : \lim_{k\to\infty} \frac{\nu_i(t_k^n)}{\psi(t_k^n)} = r_i \right\},$$

where $\{t_k^n\}_{k\in\mathbb{N}}$ is a strictly increasing sequence converging to ω'. We now construct a strictly increasing sequence $\{t_k^{n+1}\}$ converging to ω' as $k \to \infty$ such that the corresponding set $\mathcal{J}_{n+1}$ contains $\mathcal{J}_n$ and there is $j_{n+1} \in \mathcal{I}_+$ which is in $\mathcal{J}_{n+1}$ but not in $\mathcal{J}_n$. Therefore we will first construct a sequence $\{s_p^{n+1}\}_{p\in\mathbb{N}}$ such that

$$\forall i \in \mathcal{J}_n : \lim_{p\to\infty} \frac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})} = r_i \tag{5}$$

and such that for each $p \in \mathbb{N}$ there is an index $j_p \in \mathcal{I}_+ \setminus \mathcal{J}_n$ with

$$\frac{\nu_{j_p}(s_p^{n+1})}{\psi(s_p^{n+1})} > r_{j_p}(1 - \delta_p); \tag{6}$$

where $\{\delta_p\}_{p\in\mathbb{N}}$ is some strictly decreasing sequence converging to zero with $\delta_0 > 0$ such that for all $i \in \mathcal{N} \setminus \mathcal{I}_+$ and all $t \in [0, \omega')$ we have $\nu_i(t)/\psi(t) \leq r_i(1 - \delta_0)$. Since $\mathcal{I}_+ \setminus \mathcal{J}_n$ is finite we find a subsequence $t_k^{n+1} := s_{p_k}^{n+1}$ and an index $j_{n+1} \in \mathcal{I}_+ \setminus \mathcal{J}_n$ such that

$$\lim_{k\to\infty} \frac{\nu_{j_{n+1}}(t_k^{n+1})}{\psi(t_k^{n+1})} = r_{j_{n+1}};$$

in other words, $j_{n+1} \in \mathcal{J}_{n+1}$, where $\mathcal{J}_{n+1}$ is defined analogously as $\mathcal{J}_n$ via the sequence $\{t_k^{n+1}\}$. Since (5) also holds for any subsequence, it follows that $\mathcal{J}_n \subsetneq \mathcal{J}_{n+1}$. Therefore, it remains to construct the sequence s_p^{n+1} such that (5) and (6) hold.

Towards this goal let

$$W_n(t) := \sum_{i\in\mathcal{J}_n} \frac{\nu_i(t)}{\psi(t)}, \quad t \in [0, \omega').$$

Then, by the definition of $\mathcal{J}_n$,

$$\lim_{k\to\infty} W_n(t_k^n) = \sum_{i\in\mathcal{J}_n} r_i =: \overline{r}^n$$

and $W_n(t) < \overline{r}^n$ for all $t \in [0, \omega')$. For a suitably chosen strictly decreasing sequence $\{\varepsilon_p^n\}_{p\in\mathbb{N}}$ with $\varepsilon_p^n \to 0$ as $p \to \infty$, we first choose a subsequence $\{t_{k_p}^n\}_{p\in\mathbb{N}}$ of $\{t_k^n\}_{k\in\mathbb{N}}$ such that

$$W_n(t_{k_p}^n) \geq \overline{r}^n - \underline{r}^n \varepsilon_p^n/2, \quad \forall p \in \mathbb{N},$$

where $\underline{r}^n := \min_{i\in\mathcal{J}_n} r_i > 0$.[2] Based on this sequence, we now define a sequence $\{s_p^{n+1}\}_{p\in\mathbb{N}}$ as follows, see also Figure 1,

$$s_p^{n+1} := \max\{s \in [0, t_{k_p}^n] : W_n(s) = \overline{r}^n - \underline{r}^n \varepsilon_p^n\}.$$

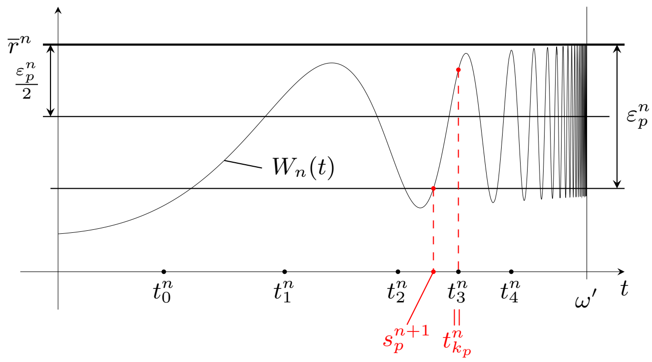

Fig. 1. Illustration of the choice of the sequence $\{s_p^{n+1}\}_{p\in\mathbb{N}}$ based on $\{t_k^n\}_{k\in\mathbb{N}}$, for simplicity $\underline{r}^n$ is assumed to be one.

By the choice of s_p^{n+1} we have that $W_n(s_p^{n+1}) \to \overline{r}^n$ and therefore (5) holds. Furthermore, the choice of s_p^{n+1} also implies $\dot{W}_n(s_p^{n+1}) \geq 0$. Assume now that (6) does *not* hold; we will show in the following that we then arrive at the contradiction $0 \leq \psi(s_p^{n+1})\dot{W}_n(s_p^{n+1}) < 0$.

The derivative of W_n can be bounded as follows

$$\begin{aligned}
\dot{W}_n(t) &= -\frac{\dot{\psi}(t)}{\psi(t)^2} \sum_{i\in\mathcal{J}_n} \nu_i(t) \\
&\quad + \frac{1}{\psi(t)} \sum_{i\in\mathcal{J}_n} \sum_{j\in\mathcal{N}} \alpha_{ij}(f_j(t, x_j(t)) - f_i(t, x_i(t))) \\
&\quad + \frac{1}{\psi(t)} \sum_{i\in\mathcal{J}_n} \sum_{j\in\mathcal{N}} \alpha_{ij} \left[\mu_j\left(\tfrac{\nu_j(t)}{\psi(t)}\right) - \mu_i\left(\tfrac{\nu_i(t)}{\psi(t)}\right) \right] \\
&\leq \frac{M_0}{\psi(t)} + \frac{1}{\psi(t)} \sum_{i\in\mathcal{J}_n} \sum_{j\in\mathcal{N}} \alpha_{ij} \left[\mu_j\left(\tfrac{\nu_j(t)}{\psi(t)}\right) - \mu_i\left(\tfrac{\nu_i(t)}{\psi(t)}\right) \right]
\end{aligned}$$

with (recalling that $\mathcal{J}_n \subseteq \mathcal{I}^+$)

$$M_0 := \theta_\psi \sum_{i\in\mathcal{I}^+} r_i + \sum_{i\in\mathcal{I}^+} \sum_{j\in\mathcal{N}} \alpha_{ij} \overline{M}_f,$$

[2] Without loss of generality we assume $W_n(0) < \overline{r}^n - \underline{r}^n \varepsilon_0^n$.

according to the assumption. Invoking now that the graph is undirected, we get [3]

$$\psi(s_p^{n+1})\dot{W}_n(s_p^{n+1}) \leq$$
$$M_0+\sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}\setminus\mathcal{J}_n}\alpha_{ij}\Big[\mu_j\Big(\tfrac{\nu_j(s_p^{n+1})}{\psi(s_p^{n+1})}\Big)-\mu_i\Big(\tfrac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})}\Big)\Big]. \quad (7)$$

To arrive at the sought contradiction $\psi(s_p^{n+1})\dot{W}_n(s_p^{n+1}) < 0$ we will exploit that due to (5) the terms $\mu_i\left(\frac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})}\right)$ will be very large for each $i \in \mathcal{J}_n$ while $\mu_j\left(\frac{\nu_j(s_p^{n+1})}{\psi(s_p^{n+1})}\right)$ for $j \notin \mathcal{J}_n$ will not be very large. We will now choose the sequence $\{\varepsilon_p^n\}$ (used above to define the sequence $\{s_p^{n+1}\}$) in a suitable way to make this intuition precise; indeed, let each $\varepsilon_p^n > 0$ be so small that

$$1-\varepsilon_p^n \geq \max_{i,j\in\mathcal{N}}\frac{1}{r_i}\mu_i^{-1}\left(\frac{\mu_j(r_j(1-\delta_p))}{1-\delta_p}\right) \quad \text{and}$$
$$1-\varepsilon_p^n \geq \max_{i\in\mathcal{N}}\frac{1}{r_i}\mu_i^{-1}\left(\frac{M_0+1}{\underline{\alpha}\delta_p}\right)$$

where $\underline{\alpha} := \min_{i\in\mathcal{N},\, j\in\mathcal{N}_i}\alpha_{ij} > 0$. This choice is possible because $\mu_i : (-r_i, r_i) \to \mathbb{R}$ is by the properties for γ-functions strictly increasing and bijective.

By rewriting the definition of W_n we get for each $i \in \mathcal{J}_n$

$$\frac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})} > W_n(s_p^{n+1}) - \sum_{j\neq i, j\in\mathcal{J}_n} r_j$$
$$= \overline{r}^n - \underline{r}^n\varepsilon_p^n - (\overline{r}^n - r_i) \geq r_i(1-\varepsilon_p^n), \quad (8)$$

and hence, by monotonicity of the μ-functions and the choice of ε_p^n and δ_0 we have for each $j \in \mathcal{N}\setminus\mathcal{I}_+$ that

$$\mu_j\left(\tfrac{\nu_j(s_p^{n+1})}{\psi(s_p^{n+1})}\right) \leq \mu_j(r_j(1-\delta_0)) \leq \mu_j(r_j(1-\delta_p))$$
$$\leq (1-\delta_p)\mu_i(r_i(1-\varepsilon_p^n)) \leq (1-\delta_p)\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})}\right),$$

and due to assuming that (6) does *not* hold, we can conclude $\mu_j\left(\frac{\nu_j(s_p^{n+1})}{\psi(s_p^{n+1})}\right) \leq \mu_j(r_j(1-\delta_p))$ and hence the same outer inequality also for all $j \in \mathcal{I}_+\setminus\mathcal{J}_n$. This allows us to bound, in (7), each μ_j term by $(1-\delta_p)\mu_i$ term, i.e.,

$$\psi(s_p^{n+1})\dot{W}_n(s_p^{n+1}) \leq M_0 - \sum_{i\in\mathcal{J}_n}\sum_{j\in\mathcal{N}\setminus\mathcal{J}_n}\alpha_{ij}\delta_p\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})}\right)$$
$$\leq M_0 - \underline{\alpha}\delta_p\min_{i\in\mathcal{J}_n}\mu_i\left(\tfrac{\nu_i(s_p^{n+1})}{\psi(s_p^{n+1})}\right)$$
$$\overset{(8)}{<} M_0 - \underline{\alpha}\delta_p\min_{i\in\mathcal{J}_n}\mu_i(r_i(1-\varepsilon_p^n)) \leq -1,$$

which is the sought contradiction and the proof is complete. □

[3] For undirected graphs, we have for any $\chi \in \mathbb{R}^N$ and index set $\mathcal{J} \subsetneq \mathcal{N}$ the following identity:

$$\sum_{i\in\mathcal{J}}\sum_{j\in\mathcal{N}}\alpha_{ij}(\chi_j-\chi_i) = \sum_{i\in\mathcal{J}}\sum_{j\in\mathcal{N}\setminus\mathcal{J}}\alpha_{ij}(\chi_j-\chi_i)$$

where the fact that $\sum_{i\in\mathcal{J}}\sum_{j\in\mathcal{J}}\alpha_{ij}(\chi_j-\chi_i) = 0$ is used.

Acknowledging the advantages of the funnel coupling law (2) guaranteed by Theorem 2, we have to be however careful about its nonlinear structure. Since the funnel coupling law doesn't have a memory and the size of it can increase arbitrarily large, the input used for the synchronization might grow unboundedly as time flows. Therefore, in addition to proving that the diffusive term resides inside the funnel, we have to ensure separately that the input used in our system is uniformly bounded on the time interval $[0, \omega)$. Luckily, Lemma 3 provides us with some sufficient conditions, which guarantee boundedness of the input as follows.

Corollary 4 *In addition to the assumptions of Theorem 2, assume that one of the following conditions hold, where* $\omega := \sup\{t : \psi(\tau) > 0\ \forall\tau \in [0, t)\}$.

(a) $\omega < \infty$.
(b) $\omega = \infty$ *and* $f_i(t,x) \equiv F(t,x) + g_i(t,x)$ *where* $F(t,x)$ *is globally Lipschitz with respect to* x *uniformly in* t *and there exists* $\overline{M}$ *such that* $|g_i(t,x)| \leq \overline{M}$ *for all* $i \in \mathcal{N}$, $t \geq 0$, *and* $x \in \mathbb{R}$.
(c) $\omega = \infty$ *and there exists* $\overline{M}$ *such that* $|x_i(t)| \leq \overline{M}$ *for all* $i \in \mathcal{N}$ *and* $t \geq 0$.

Then the input $u_i(t, \nu_i(t)) \equiv \mu_i(\nu_i(t)/\psi(t))$ *is uniformly bounded on* $[0, \omega)$, *i.e., there exists* $\overline{M}_u > 0$ *such that for all* $t \in [0, \omega)$ *and* $i \in \mathcal{N}$, *we have* $|u_i(t, \nu_i(t))| \leq \overline{M}_u$. □

PROOF. Note first that, by Lemma 1, condition (a) implies that the solution trajectory is uniformly bounded on the time interval $[0, \omega)$. Then, for conditions (a) and (c), boundedness of the solution trajectory on the time interval $[0, \omega)$ and the properties of the vector fields guarantee that the condition in Lemma 3 is satisfied when $\omega' = \omega$. Now, since condition (b) also implies that the condition in Lemma 3 is satisfied when $\omega' = \omega$, the proof concludes from the fact that the index sets $\mathcal{I}_+$ and $\mathcal{I}_-$ in Lemma 3 are empty. □

Remark 5 *It is interesting to note that the stability of the agents might not be necessary to guarantee boundedness of the input. In particular, in the condition (b) of*

Corollary 4, there is no restriction on the homogeneous part $F(t,x)$, hence the dynamics $\dot{x} = F(t,x)$ might even be unstable. The utility of this remark can be found, for instance, when synchronizing heterogeneous oscillators, where it is expected that the homogeneous part is of the form $\dot{x}(t) = \Omega(t)$ which is unstable in many cases, where $\Omega(\cdot)$ is a time-varying angular velocity. Furthermore, recall that the time-varying vector fields $f_i(t,x_i)$ may depend on an external input; this external input could be used to ensure boundedness of the agents' states, so that condition (c) is ensured. □

Before concluding Section 2, we want to mention that there are some cases, where it can be explicitly shown that the solution trajectory is uniformly bounded, for instance when the dynamics introduced in Assumption 1 generates a uniformly bounded solution for the infinite time interval. This happens if, for instance, the dynamics $\dot{x} = f_i(t,x)$ are contractive for all $i \in \mathcal{N}$, i.e., for each $i \in \mathcal{N}$ there exists $c_i > 0$ such that $(\partial f_i/\partial x)(t,x) \le -c_i$ for all $t \ge 0$ and $x \in \mathbb{R}$ (the utility of this case can be seen in Section 5). In this case, the arguments in the proof of Lemma 1 ensure that the condition (c) of Corollary 4 holds.

3 Emergent behavior under funnel coupling

The system (1) is now proven to achieve synchronization with respect to the performance function ψ by the node-wise funnel coupling law (2) under only mild assumptions on the individual vector field f_i and under only the connectivity of the undirected network. This implies that for the same system (1), any performance function ψ can be utilized if they have a bounded derivative. Therefore, we can enforce arbitrary precision synchronization by making the performance function sufficiently narrow, and in this section, we will analyze what is the emergent collective behavior that arises then. In particular, in Theorem 2 it is shown that we have $|\nu_i(t)| < r_i\psi(t)$ for all $t \ge 0$ and $i \in \mathcal{N}$ (when $\psi(t) > 0$ for all $t \ge 0$), and thus, we can enforce synchronization by choosing $t_0 > 0$ and by making the sequence of performance functions $\{\psi_\varepsilon^{t_0}\}$ to satisfy $\psi_\varepsilon^{t_0}(t) \in (0, \varepsilon]$ for all $t \ge t_0$ and $\varepsilon > 0$.

Then, it is expected that as ε goes to zero, synchronization with arbitrary precision is achieved after time t_0. This, in other words, means that as ε goes to zero, $\mu_i(\nu_i/\psi_\varepsilon^{t_0}(t))$ term generates a compensation that can resolve the heterogeneity among the agents, and thus, the vector field of the network are aligned, i.e., $f_i(t,x_i) + \mu_i(\nu_i/\psi_\varepsilon^{t_0}(t)) = f_j(t,x_j) + \mu_j(\nu_j/\psi_\varepsilon^{t_0}(t))$ for all $i,j \in \mathcal{N}$. In particular, when the states are synchronized to ξ at time $t \ge t_0$, the vector fields which are also synchronized to say h_μ, i.e., $f_i(t,\xi) + \mu_i(\nu_i/\psi_\varepsilon^{t_0}(t)) = h_\mu$ for all $i \in \mathcal{N}$, should satisfy

$$\sum_{i=1}^{N} \mu_i^{-1}\left(h_\mu - f_i(t,\xi)\right) \equiv 0,$$

by the algebraic constraint $\sum_{i=1}^N \nu_i \equiv 0$. The solution $h_\mu(f_1(t,\xi),\dots,f_N(t,\xi))$ of the above algebraic equation exists uniquely for each $(f_1(t,\xi),\dots,f_N(t,\xi))$, as proven by the following lemma, and thus, we could guess that the synchronized behavior of the whole network can be illustrated by the solution trajectory of a single scalar dynamics given as

$$\dot{\xi} = h_\mu(f_1(t,\xi),\dots,f_N(t,\xi)) \tag{9}$$

which we call 'emergent dynamics.'

Remark 6 *We want to emphasize that this methodology of finding emergent behavior that arises as we enforce synchronization to a heterogeneous network is universal. In particular, when high-gain linear coupling $u_i(t,\nu_i) = k\nu_i$ is used to achieve arbitrary precision synchronization as illustrated in the Introduction, if the states are synchronized to ξ at time $t > 0$ and if the vector fields are also synchronized to f_s, i.e., $f_i(t,\xi) + k\nu_i = f_s$ for all $i \in \mathcal{N}$, then f_s should satisfy $\sum_{i=1}^N (f_s - f_i(t,\xi)) \equiv 0$, by the algebraic constraint $\sum_{i=1}^N \nu_i \equiv 0$. The solution of this algebraic equation is the average among the vector fields, and thus, we can guess that the synchronized behavior can be illustrated by the solution trajectory of the blended dynamics* (3) *introduced in the Introduction.* □

Lemma 7 *For any fixed collection of $\mu_i(\cdot)$ as in* (2)*, we have for each $\mathrm{col}(f_1,\dots,f_N) \in \mathbb{R}^N$ a unique value $h_\mu(f_1,\dots,f_N)$ which satisfies the algebraic equation*

$$\sum_{i=1}^{N} \mu_i^{-1}\left(h_\mu(f_1,\dots,f_N) - f_i\right) \equiv 0. \tag{10}$$

Furthermore, the map $(f_1,\dots,f_N) \mapsto h_\mu(f_1,\dots,f_N)$ is continuous. □

PROOF. Note from its definition that μ_i^{-1} is a continuous function defined over $\mathbb{R}$ which is strictly increasing. This implies that $\sum_{i=1}^N \mu_i^{-1}(h - f_i)$ increase (decrease) as h increases (decreases). By noting also that the value is positive (negative) when h is bigger than $\max_i f_i$ (smaller than $\min_i f_i$), the result follows.

For the proof of continuity, consider $\mathrm{col}(f_1,\dots,f_N) \in \mathbb{R}^N$ and $\mathrm{col}(\tilde{f}_1,\dots,\tilde{f}_N) \in \mathbb{R}^N$ such that $|\tilde{f}_i| < \delta$ holds for all $i \in \mathcal{N}$. Then, if the value $h_\mu(f_1+\tilde{f}_1,\dots,f_N+\tilde{f}_N) =: \tilde{H}$ is smaller than or equal to $h_\mu(f_1,\dots,f_N) - \delta =: H - \delta$, it is a contradiction since

$$0 \equiv \sum_{i=1}^{N} \mu_i^{-1}(\tilde{H} - f_i - \tilde{f}_i) < \sum_{i=1}^{N} \mu_i^{-1}(\tilde{H} - f_i + \delta) \leq \sum_{i=1}^{N} \mu_i^{-1}(H - f_i) \equiv 0.$$

Hence, by a similar argument, we also have that $\tilde{H} \geq H + \delta$ is a contradiction, and thus,

$$|h_\mu(f_1, \ldots, f_N) - h_\mu(f_1 + \tilde{f}_1, \ldots, f_N + \tilde{f}_N)| < \delta$$

holds, which proves continuity. □

Before proving the main claim, we develop first a technical result that guarantees the existence of a limit at time t_0, under the following assumption.

Assumption 3 *The sequence of performance functions $\{\psi_\varepsilon^{t_0}\}$ parametrized by $\varepsilon > 0$ satisfies the following:*[4]

(i) $\psi_\varepsilon^{t_0}(t) \in (0, \varepsilon]$ *for all* $t \geq t_0$ *and* $\varepsilon > 0$.
(ii) There exists $\psi_0 > 0$ *such that* $\psi_\varepsilon^{t_0}(0) = \psi_0$ *for all* ε.
(iii) There exists $\overline{\psi}^{t_0} : [0, t_0) \to \mathbb{R}_{>0}$ *such that for each* $t \in [0, t_0)$, *we have* $\lim_{\varepsilon \to 0} \psi_\varepsilon^{t_0}(t) = \overline{\psi}^{t_0}(t)$.
(iv) $\lim_{t \to t_0} d\overline{\psi}^{t_0}(t)/dt = 0$.
(v) There exist $\overline{\varepsilon} > 0$ *and* θ_ψ *such that* $|d\psi_\varepsilon^{t_0}(t)/dt| \leq \theta_\psi \psi_\varepsilon^{t_0}(t)$ *for all* $t \in [t_0, \infty)$ *and* $\varepsilon \in (0, \overline{\varepsilon})$.

□

Lemma 8 *In addition to Assumptions 1, 2, 3, let us assume that the initial condition $x_i^0 \in \mathbb{R}$, $i \in \mathcal{N}$ satisfies $|\sum_{j \in \mathcal{N}_i} \alpha_{ij}(x_j^0 - x_i^0)| < r_i \psi_0$ for all $i \in \mathcal{N}$. Then, there exists $x^*(t_0) \in \mathbb{R}$ and $\nu_i^*(t_0) \in (-r_i, r_i)$, $i \in \mathcal{N}$, such that*

$$\lim_{\varepsilon \to 0} x_i(t_0, t_0, \varepsilon) = x^*(t_0), \quad \lim_{\varepsilon \to 0} \frac{\nu_i(t_0, t_0, \varepsilon)}{\psi_\varepsilon^{t_0}(t_0)} = \nu_i^*(t_0),$$

for all $i \in \mathcal{N}$, where $x_i(t, t_0, \varepsilon)$ is the solution of (1) *under the funnel coupling law* (2) *with the performance function $\psi_\varepsilon^{t_0}$ and initial condition $x_i(0, t_0, \varepsilon) = x_i^0$, $i \in \mathcal{N}$. In particular, $x^*(t_0)$ and $\nu_i^*(t_0)$, $i \in \mathcal{N}$, satisfy*

$$\mu_i(\nu_i^*) = h_\mu(f_1(t_0, x^*), \ldots, f_N(t_0, x^*)) - f_i(t_0, x^*),$$

for all $i \in \mathcal{N}$, where h_μ is given by Lemma 7. □

[4] An example of $\{\psi_\varepsilon^{t_0}\}$ satisfying these assumptions is

$$\psi_\varepsilon^{t_0}(t) = \begin{cases} (1 - \varepsilon) \left(\frac{t - t_0}{t_0}\right)^2 + \varepsilon, & \text{if } t \in [0, t_0), \\ \varepsilon, & \text{if } t \in [t_0, \infty). \end{cases}$$

A brief proof of this lemma is found in Appendix A.4, which will be however omitted in the final version (if accepted) due to the page limit.

We are now ready for the proof of the main claim that says the solution trajectory of the emergent dynamics illustrates the collective behavior of the whole network.

Theorem 9 *Under the assumptions of Lemma 8, for arbitrary $\eta > 0$ and $T > 0$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the solution $x_i(t, t_0, \varepsilon)$ of* (1) *under the funnel coupling law* (2) *with the performance function $\psi_\varepsilon^{t_0}$ and initial condition $x_i(0, t_0, \varepsilon) = x_i^0$, $i \in \mathcal{N}$, exists on $[t_0, t_0 + T]$, and satisfies*

$$\left| \frac{1}{N} \sum_{i=1}^{N} x_i(t, t_0, \varepsilon) - \xi(t) \right| =: |a(t, t_0, \varepsilon) - \xi(t)| \leq \eta,$$
$$\left| \frac{\nu_i(t, t_0, \varepsilon)}{\psi_\varepsilon^{t_0}(t)} - \mu_i^{-1} \left(h_\mu(f_1^a, \ldots, f_N^a) - f_i^a \right) \right| \leq \eta, \quad (11)$$

for all $t \in [t_0, t_0 + T]$, where $f_i^a := f_i(t, a(t, t_0, \varepsilon))$ and $\xi(\cdot)$ is the solution of the emergent dynamics (9) *with the initial condition $\xi(t_0) = x^*(t_0)$, which is given by Lemma 8. In particular, we have $\lim_{\varepsilon \to 0} x_i(t, t_0, \varepsilon) = \xi(t)$ for all $i \in \mathcal{N}$ and $t \geq t_0$.* □

The proof of this theorem is given in Appendix A.1.

While the convergence in Theorem 9 is point-wise in time t, stability assumptions for the emergent dynamics, for instance, contraction, can make this convergence uniform in time. Here, it is emphasized that we require stability for the emergent dynamics but not for the individual agents. The following theorem is about this.

Theorem 10 *In addition to the assumptions of Lemma 8, assume that the emergent dynamics* (9) *is contractive, i.e., there exists $c > 0$ such that*

$$\frac{\partial h_\mu(f_1(t, \xi), \ldots, f_N(t, \xi))}{\partial \xi} \leq -c, \quad \forall t \geq 0, \quad \xi \in \mathbb{R}.$$

Then, the convergence in Theorem 9 is uniform on $[t_0, \infty)$, i.e., for any $\eta > 0$, there exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, the solution $x_i(t, t_0, \varepsilon)$ of (1) *under the funnel coupling law* (2) *with the performance function $\psi_\varepsilon^{t_0}$ and initial condition $x_i(0, t_0, \varepsilon) = x_i^0$, $i \in \mathcal{N}$, exists on $[t_0, \infty)$, and satisfies* (11) *for all $t \in [t_0, \infty)$, where $\xi(\cdot)$ is the solution of the emergent dynamics* (9) *with the initial condition $\xi(t_0) = x^*(t_0)$, which is given by Lemma 8.* □

The proof of this theorem is given in Appendix A.2.

Now, by the characterization of the emergent dynamics, and by the analysis that heterogeneous agents be-have like the solution trajectory of the emergent dynamics when the performance function is sufficiently narrow, we can construct a heterogeneous network achieving a specific purpose as noted in the Introduction, if the emergent dynamics is contractive. Moreover, when this happens, the solution trajectory of each agent is uniformly bounded, and hence, the inputs are also uniformly bounded. In fact, Theorem 10 not only offers the argument of uniform convergence of $x_i(t, t_0, \varepsilon)$ to $\xi(t)$ but also ensures that we can find $\overline{M}_u > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ we have

$$\left|\mu_i\left(\frac{\nu_i(t, t_0, \varepsilon)}{\psi_\varepsilon^{t_0}(t)}\right)\right| < \overline{M}_u, \quad \forall t \geq 0, \quad i \in \mathcal{N}.$$

This is because we have a bounded averaged trajectory $a(t, t_0, \varepsilon)$, which implies that $h_\mu(f_1(t, a), \ldots, f_N(t, a)) - f_i(t, a)$ is uniformly bounded on $[t_0, \infty)$ for all $i \in \mathcal{N}$.

Note that the emergent dynamics only depends on the individual vector field f_i and the μ_i function for all $i \in \mathcal{N}$, hence can be designed prior without knowing the performance function and the network topology if it is undirected and connected. Meanwhile, the solution of the emergent dynamics, which illustrates the emergent collective behavior that arises after time t_0, depends on its initial condition $x^*(t_0)$, and this can depend on the network topology and the performance function, especially $\overline{\psi}^{t_0}$ in Assumption 3. In fact, $x^*(t_0)$ is the synchronized state at time t_0 of the system (1) under the funnel coupling law (2) with the performance function $\overline{\psi}^{t_0}$ and initial condition x_i^0, $i \in \mathcal{N}$, where finite-time synchronization is achieved because $\lim_{t \to t_0} \overline{\psi}^{t_0}(t) = 0$. Therefore, it is hard to characterize $x^*(t_0)$, in general. However, characterization of $x^*(t_0)$ is important only when we are interested in the approximation of the transient behavior because we already assumed that the emergent dynamics is contractive, and thus, the steady-state behavior of the heterogeneous network can be still illustrated by that of the emergent dynamics.

Meanwhile, it is conjectured that the value $\lim_{t_0 \to 0} x^*(t_0)$ exists and is a weighted median of a collection χ^0 of the initial values x_i^0 with the weights r_i, defined as a real number that belongs to the set

$$\mathcal{M}_r(\chi^0) = \begin{cases} \{x_{j_s}^0\}, & \text{if } \exists j \in \mathcal{N}, \sum_{k=1}^{j} r_{k_s} > r_{\text{thr}} \\ & \text{and } \sum_{k=1}^{j-1} r_{k_s} < r_{\text{thr}}, \\ [x_{j_s}^0 \;\; x_{(j+1)_s}^0], & \text{if } \exists j \in \mathcal{N}, \sum_{k=1}^{j} r_{k_s} = r_{\text{thr}}, \end{cases}$$

where $r_{\text{thr}} := (1/2)\sum_{i=1}^N r_i$ and $\{j_s\}$ is the rearrangement of the sequence $\{1, \ldots, N\}$ such that

$$x_{1_s}^0 \leq x_{2_s}^0 \leq \cdots \leq x_{N_s}^0.$$

In fact, for particular cases, for instance when N is odd, $r_i = \bar{r}$ for all $i \in \mathcal{N}$, and the graph is complete and unitary, the conjecture can be proved. This is because now $\mathcal{M}_r(\chi^0)$ is just a singleton which consists of the median, and thus, by letting $V(t) := \text{med}_i x_i(t)$, we have

$$\left|\dot{V}(t)\right| \leq |f_{J(t)}(t, V(t))| + \left|\mu_{J(t)}\left(\frac{\nu_{J(t)}(t)}{\psi(t)}\right)\right|,$$

where $J(t) \in \mathcal{N}$ is such that $x_{J(t)}(t) = V(t)$ and $|\dot{x}_{J(t)}(t)| \geq |\dot{x}_j(t)|$ for any $j \in \mathcal{N}$ satisfying $x_j(t) = V(t)$. The second term is bounded by a constant, which is independent of the function ψ since we know that

$$\nu_i = \sum_{j=1}^N x_j - N x_i,$$

which implies $\nu_{J(t)}(t) = \text{med}_i \nu_i(t)$, and thus, we have

$$\left|\frac{\nu_{J(t)}(t)}{\psi(t)}\right| \leq \frac{N-1}{N+1}\bar{r} < \bar{r},$$

by the algebraic constraint $\sum_{i=1}^N \nu_i(t) \equiv 0$.[5] Therefore, regardless of the choice of the performance function ψ, we have

$$\lim_{t \to 0} |V(t)|_{\mathcal{M}_r(\chi^0)} = 0,$$

and therefore, by recalling that $x^*(t_0)$ is the synchronized state at time t_0 with the performance function $\overline{\psi}^{t_0}$, we can conclude that $\lim_{t_0 \to 0} |x^*(t_0)|_{\mathcal{M}_r(\chi^0)} = \lim_{t_0 \to 0} |V(t_0)|_{\mathcal{M}_r(\chi^0)} = 0$.

Now, if the conjecture is correct for any cases, then by the continuous dependence on initial conditions, we can re-state the result of Theorems 9 and 10 with the solution of the emergent dynamics

$$\dot{\xi} = h_\mu(f_1(t, \xi), \ldots, f_N(t, \xi)), \quad \xi(0) = \mathcal{M}_r(\chi^0),$$

where now the trajectory ξ is independent of t_0, and the approximation is valid for sufficiently small t_0. However, even though the existence of the limit $\lim_{t_0 \to 0} x^*(t_0)$ is not proved and the characterization is only a conjecture, we can still prove that for any $\eta > 0$ there exists $t_0^* > 0$ such that for all $t_0 \in (0, t_0^*)$ we have $\min_i x_i^0 - \eta \leq x^*(t_0) \leq \max_i x_i^0 + \eta$ according to the arguments in the proof of Lemma 1. From this, we can ensure a reasonable estimate because the stability of the emergent dynamics gives us the property of forgetting the initial condition and since the initial condition is contained in a compact interval. In particular, we can

[5] This is because, if $\nu_{J(t)}(t)/\psi(t) > (N-1)\bar{r}/(N+1)$, then

$$0 \equiv \sum_{i=1}^N \frac{\nu_i(t)}{\psi(t)} > \frac{N+1}{2}\frac{N-1}{N+1}\bar{r} - \frac{N-1}{2}\bar{r} = 0.$$

make a transient error arbitrary small after an arbitrarily short time by making the stability of the emergent dynamics sufficiently strong.

4 Discussions on the emergent dynamics

In this section, we will discuss emergent dynamics (9). Since the dynamics consist of the mapping obtained from the algebraic equation (10), it is at first glance hard to identify what it is. By starting to provide some algorithms to find the emergent dynamics for some special cases, we offer another formulation of the emergent dynamics, which is more eligible and can be simulated. Finally, by analyzing some extreme cases, we will discuss when the vector field of the emergent dynamics become some specific individual vector field f_i, the weighted average among the vector fields, or the weighted median among the vector fields.

4.1 Special cases

Consider first a funnel coupling law, which is well used in the practice, i.e., $\mu_i(t) = t/(1-|t|)$, $i \in \mathcal{N}$. Then, its inverse can be calculated as $\mu_i^{-1}(s) = s/(1+|s|)$, and the algebraic equation (10) follows as

$$\sum_{i=1}^{N} \frac{h_\mu(f_1,\ldots,f_N) - f_i}{1 + |h_\mu(f_1,\ldots,f_N) - f_i|} \equiv 0.$$

For this special case, this is equivalent to solving a piecewise $(N+1)$-th order polynomial, and the solution can be found by following the steps of the algorithm given by

1. Find an index set $\{i_1,\ldots,i_N\}$ such that $f_{i_j} \le f_{i_{j+1}}$ for all $j = 1,\ldots,N-1$.
2. Set $j = 1$.
3. Solve $\sum_{k=1}^{j} \frac{h-f_{i_k}}{1+h-f_{i_k}} + \sum_{k=j+1}^{N} \frac{h-f_{i_k}}{1-h+f_{i_k}} = 0$ which is a polynomial of order at most $(N+1)$.
4. If there is h such that $f_{i_j} \le h \le f_{i_{j+1}}$ then return h.
5. If not, increase j by 1 and go back to Step 3.

Another example of funnel coupling law is given by

$$\mu_i(t) = \begin{cases} \ln(1/(1-t)) & \text{if } t \ge 0, \\ \ln(1+t) & \text{if } t < 0. \end{cases}$$

Then, the inverse can be calculated as

$$\mu_i^{-1}(s) = \begin{cases} 1 - e^{-s} & \text{if } s \ge 0, \\ -1 + e^{s} & \text{if } s < 0. \end{cases}$$

For this special case, we only have to solve a piecewise 2nd order polynomial, and the solution can be found by following the steps of the algorithm given by

1. Find an index set $\{i_1,\ldots,i_N\}$ such that $f_{i_j} \le f_{i_{j+1}}$ for all $j = 1,\ldots,N-1$.
2. Set $j = 1$.
3. Solve $\sum_{k=1}^{j}(1-e^{-h+f_{i_k}}) + \sum_{k=j+1}^{N}(-1+e^{h-f_{i_k}}) = 0$ which is equivalent to solving a 2nd order polynomial given by ($H = e^h$)

$$\left[\sum_{k=j+1}^{N} e^{-f_{i_k}}\right] H^2 + (2j-N)H - \sum_{k=1}^{j} e^{f_{i_k}} = 0.$$

4. If there is h such that $f_{i_j} \le h \le f_{i_{j+1}}$ then return h.
5. If not, increase j by 1 and go back to Step 3.

However, even when such an algorithm is derived, simulating its solution requires some computational power, because, for each time step, the vector field must be obtained by running an algorithm. Therefore, we provide another formulation of the emergent dynamics which makes easier to simulate the solution, when f_i's are continuously differentiable with respect to its arguments.

4.2 Different formulation

The key observation in this subsection is that the partial derivative of $h_\mu(f_1,\ldots,f_N)$ with respect to their arguments can be obtained rather easily. In particular, let's compute the partial derivative of the algebraic equation (10) with respect to f_i. Then, we obtain

$$\sum_{j=1}^{N} (\mu_j^{-1})'(h_\mu - f_j)\left[\frac{\partial h_\mu}{\partial f_i} - \delta_{ij}\right] \equiv 0,$$

where δ_{ij} is the kronecker delta function. Now, this gives

$$\frac{\partial h_\mu}{\partial f_i}(f_1,\ldots,f_N) = \frac{(\mu_i^{-1})'(h_\mu - f_i)}{\sum_{j=1}^{N}(\mu_j^{-1})'(h_\mu - f_j)}.$$

Therefore, the solution of the emergent dynamics can be generated by a two-dimensional dynamical system given by

$$\begin{aligned} \dot{\xi} &= \chi = h_\mu(f_1(t,\xi),\ldots,f_N(t,\xi)) \\ \dot{\chi} &= \frac{\sum_{j=1}^{N}(\mu_j^{-1})'(\chi - f_j(t,\xi))\left[\frac{\partial f_j}{\partial t}(t,\xi) + \frac{\partial f_j}{\partial \xi}(t,\xi)\chi\right]}{\sum_{j=1}^{N}(\mu_j^{-1})'(\chi - f_j(t,\xi))} \\ \chi(0) &= h_\mu(f_1(0,\xi(0)),\ldots,f_N(0,\xi(0))). \end{aligned}$$

For example, the synchronous behavior of a heterogeneous network under funnel coupling law given by

$$\dot{x}_i = f_i(t,x_i) + \tan\left(\frac{\nu_i}{\psi(t)}\right),$$

can be approximated by the solution trajectory of a two-dimensional emergent dynamics given by

$$\dot{\xi} = \chi$$
$$\dot{\chi} = \frac{\sum_{j=1}^{N} \left[\frac{\partial f_j}{\partial t}(t,\xi) + \frac{\partial f_j}{\partial \xi}(t,\xi)\chi\right]/(1+(\chi - f_j(t,\xi))^2)}{\sum_{j=1}^{N} 1/(1+(\chi - f_j(t,\xi))^2)}.$$

Another example is taken from (Shim & Trenn, 2015), where the i-th agent is given by

$$\dot{x}_i = (-1+\delta_i)x_i + c_i(t) + \frac{1}{1-|\nu_i|/\psi(t)} \frac{\nu_i}{\psi(t)}$$
$$c_i(t) = 10\sin t + 10m_i^1 \sin(0.1t + \theta_i^1) + 10m_i^2 \sin(10t + \theta_i^2)$$

where $\psi(t) = 2 + 38e^{-t}$ and $N = 5$. Then, the synchronized behavior of the given network can be approximated by the solution trajectory of a two-dimensional emergent dynamics given by

$$\dot{\xi} = \chi \tag{12}$$
$$\dot{\chi} = \frac{\sum_{i=1}^{N} [(-1+\delta_i)\chi + \dot{c}_i]/(1+|\chi + (1-\delta_i)\xi - c_i|)^2}{\sum_{i=1}^{N} 1/(1+|\chi + (1-\delta_i)\xi - c_i|)^2}.$$

It is now also clear that this emergent behavior is invariant under the change of a graph topology if it is still undirected and also under the change of performance function $\psi(\cdot)$, as numerically shown in Figures 4, 6, and 7 of (Shim & Trenn, 2015), see also Figure 2.

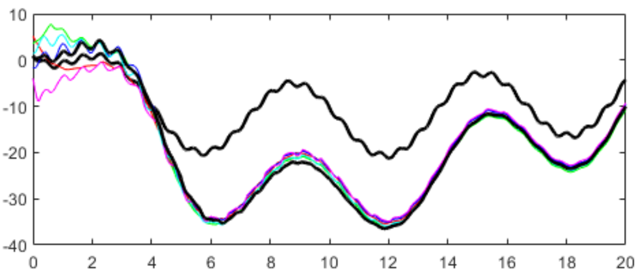

Fig. 2. Simulation result in (Shim & Trenn, 2015) has been revisited. Now, the unsolved question of synchronized behavior is answered as a lower black line, which illustrates the solution trajectory of the emergent dynamics (12).

We thus have seen some explicit characterization of the emergent dynamics for their possibility of simulation. However, by considering some extreme cases in the following, we can see more easily what the emergent dynamics will become.

4.3 Extreme cases

1) Specific individual vector field: First of all, for arbitrary index $i \in \mathcal{N}$, we can make the emergent dynamics similar to the individual vector field f_i. In particular, for any fixed continuously differentiable function $\bar{\gamma}_i : [0,1) \to [0,\infty)$ such that $\bar{\gamma}_i$ is strictly increasing and $\lim_{s\to 1} \bar{\gamma}_i(s) = \infty$, by letting $\mu_i(s) := \gamma_i(|s|)s := \bar{\gamma}_i(|s|/r_i)s/r_i$, we achieve

$$\lim_{r_i\to\infty} h_\mu(f_1,\dots,f_N) = f_i.$$

The underlying intuition is that the agent i gets less affected by the others because the coupling term dominates the dynamics of agent i only when the diffusive term approaches the funnel boundary $r_i\psi(t)$, which has become sufficiently large. On the other hand, the agent $j \neq i$ has to follow the trajectory of the agent i as the performance function becomes narrower because r_j is fixed and bounded. In fact, for any performance function ψ, we have

$$\lim_{r_i\to\infty} x_i(t) = \overline{x}_i(t), \quad \forall t \geq 0,$$

where $\overline{x}_i$ is the solution of $\dot{\overline{x}}_i(t) = f_i(t, \overline{x}_i(t))$ with the initial condition $\overline{x}_i(0) = x_i(0)$.

This is because, when r_i is larger than $\sum_{j\neq i} r_j$, we have, from the algebraic constraint $\sum_{i=1}^{N} \nu_i \equiv 0$, that

$$\frac{|\nu_i(t)|}{\psi(t)} \leq \sum_{j\neq i} \frac{|\nu_j(t)|}{\psi(t)} < \sum_{j\neq i} r_j < r_i, \quad \forall t \geq 0,$$

which implies that the error variable $e_i := x_i - \bar{x}_i$ satisfies

$$|\dot{e}_i| \leq |f_i(t, e_i + \bar{x}_i) - f_i(t, \bar{x}_i)| + \left|\mu_i\left(\frac{\nu_i(t)}{\psi(t)}\right)\right|$$
$$\leq |f_i(t, e_i + \bar{x}_i) - f_i(t, \bar{x}_i)| + \bar{\gamma}_i\left(\frac{\sum_{j\neq i} r_j}{r_i}\right)\frac{\sum_{j\neq i} r_j}{r_i},$$

where the second term can be made arbitrarily small by increasing r_i sufficiently large.

Remark 11 *We want to note that by the similar arguments as above, it can be proved that, when there exists $i \in \mathcal{N}$ such that $\sum_{j\neq i} r_j < r_i$, and when the dynamics $\dot{x} = f_i(t,x)$ is contractive, i.e., there exists $c > 0$ such that $\partial f_i(t,x)/\partial x \leq -c$ for all $t \geq 0$ and $x \in \mathbb{R}$, the solution trajectory is uniformly bounded on $[0,\infty)$, hence also the inputs. This is because, $V := |e_i|$ satisfies*

$$\dot{V} = \frac{e_i}{|e_i|}\dot{e}_i \leq -cV + \bar{\gamma}_i\left(\frac{\sum_{j\neq i} r_j}{r_i}\right)\frac{\sum_{j\neq i} r_j}{r_i},$$

which gives the boundedness of x_i from the boundedness of $\bar{x}_i$ (which comes from the fact that the dynamics is contractive), and the rest follows from the fact that $|\nu_j(t)| < r_j\psi(t)$ for all $t \geq 0$ and $j \in \mathcal{N}$. On the other hand, by noting that the coupling term is bounded independent to the performance function ψ, we can also prove that, by a similar argument, $x^*(t_0)$ in Lemma 8 satisfies $\lim_{t_0\to 0} x^*(t_0) = x_i(0) = \mathcal{M}_r(\chi^0)$. □

2) Weighted average among the vector fields: On the other hand, by enlarging the coupling gain $\gamma_i(|\nu_i(t)|/\psi(t))$ uniformly, for instance by using $\mu_i(s) := \kappa\bar{\gamma}_i(|s|)s$, we can make the emergent dynamics similar to the weighted averaged dynamics written as

$$\dot{\xi}(t) = \frac{\sum_{i=1}^{N} r_i f_i(t, \xi(t))}{\sum_{i=1}^{N} r_i}.$$

This is because the solution of the algebraic equation (10) converges to the weighted average as κ goes to infinity. Meanwhile, the convergence of $h_\mu(f_1, \dots, f_N)$ to the weighted average among the vector fields is only point-wise in $\mathrm{col}(f_1, \dots, f_N)$, and in fact, it can be proved that for any collection of μ_i functions, the solution of the algebraic equation (10) can never be linear, i.e., for each $\mathrm{col}(a_1, \dots, a_N) \in \mathbb{R}^N$ there exists $\mathrm{col}(f_1, \dots, f_N) \in \mathbb{R}^N$ such that $h_\mu(f_1, \dots, f_N) \neq \sum_{i=1}^{N} a_i f_i$.

However, one might be interested in recovering the weighted average at least locally, due to the utility of the blended dynamics introduced in the Introduction. For the interested readers, we want to mention that, it is possible to maintain the utility of the design method based on the blended dynamics while making it fully decentralized. In fact, when the blended dynamics is utilized for the synthesis of a network with some specific purposes, in many cases the blended dynamics is stable and there exists a region of interest, which is compact say $K \subseteq \mathbb{R}$. Then, we can recover the desired collective behavior, by the emergent dynamics, in the region of interest, for instance, by making μ_i^{-1} linear on K. By designing the emergent dynamics to be globally stable, we then achieve our goal.

Remark 12 *This also explains now why the behavior of the blended dynamics has been recovered by increasing κ in the numerical study in (Shim & Trenn, 2015).* □

3) Weighted median among the vector fields: The most interesting case is when the inverse of the μ_i function becomes the form of the signum function, i.e., the function defined by

$$\mathrm{sgn}(s, r_i) := \begin{cases} r_i & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -r_i & \text{if } s < 0. \end{cases}$$

For this special case, the solution of the algebraic equation (10) becomes a weighted median. In particular, if the function μ_i satisfies

$$|\mu_i^{-1}(s)| \geq r_i(1 - \varepsilon), \quad \forall s \text{ s.t. } |s| \geq \eta > 0, \tag{13}$$

for sufficiently small ε, then we can prove that

$$|h_\mu(f_1, \dots, f_N)|_{\mathcal{M}_r(\mathcal{F})} \leq \eta,$$

where $\mathcal{F}$ is a collection of real numbers f_i, $i \in \mathcal{N}$ and $\mathcal{M}_r(\mathcal{F})$ is defined at the end of Section 3.

Lemma 13 *For any fixed collection of $\mu_i(\cdot)$ as in (2), let us assume that (13) is satisfied for some $\eta > 0$ and $\varepsilon < 4\delta/(2\delta + 1)$, where $\delta > 0$ is such that $\sum_{i\in\mathcal{K}} r_i \geq (\frac{1}{2} + \delta)\sum_{i\in\mathcal{N}} r_i$ for any $\mathcal{K} \subseteq \mathcal{N}$ satisfying $\sum_{i\in\mathcal{K}} r_i > \frac{1}{2}\sum_{i\in\mathcal{N}} r_i$. Then, we have $|h_\mu(f_1, \dots, f_N)|_{\mathcal{M}_r(\mathcal{F})} \leq \eta$.*□

PROOF. By the constraint (13), we have either

$$\left|\mu_i^{-1}(h_\mu(f_1, \dots, f_N) - f_i)\right| \geq r_i(1 - \varepsilon), \tag{14}$$

or

$$|h_\mu(f_1, \dots, f_N) - f_i| < \eta. \tag{15}$$

Now, let us consider two separate situations, (i) when there exists $j \in \mathcal{N}$ such that $\mathcal{M}_r(\mathcal{F}) = \{f_{j_s}\}$ and (ii) when there exists $j \in \mathcal{N}$ such that $\mathcal{M}_r(\mathcal{F}) = [f_{j_s}, f_{(j+1)_s}]$, where $\{j_s\}$ is the rearrangement of the sequence $\{1, \dots, N\}$ such that

$$f_{1_s} \leq f_{2_s} \leq \cdots \leq f_{N_s}.$$

For the case (ii), if (15) is satisfied for either j_s or $(j+1)_s$, then we are done. However, if this is not the case, we have

$$\mu_{j_s}^{-1}(h_\mu(f_1, \dots, f_N) - f_{j_s}) \geq r_{j_s}(1 - \varepsilon) \tag{16}$$
$$\mu_{(j+1)_s}^{-1}(h_\mu(f_1, \dots, f_N) - f_{(j+1)_s}) \leq -r_{(j+1)_s}(1 - \varepsilon)$$

or there exists an index set $\mathcal{K} \subseteq \mathcal{N}$ such that $\sum_{i\in\mathcal{K}} r_i > (1/2)\sum_{i\in\mathcal{N}} r_i$, and that

$$\mu_i^{-1}(h_\mu(f_1, \dots, f_N) - f_i) \geq r_i(1 - \varepsilon), \quad \forall i \in \mathcal{K}, \tag{17}$$

or

$$\mu_i^{-1}(h_\mu(f_1, \dots, f_N) - f_i) \leq -r_i(1 - \varepsilon), \quad \forall i \in \mathcal{K}.$$

Also for the case (i), if (15) is satisfied for j_s, then we are done, and if this is not the case, we have $\mathcal{K} \subseteq \mathcal{N}$ as above. Since (16) implies that $h_\mu(f_1, \dots, f_N) \in \mathcal{M}_r(\mathcal{F})$, we now only have to consider the case when there exists an index set $\mathcal{K}$ with the aforementioned properties. However, this yields a contradiction, because, if without loss of generality assume that (17) hold, then we have

$$\begin{aligned} 0 &= \sum_{i=1}^{N} \mu_i^{-1}(h_\mu(f_1, \dots, f_N) - f_i) \\ &\geq \sum_{i\in\mathcal{K}} r_i(1 - \varepsilon) - \sum_{i\in\mathcal{N}\setminus\mathcal{K}} r_i = (2 - \varepsilon)\sum_{i\in\mathcal{K}} r_i - \sum_{i\in\mathcal{N}} r_i \\ &\geq \left[(2 - \varepsilon)\left(\frac{1}{2} + \delta\right) - 1\right]\sum_{i\in\mathcal{N}} r_i > 0 \end{aligned}$$

where the last term is positive by the definition of ε. $\square$

In particular, when all r_i's are identical, we find $\delta \geq 1/(2N)$, and thus $\varepsilon < 2/(N+1)$ is enough.

5 Application: Distributed median solver

The application we provide in this section is a distributed median solver. In fact, it is a simple application of our theorems, and thus, the proposed solver has the form of

$$\dot{x}_i(t) = f_i^* - x_i(t) + \mu_i\left(\frac{\nu_i(t)}{\psi(t)}\right), \quad x_i(0) = x_i^0, \quad (18)$$

where $\lim_{t\to\infty}\psi(t) = 0$, $\psi(t) > 0$ for all $t \geq 0$, and there exists $\theta_\psi > 0$ such that $|d\psi(t)/dt| \leq \theta_\psi\psi(t)$ for all $t \geq 0$. Then, by Theorem 2 it will achieve asymptotic synchronization. However, in this special case, we can show that the steady-state behavior of this synchronized network is equivalent to that of the emergent dynamics, which is

$$\dot{\xi}(t) = h_\mu(f_1^*, \ldots, f_N^*) - \xi(t) \quad (19)$$

in this case, and thus, contractive, and converges exponentially fast to the constant $h_\mu(f_1^*, \ldots, f_N^*)$. Since as shown earlier in Lemma 13, we can make this constant arbitrary close to a weighted median among the constants f_i^*, the proposed scalar network finds a weighted median with arbitrary precision. Note that the error is independent to the collection $\{f_1^*, \ldots, f_N^*\}$, and only depends on the characteristics of the μ_i function as illustrated in Lemma 13. A brief proof of this claim is found in Appendix A.3.

Remark 14 *In particular, the analysis conducted in Appendix A.3 can be done for any network, which has the same properties for the performance function ψ, if the solution is proved to be bounded explicitly.* $\square$

Such median solver can be used to extract outliers, hence with the majority of good samples, we can specify by observing the whole, what is good. This may when extended to vector counterpart be the distributed solution of extracting malicious attacks in a cyber-physical system as in (Lee, Kim, & Shim, 2019b). Final emphasis is made that the design can be done in a fully decentralized manner, with the only prior agreement on ε and η in Lemma 13, and the agreement on ψ.

6 Conclusion

This paper introduces funnel coupling law which guarantees synchronization for a heterogeneous multi-agent system under only mild assumptions. Some sufficient conditions which guarantee boundedness of the inputs are also provided, and the analysis on the emergent collective behavior that appears as we enforce synchronization by the proposed funnel coupling law has been conducted. In fact, the paper introduced emergent dynamics that can illustrate the synchronized behavior of the whole network, and from its nonlinear structure, some new applications have been discovered, e.g., distributed median solver. Our future work is to extend our result to its vector counterpart, hence utilizing its interesting features, and to further derive useful applications.

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A Proofs

A.1 Proof of Theorem 9

By Theorem 2 we know that the solution exists and by Lemma 1 we know that there exists $\overline{M} > 0$ such that

$$|x_i(t, t_0, \varepsilon)| \le \overline{M}, \quad \forall t \in [0, t_0 + T], \quad i \in \mathcal{N},\ \varepsilon > 0.$$

Then, let us denote L_f as a maximum among the Lipschitz constants of f_i on the compact set $[-\overline{M}, \overline{M}]$, i.e., $|f_i(t, a) - f_i(t, b)| \le L_f|a - b|$ for all $i \in \mathcal{N}$, $t \ge 0$, and $a, b \in [-\overline{M}, \overline{M}]$. Moreover, denote L_h and M_h as a Lipschitz constant and a norm bound of $h_\mu(f_1(t, a), \ldots, f_N(t, a))$ on the same compact set respectively. Now, we also know that there exists $\delta > 0$ such that

$$\left|\mu_i^{-1}(h_\mu(f_1(t, a), \ldots, f_N(t, a)) - f_i(t, a))\right| \le r_i(1 - 2\delta), \tag{A.1}$$

for all $i \in \mathcal{N}$, $t \ge 0$, and $a \in [-\overline{M}, \overline{M}]$. Let us finally denote L_μ as a maximum among the Lipschitz constants of μ_i on the compact set $[-r_i(1-\delta), r_i(1-\delta)] \subseteq (-r_i, r_i)$ for $i \in \mathcal{N}$.

From Assumption 2, we can find a diagonal matrix $\Lambda = \text{diag}(\lambda_2, \ldots, \lambda_N)$ with $0 < \lambda_2 < \cdots < \lambda_N$ and a matrix $R \in \mathbb{R}^{N\times(N-1)}$ such that $\left[\frac{1}{\sqrt{N}}1_N \ R\right]$ is an orthogonal matrix and satisfy $\mathcal{L} = R\Lambda R^T$.

Now, recall that $a(t, t_0, \varepsilon) := (1/N)\sum_{i=1}^N x_i(t, t_0, \varepsilon)$, and let

$$y(t, t_0, \varepsilon) := -\frac{1}{\psi_\varepsilon^{t_0}(t)}\Lambda R^T \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} - R^T \begin{bmatrix} \mu_1^{-1}(h_\mu^a - f_1^a) \\ \vdots \\ \mu_N^{-1}(h_\mu^a - f_N^a) \end{bmatrix},$$

where h_μ^a stands for $h_\mu(f_1^a, \ldots, f_N^a)$ and f_i^a stands for $f_i(t, a(t, t_0, \varepsilon))$. Then, we have

$$\frac{\nu_i}{\psi_\varepsilon^{t_0}} - \mu_i^{-1}\left(h_\mu^a - f_i^a\right) \equiv \mathfrak{r}_i y \tag{A.2}$$

where $\mathfrak{r}_i \in \mathbb{R}^{1\times(N-1)}$ is the i-th row of R, and thus,

$$\begin{aligned} \dot{a} &= \frac{1}{N}\sum_{i=1}^N \left[f_i(t, x_i) + \mu_i\left(\frac{\nu_i}{\psi_\varepsilon^{t_0}}\right) + (h_\mu^a - f_i^a) - (h_\mu^a - f_i^a)\right] \\ &= h_\mu^a + \frac{1}{N}\sum_{i=1}^N [f_i(t, x_i) - f_i^a] \\ &\quad + \frac{1}{N}\sum_{i=1}^N \left[\mu_i\left(\mathfrak{r}_i y + \mu_i^{-1}(h_\mu^a - f_i^a)\right) - \mu_i(\mu_i^{-1}(h_\mu^a - f_i^a))\right]. \end{aligned}$$

Therefore, we know that $V(t, t_0, \varepsilon) := |a(t, t_0, \varepsilon) - \xi(t)|$

satisfies [6]

$$\begin{aligned}\dot V \le |\dot a - \dot\xi| &\overset{\text{(A.1)}}{\le} \left|h_\mu^a - h_\mu(f_1(t,\xi),\dots,f_N(t,\xi))\right| \\ &\quad + \frac{1}{N}\sum_{i=1}^N L_f|x_i - a| + \frac{1}{N}\sum_{i=1}^N L_\mu\,|\mathfrak{r}_i y| \\ &\le L_h V + M_V\psi_\varepsilon^{t_0} + L_\mu|y| \qquad \text{(A.3)}\end{aligned}$$

whenever $|y| < \delta$, where $M_V := \sqrt{N}L_f(\max_i r_i)/\lambda_2$.

Now, note that y satisfies

$$\begin{aligned}\dot y &= \frac{\dot\psi_\varepsilon^{t_0}}{(\psi_\varepsilon^{t_0})^2}\Lambda R^T\begin{bmatrix}x_1\\ \vdots \\ x_N\end{bmatrix} - \frac{1}{\psi_\varepsilon^{t_0}}\Lambda R^T\begin{bmatrix}f_1(t,x_1)-f_1^a\\ \vdots\\ f_N(t,x_N)-f_N^a\end{bmatrix} \\ &\quad - \frac{1}{\psi_\varepsilon^{t_0}}\Lambda R^T\begin{bmatrix}\mu_1\big(\mathfrak{r}_1 y + \mu_1^{-1}(h_\mu^a - f_1^a)\big) - (h_\mu^a - f_1^a)\\ \vdots \\ \mu_N\big(\mathfrak{r}_N y + \mu_N^{-1}(h_\mu^a - f_N^a)\big) - (h_\mu^a - f_N^a)\end{bmatrix} \\ &\quad - R^T\begin{bmatrix}(\mu_1^{-1})'(h_\mu^a - f_1^a)\frac{d}{dt}(h_\mu^a - f_1^a)\\ \vdots \\ (\mu_N^{-1})'(h_\mu^a - f_N^a)\frac{d}{dt}(h_\mu^a - f_N^a)\end{bmatrix}. \qquad \text{(A.4)}\end{aligned}$$

Then, $W(t,t_0,\varepsilon) := y(t,t_0,\varepsilon)^T\Lambda^{-1}y(t,t_0,\varepsilon)$ satisfies

$$\begin{aligned}\dot W &\le 2\left|\frac{\dot\psi_\varepsilon^{t_0}}{\psi_\varepsilon^{t_0}}\right|\left[y^T\Lambda^{-1}y + \left|y^T\Lambda^{-1}R^T\begin{bmatrix}\mu_1^{-1}(h_\mu^a - f_1^a)\\ \vdots \\ \mu_N^{-1}(h_\mu^a - f_N^a)\end{bmatrix}\right|\right] \\ &\quad + \frac{2}{\psi_\varepsilon^{t_0}}|y|NL_f\frac{\max_i r_i}{\lambda_2}\psi_\varepsilon^{t_0} \\ &\quad - \frac{2}{\psi_\varepsilon^{t_0}}\sum_{i=1}^N(\mathfrak{r}_i y)\left[\mu_i\big(\mathfrak{r}_i y + \mu_i^{-1}(h_\mu^a - f_i^a)\big) - (h_\mu^a - f_i^a)\right] \\ &\quad + 2|\Lambda^{-1}y|N\max_i\left|(\mu_i^{-1})'(h_\mu^a - f_i^a)\frac{d}{dt}(h_\mu^a - f_i^a)\right|, \qquad \text{(A.5)}\end{aligned}$$

where the identity

$$\begin{aligned}&-\frac{1}{\psi_\varepsilon^{t_0}}y^TR^Tx = y^T\Lambda^{-1}\left(-\frac{1}{\psi_\varepsilon^{t_0}}\Lambda R^T x\right) \\ &= y^T\Lambda^{-1}[y + R^T\mathrm{col}(\mu_1^{-1}(h_\mu^a - f_1^a),\dots,\mu_N^{-1}(h_\mu^a - f_N^a))]\end{aligned}$$

has been utilized for the derivation. Thus, by noting that we have

$$|\dot a| \le M_h + M_V\overline{\psi} + L_\mu\delta =: M_a$$

whenever $|y| < \delta$ where $\overline{\psi} := \sup_{t\ge 0,\,\varepsilon\in(0,\bar\varepsilon)}\psi_\varepsilon^{t_0}(t)$ and $\bar\varepsilon$ is given in Assumption 3, and by also noting that we always have

$$(b-a)(\mu_i(b)-\mu_i(a)) \ge \underline{\mu}(b-a)^2, \quad -\infty < \forall a \le \forall b < \infty,$$

where $\underline{\mu} := \min_i\gamma_i(0)$, we obtain for all $\varepsilon\in(0,\bar\varepsilon)$,

$$\begin{aligned}\dot W &\le M_W|y| + 2\theta_\psi W - \frac{2}{\psi_\varepsilon^{t_0}}\sum_{i=1}^N\underline{\mu}(\mathfrak{r}_i y)^2 \\ &= M_W|y| + 2\theta_\psi W - \frac{2}{\psi_\varepsilon^{t_0}}\underline{\mu}|y|^2 \qquad \text{(A.6)}\end{aligned}$$

where θ_ψ is given in Assumption 3 and M_W is defined as

$$\frac{2N}{\lambda_2}\Big[(L_f + \theta_\psi)\max_i r_i + (L_f + L_h)M_a\max_i(\mu_i^{-1})'(0)\Big].$$

Now, let $\varepsilon^* > 0$ be such that

$$\varepsilon^* \le \min\left\{\frac{L_h\eta}{4M_Ve^{L_hT}},\frac{\underline{\mu}\delta_\eta\lambda_2}{M_W\sqrt{\lambda_2\lambda_N} + \theta_\psi\delta_\eta},\bar\varepsilon\right\}$$

and satisfies for all $\varepsilon\in(0,\varepsilon^*)$

$$V(t_0,t_0,\varepsilon) < \frac{\eta}{4e^{L_hT}} \le \frac{\eta}{4},\quad W(t_0,t_0,\varepsilon) < \frac{\delta_\eta^2}{4\lambda_N}, \quad \text{(A.7)}$$

which is always possible by Lemma 8, where

$$\delta_\eta := \min\left\{\frac{L_h\eta}{4L_\mu e^{L_hT}},\delta,\frac{3}{4}\eta\right\}.$$

In the following, we will show that V and W satisfy

$$V(t,t_0,\varepsilon) < \eta,\quad W(t,t_0,\varepsilon) < \frac{\delta_\eta^2}{\lambda_N}, \qquad \text{(A.8)}$$

for all $t\in[t_0,t_0+T]$ and $\varepsilon\in(0,\varepsilon^*)$, which then completes the proof because we have from (A.2):

$$\left|\frac{\nu_i}{\psi_\varepsilon^{t_0}} - \mu_i^{-1}\left(h_\mu^a - f_i^a\right)\right| = |\mathfrak{r}_i y| \le |y| \le \sqrt{\lambda_N W} < \delta_\eta < \eta$$

for all $i\in\mathcal{N}$.

For this purpose, fix $\varepsilon\in(0,\varepsilon^*)$ and let

$$\begin{aligned}\Omega_\varepsilon := \{&(t,x)\in\mathbb{R}_{\ge t_0}\times\mathbb{R}^N : \\ &|\mathcal{L}_r x|_\infty < \psi_\varepsilon^{t_0}(t),\ V < \eta,\ W < \delta_\eta^2/\lambda_N\},\end{aligned}$$

[6] Here, the following inequality is used.

$$\begin{aligned}&\max_{i\in\mathcal{N}}|x_i - a| = |RR^Tx|_\infty = |R\Lambda^{-1}R^T\mathrm{diag}(r_1,\dots,r_N)\mathcal{L}_r x|_\infty \\ &\le \sqrt{N}|R\Lambda^{-1}R^T|_2|\mathrm{diag}(r_1,\dots,r_N)|_\infty|\mathcal{L}_r x|_\infty \le \sqrt{N}\frac{\max_i r_i}{\lambda_2}\psi_\varepsilon^{t_0}\end{aligned}$$

and choose a maximal $\omega > t_0$ such that $x(t, t_0, \varepsilon) \in \Omega_\varepsilon$ for all $t \in [t_0, \omega)$. Seeking a contradiction suppose that $\omega \le t_0 + T$. Then, there is a time sequence $\{t_k\}$ satisfying $\lim_{k\to\infty} t_k = \omega$ and $\lim_{k\to\infty} V(t_k) = \eta$ or $\lim_{k\to\infty} W(t_k) = \delta_\eta^2/\lambda_N$, because, in Lemma 3, we have already shown that there does not exist a time sequence $\{t_k\}$ satisfying $\lim_{k\to\infty} t_k = \omega$ and $\lim_{k\to\infty} |\mathcal{L}_r x(t_k, t_0, \varepsilon)|_\infty = \psi_\varepsilon^{t_0}(\omega)$.

However, for $t \in [t_0, \omega)$, we have from $W < \delta_\eta^2/\lambda_N$ that $|y| < \delta_\eta < \delta$, and thus, by (A.3), V is bounded by

$$V(t) \le V(t_0)e^{L_h(t-t_0)} + \int_{t_0}^{t} e^{L_h(t-\tau)}(M_V\psi_\varepsilon^{t_0} + L_\mu\delta_\eta)d\tau$$
$$\le V(t_0)e^{L_hT} + \frac{1}{L_h}(M_V\varepsilon + L_\mu\delta_\eta)e^{L_hT} \le \frac{3\eta}{4},$$

and W is bounded by $\delta_\eta^2/(4\lambda_N)$ because $W \ge \delta_\eta^2/(4\lambda_N)$ implies $|y| \ge (\delta_\eta/2)\sqrt{\lambda_2/\lambda_N}$, and thus

$$M_W \le \left(\frac{2\underline{\mu}}{\varepsilon^*} - \frac{2\theta_\psi}{\lambda_2}\right)\frac{\delta_\eta}{2}\sqrt{\frac{\lambda_2}{\lambda_N}} < \left(\frac{2\underline{\mu}}{\psi_\varepsilon^{t_0}} - \frac{2\theta_\psi}{\lambda_2}\right)|y|,$$

which implies by (A.6) that $\dot{W} < 0$, and hence the set $\{W < \delta_\eta^2/(4\lambda_N)\}$ is positively invariant on $[t_0, \omega)$. Now, this yields the desired contradiction.

A.2 Proof of Theorem 10

First of all, we can find M_ξ such that $|\xi(t)| \le M_\xi$ for all $t \ge t_0$ because the emergent dynamics is contractive. Then, let $\overline{M}$ be such that $\overline{M} \ge M_\xi + \eta$ and

$$|x_i(t, t_0, \varepsilon)| \le \overline{M}, \quad \forall t \in [0, t_0], \quad i \in \mathcal{N}, \ \varepsilon > 0.$$

Now, proceed as in Appendix A.1 with this new $\overline{M}$ to obtain (A.3) and (A.6), and let $\varepsilon^* > 0$ be such that

$$\varepsilon^* \le \min\left\{\frac{c\eta}{4M_V}, \frac{\underline{\mu}\delta_\eta\lambda_2}{M_W\sqrt{\lambda_2\lambda_N} + \theta_\psi\delta_\eta}, \bar{\varepsilon}\right\}$$

and satisfies (A.7) for all $\varepsilon \in (0, \varepsilon^*)$, where $\delta_\eta := \min\{c\eta/(4L_\mu), \delta, 3\eta/4\}$.

In the following, we will show that V and W satisfy (A.8) for all $t \in [t_0, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, by following the proof of Theorem 9. In particular, seeking a contradiction suppose that $\omega \in (t_0, \infty)$. Then, for $t \in [t_0, \omega)$, we have $|y| < \delta_\eta < \delta$, and thus, by (A.3) (where by the assumption that the emergent dynamics is contractive, we can replace L_h by $-c$), V is bounded by

$$V(t) \le V(t_0)e^{-c(t-t_0)} + \int_{t_0}^{t} e^{-c(t-\tau)}(M_V\psi_\varepsilon^{t_0} + L_\mu\delta_\eta)d\tau$$
$$\le V(t_0) + \frac{1}{c}(M_V\varepsilon + L_\mu\delta_\eta) \le \frac{3\eta}{4}.$$

The rest of the proof is identical to Appendix A.1.

A.3 Sketch of the proof of the claim in Section 5

First of all, note that the boundedness of the solution is guaranteed, i.e., there exists $\overline{M}$ such that $|x_i(t)| \le \overline{M}$ for all $i \in \mathcal{N}$ and $t \in [0, \infty)$, because, by a similar argument as in the proof of Lemma 1, we have for all $t \ge 0$,

$$\min\left\{\min_i f_i^*, \min_i x_i^0\right\} \le x_i(t) \le \max\left\{\max_i f_i^*, \max_i x_i^0\right\}.$$

Now, proceed as in Appendix A.1 with this new $\overline{M}$ to obtain (A.3) and (A.6). By noting that, in this special case, $h_\mu^a - f_i^a$ is a constant, (A.6) holds for any y because the last terms of (A.4) and (A.5) are zero.

Therefore, we have for all $t \ge 0$,

$$\dot{W} \le M_W\sqrt{\lambda_N}\sqrt{W} - \left(\frac{2\underline{\mu}}{\lambda_N\psi} - 2\theta_\psi\right)W.$$

Now, this implies $\lim_{t\to\infty} W(t) = 0$ because, otherwise, there exists $\eta > 0$, such that $W(t) \ge \eta^2$ for all $t \ge 0$, however, there also exists $T_\eta > 0$ such that

$$M_W\sqrt{\lambda_N} < \left(\frac{2\underline{\mu}}{\lambda_N\psi(t)} - 2\theta_\psi\right)\eta, \quad \forall t \ge T_\eta,$$

which is a contradiction because $x_i(t)$ is bounded by the constant $\overline{M}$.

On the other hand, V satisfies (as in Appendix A.2)

$$\dot{V} \le -V + M_V\psi(t) + L_\mu|y|$$

whenever $|y| \le \delta$, and since $\lim_{t\to\infty}\psi(t) = 0$ and $\lim_{t\to\infty}|y(t)| = 0$, we can conclude $\lim_{t\to\infty} V(t) = 0$, by a similar argument as above. Since any solution trajectory of the emergent dynamics (9) converges to the constant $h_\mu(f_1^*, \ldots, f_N^*)$, the proof completes.

A.4 Sketch of the proof of Lemma 8

First of all, recall that $\lim_{\varepsilon\to 0}\psi_\varepsilon^{t_0}(t) =: \overline{\psi}^{t_0}(t) > 0$ for $t \in [0, t_0)$ and the result of Theorem 2. Then, there exists $\overline{x}_i : [0, t_0) \to \mathbb{R}$ such that for all $i \in \mathcal{N}$, we have

$$\dot{\overline{x}}_i(t) = f_i(t, \overline{x}_i(t)) + \mu_i\left(\frac{\overline{\nu}_i}{\overline{\psi}^{t_0}(t)}\right), \quad \forall t \in [0, t_0).$$

Moreover, by Corollary 4, the solution trajectories and the inputs are uniformly bounded. Therefore, as in Appendix A.3, we can show that $\lim_{t\to t_0} W(t) = 0$, i.e., $\lim_{t\to t_0}|\overline{y}(t)| = 0$.

Now, if we consider $\overline{x}_i(\cdot)$ as $x_i(\cdot, t_0, 0)$, by the continuous dependence, for any $\tau < t_0$ and $\eta > 0$, there exists ε^* such that for each $\varepsilon \in (0, \varepsilon^*)$ we have

$$\begin{aligned} |x_i(t, t_0, \varepsilon) - \overline{x}_i(t)| &\le \eta, \quad \forall t \in [0, \tau], \\ |y_i(t, t_0, \varepsilon) - \overline{y}_i(t)| &\le \eta, \quad \forall t \in [0, \tau]. \end{aligned}$$

Then, by a similar argument as in the proof of Theorem 9, for each $\eta > 0$, there exists ε^* and $\tau^* < t_0$ such that, for each $\varepsilon \in (0, \varepsilon^*)$, we have

$$\left| \frac{1}{N} \sum_{i=1}^{N} x_i(t, t_0, \varepsilon) - \xi(t) \right| =: |a(t, t_0, \varepsilon) - \xi(t)| \le \eta,$$
$$|\mathfrak{r}_i y(t, t_0, \varepsilon)| \le \eta,$$

for all $t \in [\tau^*, t_0]$, where $\xi(\cdot)$ is the solution of

$$\begin{aligned} \dot{\xi}(t) &= h_\mu(f_1(t, \xi(t)), \ldots, f_N(t, \xi(t))), \quad t \in [\tau^*, t_0] \\ \xi(\tau^*) &= \frac{1}{N} \sum_{i=1}^{N} \overline{x}_i(\tau^*). \end{aligned}$$

By this, $\mu_i(\nu_i/\psi_\varepsilon^{t_0}(t))$ is bounded on $[\tau^*, t_0]$ uniformly in $\varepsilon \in (0, \varepsilon^*)$, for sufficiently small ε^* and for τ^* sufficiently close to t_0.

Now, since $\overline{x}_i$ is bounded on $[0, t_0)$, for each time sequence $\{t_k\} \to t_0$, there exists a sequence $\{k_p\} \to \infty$ and $x^*(t_0)$ such that

$$\lim_{p \to \infty} t_{k_p} = t_0 \text{ and } \lim_{p \to \infty} \overline{x}_i(t_{k_p}) = x^*(t_0).$$

Then, for arbitrary $\eta > 0$, we have

$$\left| \frac{1}{N} \sum_{i=1}^{N} x_i(t_0, t_0, \varepsilon) - \xi_p(t_0) \right| \le \eta$$

for sufficiently small ε and sufficiently large p, where ξ_p is the solution of

$$\begin{aligned} \dot{\xi}_p(t) &= h_\mu(f_1(t, \xi_p(t)), \ldots, f_N(t, \xi_p(t))) \\ \xi_p(t_{k_p}) &= \frac{1}{N} \sum_{i=1}^{N} \overline{x}_i(t_{k_p}). \end{aligned}$$

Finally, noting that h_μ is globally Lipschitz uniformly in t on the compact set $[-\overline{M}, \overline{M}]$, η is arbitrary, and

$$\lim_{p \to \infty} |\xi_p(t_{k_p}) - x^*(t_0)| = 0,$$

we can conclude that

$$\lim_{\varepsilon \to 0} \frac{1}{N} \sum_{i=1}^{N} x_i(t_0, t_0, \varepsilon) = x^*(t_0).$$

This, in other words, show that for any sequence $\{t_k\}$ and any subsequence $\{t_{k_p}\}$ that has a limit, the limit is identical, which ensures

$$\lim_{t \to t_0} \overline{x}_i(t) = x^*(t_0).$$

Therefore, the rest of the claim follows.