



university of  
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# Switched differential algebraic equations: Jumps and impulses

**Stephan Trenn**

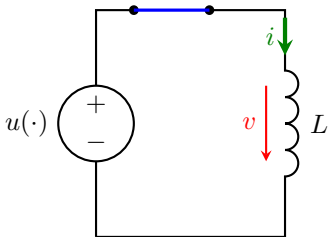
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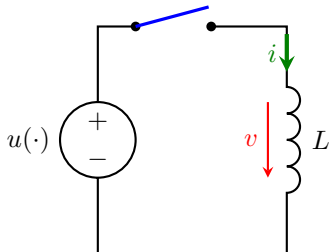
Mathematisches Kolloquium, Universität Paderborn, 17 June 2019, 16:45

# Motivating example

$t < 0$



$t \geq 0$



inductivity law:

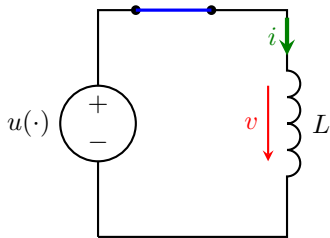
$$L \frac{d}{dt} i = v$$

switch dependent:  $0 = v - u$

$$0 = i$$

# Motivating example

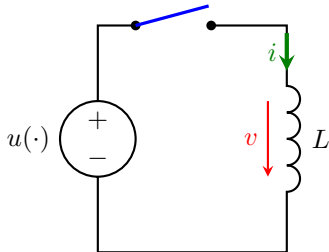
$t < 0$



$$x = [i, v]^T$$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$t \geq 0$

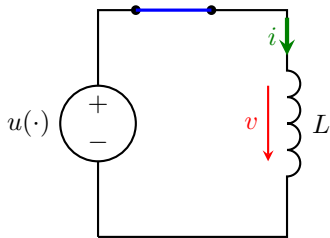


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# Motivating example

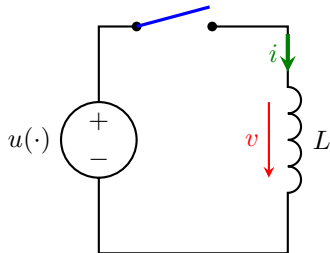
$t < 0$



$$E_1 \dot{x} = A_1 x + B_1 u$$

on  $(-\infty, 0)$

$t \geq 0$



$$E_2 \dot{x} = A_2 x + B_2 u$$

on  $[0, \infty)$

→ switched differential-algebraic equation

# Solution of circuit example

$$t < 0$$

$$v = u$$

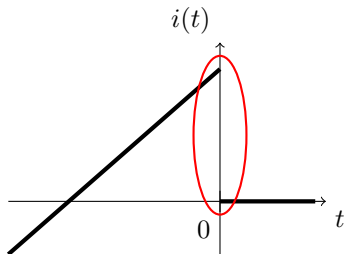
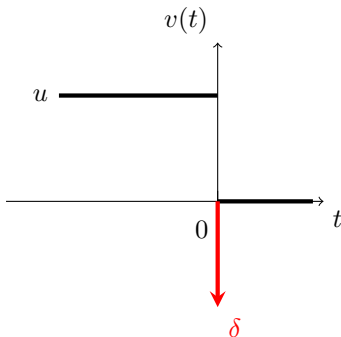
$$L \frac{d}{dt} i = v$$

$$t \geq 0$$

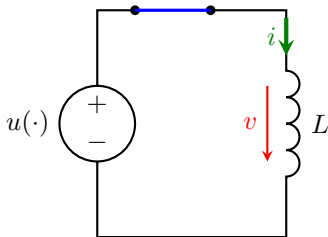
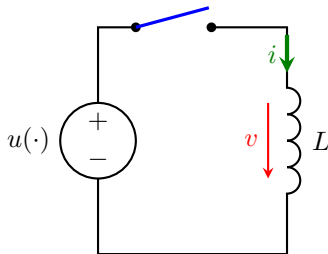
$$i = 0$$

$$v = L \frac{d}{dt} i$$

Solution (assume constant input  $u$ ):



# Observations

 $t < 0$ 

 $t \geq 0$ 


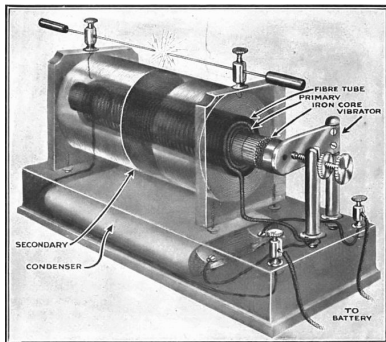
## Observations

- ›  $x(0^-) \neq 0$  **inconsistent** for  $E_2\dot{x} = A_2x + B_2u$
- › **unique jump** from  $x(0^-)$  to  $x(0^+)$
- › derivative of jump = **Dirac impulse** appears in solution

# Dirac impulse is “real”

## Dirac impulse

Not just a mathematical artifact!



Drawing: Harry Winfield Secor, public domain

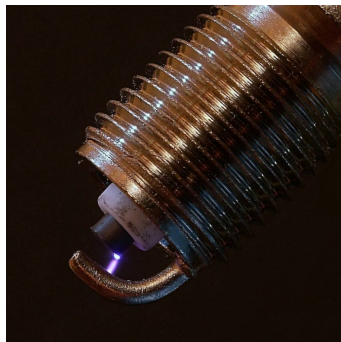


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Calculation of the four subspaces

$\mathcal{E}_-$

$\mathcal{O}_-$  and  $\mathcal{O}_+$

$\mathcal{O}_+^{\text{imp}}$

## Summary



# Definition

Switch → Different DAE models (=modes)  
 depending on **time-varying** position of switch

## Definition (Switched DAE)

Switching signal  $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$  picks mode at each time  $t \in \mathbb{R}$ :

$$\begin{aligned} E_{\sigma(t)} \dot{x}(t) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \\ y(t) &= C_{\sigma(t)} x(t) + D_{\sigma(t)} u(t) \end{aligned} \quad (\text{swDAE})$$

## Attention

Each mode might have **different consistency spaces**  
 ⇒ inconsistent initial values at each switch  
 ⇒ Dirac impulses, in particular **distributional solutions**

# Definition

Switch  $\rightarrow$  Different DAE models (=modes)  
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 $\Rightarrow$  inconsistent initial values at each switch  
 $\Rightarrow$  Dirac impulses, in particular **distributional solutions**

# Distribution theory - basic ideas

## Distributions - overview

- › Generalized functions
- › Arbitrarily often differentiable
- › Dirac-Impulse  $\delta$  is “derivative” of Heaviside step function  $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1. Functional analytical: Dual space of the space of test functions  
(L. Schwartz 1950)
2. Axiomatic: Space of all “derivatives” of continuous functions  
(J. Sebastião e Silva 1954)

# Distributions - formal

## Definition (Test functions)

$$\mathcal{C}_0^\infty := \{\varphi : \mathbb{R} \rightarrow \mathbb{R} \mid \varphi \text{ is smooth with compact support}\}$$

## Definition (Distributions)

$$\mathbb{D} := \{D : \mathcal{C}_0^\infty \rightarrow \mathbb{R} \mid D \text{ is linear and continuous}\}$$

## Definition (Regular distributions)

$$f \in \mathcal{L}_{1,\text{loc}}(\mathbb{R} \rightarrow \mathbb{R}): f_{\mathbb{D}} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)dt \in \mathbb{D}$$

## Definition (Derivative)

$$D'(\varphi) := -D(\varphi')$$

## Dirac Impulse at $t_0 \in \mathbb{R}$

$$\delta_{t_0} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}, \varphi \mapsto \varphi(t_0)$$

$$(\mathbb{1}_{[0,\infty)}_{\mathbb{D}})'(\varphi) = -\int_{\mathbb{R}} \mathbb{1}_{[0,\infty)}\varphi' = -\int_0^\infty \varphi' = -(\varphi(\infty) - \varphi(0)) = \varphi(0)$$

# Multiplication with functions

## Definition (Multiplication with smooth functions)

$$\alpha \in C^\infty : (\alpha D)(\varphi) := D(\alpha\varphi)$$

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

## Coefficients not smooth

Problem:  $E_\sigma, A_\sigma, C_\sigma \notin C^\infty$

Observation, for  $\sigma_{[t_i, t_{i+1})} \equiv p_i, i \in \mathbb{Z}$ :

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \Leftrightarrow \forall i \in \mathbb{Z} : \begin{aligned} (E_{p_i} \dot{x})_{[t_i, t_{i+1})} &= (A_{p_i} x + B_{p_i} u)_{[t_i, t_{i+1})} \\ y_{[t_i, t_{i+1})} &= (C_{p_i} x + D_{p_i} u)_{[t_i, t_{i+1})} \end{aligned}$$

New question: **Restriction of distributions**

# Desired properties of distributional restriction

Distributional restriction:

$$\{M \subseteq \mathbb{R} \mid M \text{ interval}\} \times \mathbb{D} \rightarrow \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval  $M \subseteq \mathbb{R}$

1.  $D \mapsto D_M$  is a **projection** (linear and idempotent)

2.  $\forall f \in \mathcal{L}_{1,\text{loc}} : (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$

3.  $\forall \varphi \in \mathcal{C}_0^\infty : \left[ \begin{array}{ll} \text{supp } \varphi \subseteq M & \Rightarrow D_M(\varphi) = D(\varphi) \\ \text{supp } \varphi \cap M = \emptyset & \Rightarrow D_M(\varphi) = 0 \end{array} \right]$

4.  $(M_i)_{i \in \mathbb{N}}$  pairwise disjoint,  $M = \bigcup_{i \in \mathbb{N}} M_i$ :

$$D_M = \sum_{i \in \mathbb{N}} D_{M_i}, \quad D_{M_1 \dot{\cup} M_2} = D_{M_1} + D_{M_2}, \quad (D_{M_1})_{M_2} = 0$$

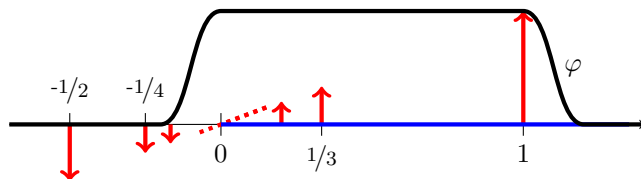
Theorem ([T. 2009])

*Such a distributional restriction **does not exist**.*

# Proof of non-existence of restriction

Consider the following (well defined!) distribution:

$$D := \sum_{i \in \mathbb{N}} d_i \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Restriction should give

$$D_{[0, \infty)} = \sum_{k \in \mathbb{N}} d_{2k} \delta_{d_{2k}}$$

Choose  $\varphi \in C_0^\infty$  such that  $\varphi_{[0,1]} \equiv 1$ :

$$D_{[0, \infty)}(\varphi) = \sum_{k \in \mathbb{N}} d_{2k} = \sum_{k \in \mathbb{N}} \frac{1}{2k+1} = \infty$$

# Dilemma

## Switched DAEs

- › Examples: distributional solutions
- › Multiplication with non-smooth coefficients
- › Or: Restriction on intervals

## Distributions

- › Distributional restriction not possible
- › Multiplication with non-smooth coefficients not possible
- › *Initial value problems cannot be formulated*

## Underlying problem

Space of distributions **too big**.

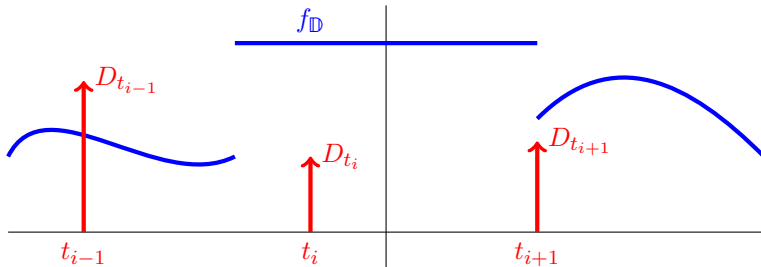


# Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions  $\mathbb{D}_{pwC^\infty}$ , [T. 2009])

$$\mathbb{D}_{pwC^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \mid \begin{array}{l} f \in C_{pw}^\infty, \\ T \subseteq \mathbb{R} \text{ locally finite,} \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



# Properties of $\mathbb{D}_{\text{pw}}\mathcal{C}^\infty$

- ›  $\mathcal{C}_{\text{pw}}^\infty$  “ $\subseteq$ ”  $\mathbb{D}_{\text{pw}}\mathcal{C}^\infty$  and  $D \in \mathbb{D}_{\text{pw}}\mathcal{C}^\infty \Rightarrow D' \in \mathbb{D}_{\text{pw}}\mathcal{C}^\infty$
- › **Well defined restriction**  $\mathbb{D}_{\text{pw}}\mathcal{C}^\infty \rightarrow \mathbb{D}_{\text{pw}}\mathcal{C}^\infty$

$$D = f_{\mathbb{D}} + \sum_{t \in T} D_t \quad \mapsto \quad D_M := (f_M)_{\mathbb{D}} + \sum_{t \in T \cap M} D_t$$

- › **Multiplication** with  $\alpha = \sum_{i \in \mathbb{Z}} \alpha_i [t_i, t_{i+1}) \in \mathcal{C}_{\text{pw}}^\infty$  well defined:

$$\alpha D := \sum_{i \in \mathbb{Z}} \alpha_i D_{[t_i, t_{i+1})}$$

- › **Evaluation** at  $t \in \mathbb{R}$ :  $D(t^-) := f(t^-)$ ,  $D(t^+) := f(t^+)$
- › **Impulses** at  $t \in \mathbb{R}$ :  $D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases}$

## Application to (swDAE)

$(x, u)$  solves (swDAE)  $\Leftrightarrow$  (swDAE) holds in  $\mathbb{D}_{\text{pw}}\mathcal{C}^\infty$

# Relevant questions

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

## Piecewise-smooth distributional solution framework

$$x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n, \quad u \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^m, \quad y \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^p$$

- › Existence and uniqueness of solutions?
- › Jumps and impulses in solutions?
- › Conditions for impulse free solutions?
- › Control theoretical questions
  - Stability and stabilization
  - Observability and observer design
  - Controllability and controller design

# Existence and uniqueness of solutions for (swDAE)

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \quad (\text{swDAE})$$

## Basic assumptions

- ›  $\sigma \in \Sigma_0 := \left\{ \sigma : \mathbb{R} \rightarrow \{1, \dots, N\} \mid \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma|_{(-\infty, 0)} \text{ is constant} \end{array} \right\}$ .
- ›  $(E_p, A_p)$  is **regular**  $\forall p \in \{1, \dots, N\}$ , i.e.  $\det(sE_p - A_p) \neq 0$

## Theorem (T. 2009)

Consider (swDAE) satisfying the basic assumptions. Then

$$\forall u \in \mathbb{D}_{\text{pw}}^m \mathcal{C}^\infty \quad \forall \sigma \in \Sigma_0 \quad \exists \text{ solution } x \in \mathbb{D}_{\text{pw}}^n \mathcal{C}^\infty$$

and  $x(0^-)$  **uniquely** determines  $x$ .

# Inconsistent initial values

$$E\dot{x} = Ax + Bu, \quad x(0) = x^0 \in \mathbb{R}^n$$

Inconsistent initial value = special switched DAE

$$\begin{aligned} \dot{x}_{(-\infty,0)} &= 0, & x(0^-) &= x^0 \\ (E\dot{x})_{[0,\infty)} &= (Ax + Bu)_{[0,\infty)} \end{aligned}$$

Corollary (Consistency projector)

Exist *unique* consistency projector  $\Pi_{(E,A)}$  such that

$$x(0^+) = \Pi_{(E,A)}x^0$$

$\Pi_{(E,A)}$  can easily be calculated via the **Wong sequences** [T. 2009].

# Sufficient conditions for impulse-freeness

## Question

When are **all solutions** of homogenous (swDAE)  $E_\sigma \dot{x} = A_\sigma x$  **impulse free**?

Note: Jumps are OK.

## Lemma (Sufficient conditions)

- ›  $(E_p, A_p)$  all have **index one** (i.e.  $(sE_p - A_p)^{-1}$  is proper)  
 ⇒ (swDAE) impulse free
- › all **consistency spaces** of  $(E_p, A_p)$  **coincide**  
 ⇒ (swDAE) impulse free

# Characterization of impulse-freeness

Theorem (Impulse-freeness, [T. 2009])

The switched DAE  $E_\sigma \dot{x} = A_\sigma x$  is *impulse free*  $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \dots, N\}$$

where  $\Pi_p := \Pi_{(E_p, A_p)}$ ,  $p \in \{1, \dots, N\}$  is the  $p$ -th consistency projector.

## Remark

- › Index-1-case  $\Rightarrow E_q(I - \Pi_q) = 0 \quad \forall q$
- › Consistency spaces equal  $\Rightarrow (I - \Pi_q)\Pi_p = 0 \quad \forall p, q$

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$\mathcal{E}_-$

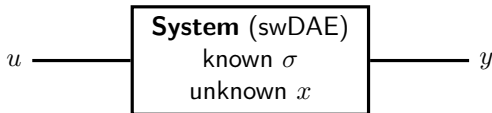
$\mathcal{O}_-$  and  $\mathcal{O}_+$

$\mathcal{O}_+^{\text{imp}}$

## Summary



# Global Observability of Switched DAEs



## Definition (Global observability)

(swDAE) with given  $\sigma$  is **(globally) observable**  $:\Leftrightarrow$

$$\forall \text{ solutions } (u_1, x_1, y_1), (u_2, x_2, y_2) : (u_1, y_1) \equiv (u_2, y_2) \Rightarrow x_1 \equiv x_2$$

## Lemma (0-distinguishability)

(swDAE) is observable if, and only if,  $y \equiv 0$  and  $u \equiv 0 \Rightarrow x \equiv 0$

Hence consider in the following (swDAE) without inputs:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x \\ y &= C_\sigma x \end{aligned}$$

and observability question:

$$y \equiv 0 \stackrel{?}{\Rightarrow} x \equiv 0$$

# Motivating example

System 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$$y = [0 \quad 0 \quad 1] x$$

$$y = x_3, \dot{y} = \dot{x}_3 = 0, x_2 = 0, \dot{x}_1 = 0$$

$$\Rightarrow x_1 \text{ unobservable}$$

$$\sigma(\cdot) : 1 \rightarrow 2$$

Jump in  $x_1$  produces impulse in  $y$   
 $\Rightarrow$  Observability

System 2:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$y = [0 \quad 0 \quad 1] x$$

$$y = x_3 = \dot{x}_1, x_1 = 0, \dot{x}_2 = 0$$

$$\Rightarrow x_2 \text{ unobservable}$$

$$\sigma(\cdot) : 2 \rightarrow 1$$

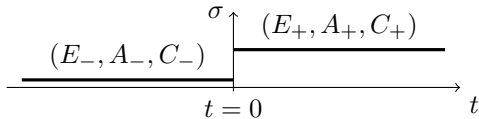
Jump in  $x_2$  no influence in  $y$   
 $\Rightarrow x_2$  remains unobservable

## Question

$$E_p \dot{x} = A_p x + B_p u \quad \text{not observable} \quad \stackrel{?}{\Rightarrow} \quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u \quad \text{observable}$$

$$y = C_p x + D_p u \quad \text{observable} \quad \Rightarrow \quad y = C_\sigma x + D_\sigma u \quad \text{observable}$$

# The single switch result



Theorem (Unobservable subspace, Tanwani & T. 2010)

For (swDAE) with **a single switch** the following equivalence holds

$$y \equiv 0 \Leftrightarrow x(0^-) \in \mathcal{M}$$

where

$$\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}$$

In particular: (swDAE) observable  $\Leftrightarrow \mathcal{M} = \{0\}$ .

## What are these four subspace?

# The four subspaces

Unobservable subspace:  $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}$ , i.e.

$$x(0^-) \in \mathcal{M} \Leftrightarrow y_{(-\infty,0)} \equiv 0 \wedge y[0] = 0 \wedge y_{(0,\infty)} \equiv 0$$

## The four spaces

- › Consistency:  $x(0^-) \in \mathfrak{C}_-$
- › Left unobservability:  $y_{(-\infty,0)} \equiv 0 \Leftrightarrow x(0^-) \in \ker O_-$
- › Right unobservability:  $y_{(0,\infty)} \equiv 0 \Leftrightarrow x(0^-) \in \ker O_+^-$
- › Impulse unobservability:  $y[0] = 0 \Leftrightarrow x(0^-) \in \ker O_+^{\text{imp}}$

## Question

How to calculate these four spaces?

# Wong sequences

## Definition

Let  $E, A \in \mathbb{R}^{m \times n}$ . The corresponding Wong sequences of the pair  $(E, A)$  are:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0, 1, 2, 3, \dots \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{j+1} &:= E^{-1}A(\mathcal{W}_j), & j &= 0, 1, 2, 3, \dots \end{aligned}$$

Note:  $M^{-1}\mathcal{S} := \{x \mid Mx \in \mathcal{S}\}$  and  $M\mathcal{S} := \{Mx \mid x \in \mathcal{S}\}$

Clearly,  $\exists i^*, j^* \in \mathbb{N}$

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \dots$$

$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \dots$$

Wong limits:

$$\mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}$$

$$\mathcal{W}^* = \bigcup_{j \in \mathbb{N}} \mathcal{W}_j = \mathcal{W}_{j^*}$$

# Wong sequences and the QWF

Theorem (QWF [Berger, Ilchmann & T. 2012])

*The following statements are equivalent for square  $E, A \in \mathbb{R}^{n \times n}$ :*

(i)  $(E, A)$  is regular

(ii)  $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$

(iii)  $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

*In particular, with  $\text{im } V = \mathcal{V}^*$ ,  $\text{im } W = \mathcal{W}^*$*

$(E, A)$  regular  $\Rightarrow T := [V, W]$  and  $S := [EV, AW]^{-1}$  invertible

*and  $S, T$  yield **quasi-Weierstrass form (QWF)**:*

$$(SET, SAT) = \left( \begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

# Calculation of Wong sequences

## Remark

Wong sequences can easily be calculated with Matlab even when the matrices still contain symbolic entries (like “R”, “L”, “C”).

```
function V=getPreImage(A,S)
% returns a basis of the preimage of A of the linear space spanned by
% the columns of S, i.e.  $\text{im } V = \{ x \mid Ax \in \text{im } S \}$ 

[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
else
    error('Both matrices must have same number of rows');
end;
```

# Consistency space

$$x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}^-} \Leftrightarrow y \equiv 0$$

## Corollary from QWF

$$\mathfrak{C}_- = \mathcal{V}_-^*$$

where  $\mathcal{V}_-^*$  is the first Wong limit of  $(E_-, A_-)$ .



# The differential projector

For regular  $(E, A)$  let  $(SET, SAT) = \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$ .

Definition (Differential “projector”)

$$\Pi_{(E,A)}^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S \quad \text{and} \quad \boxed{A^{\text{diff}} := \Pi_{(E,A)}^{\text{diff}} A}$$

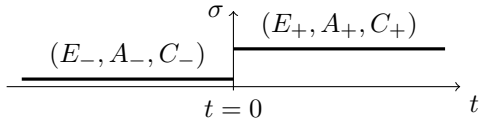
Following Implication holds:

$$x \text{ solves } E\dot{x} = Ax \quad \Rightarrow \quad \dot{x} = A^{\text{diff}}x$$

Hence, with  $y = Cx$ ,

$$y \equiv 0 \quad \Rightarrow \quad x(0) \in \ker[C/CA^{\text{diff}}/C(A^{\text{diff}})^2/\dots/C(A^{\text{diff}})^{n-1}]$$

# The spaces $O_-$ and $O_+$



Hence

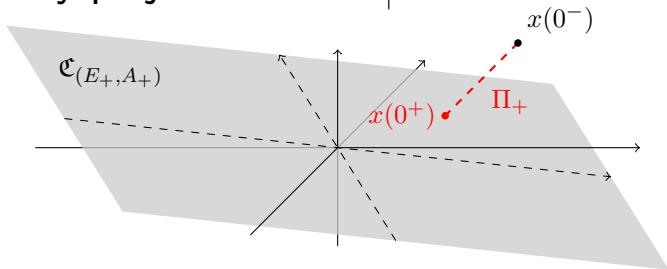
$$y_{(-\infty,0)} \equiv 0 \Rightarrow x(0^-) \in \ker \underbrace{[C_- / C_- A_-^{\text{diff}} / C_- (A_-^{\text{diff}})^2 / \dots / C_- (A_-^{\text{diff}})^{n-1}]}_{:= O_-}$$

and

$$y_{(0,\infty)} \equiv 0 \Rightarrow x(0^+) \in \ker \underbrace{[C_+ / C_+ A_+^{\text{diff}} / C_+ (A_+^{\text{diff}})^2 / \dots / C_+ (A_+^{\text{diff}})^{n-1}]}_{:= O_+}$$

Question:  $x(0^+) \in \ker O_+ \Rightarrow x(0^-) \in ?$

# Consistency projector and $O_+^-$



Assume  $(S_+ E_+ T_+, S_+ A_+ T_+) = \left( \begin{bmatrix} I & 0 \\ 0 & N_+ \end{bmatrix}, \begin{bmatrix} J_+ & 0 \\ 0 & I \end{bmatrix} \right)$ :

Consistency projector

$x(0^+) = \Pi_+ x(0^-)$  where

$$\Pi_+ := T_+ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_+^{-1}$$

$$x(0^+) \in \ker O_+$$

$$\Rightarrow x(0^-) \in \Pi_+^{-1} \ker O_+ = \ker \underbrace{O_+ \Pi_+}_{=: O_+^-}$$

# The impulsive effect

Assume  $(S_+ E_+ T_+, S_+ A_+ T_+) = \left( \begin{bmatrix} I & 0 \\ 0 & N_+ \end{bmatrix}, \begin{bmatrix} J_+ & 0 \\ 0 & I \end{bmatrix} \right)$ :

Definition (Impulse “projector”)

$$\Pi_+^{\text{imp}} := T_+ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S_+ \quad \text{and} \quad \boxed{E_+^{\text{imp}} := \Pi_+^{\text{imp}} E_+}$$

Impulsive part of solution:

$$x[0] = - \sum_{i=0}^{n-1} (E_+^{\text{imp}})^{i+1} x(0^-) \delta_0^{(i)}$$

Dirac impulses

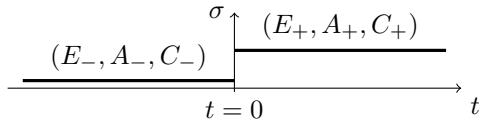
**Conclusion:**

$$y[0] = 0 \quad \Rightarrow \quad C_+ x[0] = 0 \quad \Rightarrow \quad \boxed{x(0^-) \in \ker O_+^{\text{imp}}}$$

where

$$O_+^{\text{imp}} := [C_+ E_+^{\text{imp}} / C_+ (E_+^{\text{imp}})^2 / \dots / C_+ (E_+^{\text{imp}})^{n-1}]$$

# Observability summary



$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$$

with

- ›  $\mathfrak{C}_- = \mathcal{V}_-^*$  (first Wong limit)
- ›  $O_- = [C_- / C_- A_-^{\text{diff}} / C_- (A_-^{\text{diff}})^2 / \dots / C_- (A_-^{\text{diff}})^{n-1}]$
- ›  $O_+^- = [C_+ / C_+ A_+^{\text{diff}} / C_+ (A_+^{\text{diff}})^2 / \dots / C_+ (A_+^{\text{diff}})^{n-1}] \Pi_+$
- ›  $O_+^{\text{imp}} = [C_+ E_+^{\text{imp}} / C_+ (E_+^{\text{imp}})^2 / \dots / C_+ (E_+^{\text{imp}})^{n-1}]$

# Example revisited

System 1:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$$y = [0 \ 0 \ 1] x$$

$\sigma(\cdot) : 1 \rightarrow 2$  gives

$$\mathfrak{C}_- = \text{span}\{e_1, e_3\},$$

$$\ker O_- = \text{span}\{e_1, e_2\}$$

$$\ker O_+^- = \text{span}\{e_1, e_2, e_3\},$$

$$\ker O_+^{\text{imp}} = \text{span}\{e_2, e_3\}$$

$$\Rightarrow \mathcal{M} = \{0\}$$

System 2:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$y = [0 \ 0 \ 1] x$$

$\sigma(\cdot) : 2 \rightarrow 1$  gives

$$\mathfrak{C}_- = \text{span}\{e_2\},$$

$$\ker O_- = \text{span}\{e_1, e_2\}$$

$$\ker O_+^- = \text{span}\{e_1, e_2\},$$

$$\ker O_+^{\text{imp}} = \text{span}\{e_1, e_2, e_3\}$$

$$\Rightarrow \mathcal{M} = \text{span}\{e_2\}$$

# Overall summary

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad (\text{swDAE})$$

## Piecewise-smooth distributional solution framework

$$x \in \mathbb{D}_{\text{pw}C^\infty}^n, \quad u \in \mathbb{D}_{\text{pw}C^\infty}^m, \quad y \in \mathbb{D}_{\text{pw}C^\infty}^p$$

- › Existence and uniqueness of solutions? ✓
- › Jumps and impulses in solutions? ✓
- › Conditions for impulse free solutions? ✓
- › Control theoretical questions
  - Stability ✓ and stabilization ?
  - Observability ✓ and observer design ✓
  - Controllability ✓ and controller design ?
  - Extension to nonlinear case ?