

# Delay regularity of differential-algebraic equations

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**Abstract**—We study linear time-invariant delay differential-algebraic equations (DDAEs). Such equations can arise if a feedback controller is applied to a descriptor system and the controller requires some time to measure the state and to compute the feedback resulting in the time-delay. We present an existence and uniqueness result for DDAEs within the space of piecewise-smooth distributions and an algorithm to determine whether a DDAE is delay-regular.

## I. INTRODUCTION

We consider differential-algebraic equations (DAEs), also known as descriptor systems, of the form

$$E\dot{x} = Ax + Bu + f \quad (1)$$

where  $E, A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $x$  is the  $n$ -dimensional state variable,  $u$  is the  $m$ -dimensional input and  $f$  is an  $n$ -dimensional external signal (e.g. a disturbance). It is well known that regularity of the matrix pair  $(E, A)$ , i.e.  $\det(sE - A) \neq 0$ , is necessary and sufficient for existence and uniqueness of solutions (for given sufficiently smooth  $u$  and  $f$  and consistent initial value, e.g. [1]). By considering solutions within a distributional framework (see the forthcoming Section II for details) it even holds that for any initial condition  $x(0^-) = x_0 \in \mathbb{R}^n$ , any locally integrable input and any locally integrable disturbance a unique (distributional) solution exists.

If the matrix pair  $(E, A)$  is not regular, it may be possible to *regularize* the pair by a feedback law

$$u = Fx,$$

with  $F \in \mathbb{R}^{m \times n}$ . In fact the following result holds:

*Lemma 1* ([2], [3]): Consider the DAE (1). Then there exists  $F \in \mathbb{R}^{m \times n}$  such that  $(E, A + BF)$  is regular if, and only if  $\text{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$  for some  $\lambda \in \mathbb{C}$ .

Although instantaneous (state) feedback is a convenient theoretical approach, it often suffers from the fact that signals have to be measured first, and some calculations have to be carried out, thus resulting in an intrinsically necessary *time delay*. Hence, for some time delay  $\tau > 0$ , the feedback takes the form

$$u(t) = Fx(t - \tau),$$

i.e. the closed loop system is then a *delay DAE* (DDAE) of the form

$$E\dot{x} = Ax + D\sigma_\tau x + f \quad (2)$$

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with  $D = BF$  and  $(\sigma_\tau x)(t) := x(t - \tau)$  denotes the shift operator. We arrive at the following question:

*When is the matrix triplet  $(E, A, D)$  regular, in the sense that the DDAE (2) has for all  $f$  a solution uniquely determined by the initial trajectory (at least in an appropriate distributional sense)?*

After recalling some preliminaries on the distributional solution theory for DAEs, we can formulate our first main result (Theorem 4) in Section III about the existence and uniqueness of solutions within the space of piecewise-smooth distributions when the matrix pair  $(E, A)$  is regular. In the situation when  $(E, A)$  is not regular, we first define the novel notion of delay-regularity (Definition 6). Afterwards we present in Section IV Algorithm 1, which can determine whether a given DDAE is delay-regular or not. Unfortunately, in some exceptional cases, this algorithm will not terminate, and we will discuss these issues at the end of this note.

## II. PRELIMINARIES ON DISTRIBUTIONAL SOLUTION THEORY FOR DAEs

In order to treat insufficiently smooth disturbances and/or inputs as well as inconsistent initial values for the DAE (1) it is common to interpret (1) within the space of *distributions*.

Following [4], the space of distributions  $\mathbb{D}$  consists of all linear and continuous maps with values in  $\mathbb{R}$  (functionals) on the space of test functions  $\mathcal{C}_0^\infty(\mathbb{R} \rightarrow \mathbb{R})$ , where the latter is the space of all smooth functions with bounded support equipped with a suitable locally convex topology. The space of locally integrable functions  $\mathcal{L}_{1,\text{loc}}$  can be embedded into  $\mathbb{D}$  via the following injective homomorphism:

$$\mathcal{L}_{1,\text{loc}} \ni f \mapsto f_{\mathbb{D}} := \left( \varphi \mapsto \int_{\mathbb{R}} f\varphi \right).$$

For all distributions it is possible to define a derivative via  $D'(\varphi) = -D(\varphi')$  which is consistent with the usual derivative, i.e.  $(f_{\mathbb{D}})' = (f')_{\mathbb{D}}$  if  $f$  is a differentiable (hence locally integrable) function. For notational convenience we write  $\dot{f} := \frac{d}{dt}f := f'$  and define recursively  $(\frac{d}{dt})^0 f := f$  and  $(\frac{d}{dt})^k f := \frac{d}{dt} \left( (\frac{d}{dt})^{k-1} f \right)$  for  $k \in \mathbb{N}$  and  $f \in \mathbb{D}$ . The *Dirac impulse*  $\delta_t$  at  $t \in \mathbb{R}$  is defined as the functional  $\varphi \mapsto \varphi(t)$  and it is easily seen that  $\delta_t$  is the (distributional) derivative of the Heaviside step function  $\mathbf{1}_{[t,\infty)}$ .

Let

$$\mathbb{D}^k := \left\{ \mathcal{D} : \mathcal{C}_0^\infty \rightarrow \mathbb{R}^k \mid \begin{array}{l} \mathcal{D}_i := \varphi \mapsto \mathcal{D}(\varphi)_i \in \mathbb{D}, \\ i = 1, \dots, k \end{array} \right\}$$

for any  $k \in \mathbb{N}$ , and for  $M \in \mathbb{R}^{p \times k}$  and  $\mathcal{D} \in \mathbb{D}^k$  let

$$M\mathcal{D} := (\varphi \mapsto M\mathcal{D}(\varphi)),$$

then it is easily seen, that (1) can be interpreted as an equation in  $\mathbb{D}^n$  with  $x \in \mathbb{D}^n$ ,  $u \in \mathbb{D}^m$  and  $f \in \mathbb{D}^n$ . However, embedding (1) into a distributional solution framework does *not* allow for the consideration of inconsistent initial values; the only solution of the trivial DAE  $0 = x$  is  $x = 0$  also in the distributional sense.

Considering inconsistent initial values only makes sense, when one requires that (1) should only hold on  $[0, \infty)$  (instead of the real axis); but this requires us to define a distributional restriction to the interval  $[0, \infty)$  and this is *not possible* for general distributions [5]. This problem can be resolved by considering the space of impulsive-smooth distributions [6] or by the slightly bigger space of piecewise-smooth distributions [7]. The latter is also suitable for studying the DDAE (2), therefore we will use this space in the following as the underlying solution space for (1) and (2).

*Definition 2 (Piecewise-smooth distributions):* Let  $\mathcal{C}_{\text{pw}}^\infty$  be the space of piecewise-smooth functions, where  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is called piecewise-smooth if, and only if, there exists a family of real numbers  $\{t_i \in \mathbb{R} \mid i \in \mathbb{Z}\}$  with  $t_i < t_{i+1}$  for all  $i \in \mathbb{Z}$  and  $t_{\pm k} \rightarrow \pm\infty$  as  $k \rightarrow \infty$  and smooth functions  $\alpha_i \in \mathcal{C}^\infty(\mathbb{R} \rightarrow \mathbb{R})$  such that

$$\alpha = \sum_{i \in \mathbb{Z}} \mathbf{1}_{[t_i, t_{i+1})} \alpha_i.$$

The space of *piecewise-smooth distributions* is

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ \alpha_{\mathbb{D}} + \sum_{t \in T} \mathcal{D}_t \left| \begin{array}{l} \alpha \in \mathcal{C}_{\text{pw}}^\infty, T \text{ is discrete,} \\ \mathcal{D}_t \in \text{span}\{\delta_t, \delta'_t, \delta''_t, \dots\} \end{array} \right. \right\},$$

i.e. a piecewise-smooth distribution is the sum of a piecewise-smooth function and linear combinations of Dirac impulses and their derivatives at finitely many time instants in each compact interval.

For piecewise-smooth distributions a restriction to intervals is now well defined:

$$F_{\mathcal{I}} := (\alpha_{\mathcal{I}})_{\mathbb{D}} + \sum_{t \in T \cap \mathcal{I}} \mathcal{D}_t,$$

where  $F = \alpha_{\mathbb{D}} + \sum_{t \in T} \mathcal{D}_t \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$  and  $\mathcal{I} \subseteq \mathbb{R}$  is an interval.

Within this piecewise-smooth distributional solution framework the following existence and uniqueness result for regular DAEs (1) with possible inconsistent initial values holds.

*Lemma 3 ([7]):* Consider the DAE (1) with regular matrix pair  $(E, A)$ . Then for any initial trajectory  $x^0 \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$ , any arbitrary input  $u \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^m$  and any arbitrary disturbance  $f \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$ , there exists a unique solution  $x \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$  of the *initial trajectory problem (ITP)*:

$$\begin{aligned} x_{(-\infty, 0)} &= x_{(-\infty, 0)}^0, \\ (E\dot{x})_{[0, \infty)} &= (Ax + Bu + f)_{[0, \infty)}. \end{aligned} \quad (3)$$

### III. DISTRIBUTIONAL SOLUTION THEORY FOR DDAES WITH REGULAR MATRIX PAIR $(E, A)$

To recast the DDAE (2) in the distributional framework we define for the delay time  $\tau > 0$  the distributional shift operator  $\sigma_\tau : \mathbb{D} \rightarrow \mathbb{D}$  via

$$\sigma_\tau\{f\} := (\varphi \mapsto f(\varphi(\cdot + \tau))).$$

Note that for any locally integrable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and any  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R} \rightarrow \mathbb{R})$ , we have

$$\begin{aligned} (\sigma_\tau f)_{\mathbb{D}}(\varphi) &= \int_{-\infty}^{\infty} f(t - \tau)\varphi(t)dt \\ &= \int_{-\infty}^{\infty} f(t)\varphi(t + \tau)dt = \sigma_\tau\{f_{\mathbb{D}}\}(\varphi), \end{aligned}$$

and thus  $(\sigma_\tau f)_{\mathbb{D}} = \sigma_\tau\{f_{\mathbb{D}}\}$ . As in Lemma 3, we can now consider the DDAE (2) as the ITP

$$\begin{aligned} x_{(-\infty, 0)} &= x_{(-\infty, 0)}^0, \\ (E\dot{x})_{[0, \infty)} &= (Ax + D\sigma_\tau\{x\} + f)_{[0, \infty)}, \end{aligned} \quad (4)$$

with initial trajectory  $x^0 \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$  and forcing  $f \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$ .

If the matrix pair  $(E, A)$  is regular, Lemma 3 allows us to immediately conclude existence and uniqueness of solutions of the ITP (4) for any given past trajectory and any disturbance  $f$  via integration on successive time intervals  $[i\tau, (i+1)\tau)$ , which is also known as the *method of steps* [8]. In other words we have arrived at our first main result about the solution theory of DDAEs:

*Theorem 4:* Consider the DDAE-ITP (4) with regular matrix pair  $(E, A)$ . Then for any past trajectory  $x^0 \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$  and any disturbance  $f \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$  the delay ITP (4) has a unique solution.

*Proof:* Let  $t_i := i\tau$ , for  $i \in \mathbb{N}$ . We will show that  $x = x_{(-\infty, 0)}^0 + \sum_{i=1}^{\infty} x_{[t_{i-1}, t_i)}^i$  is the solution of (4) where  $x^i \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^n$ ,  $i \in \mathbb{N}$ , is recursively defined as the unique solution of the (non-delay) ITP

$$\begin{aligned} x_{(-\infty, t_{i-1})}^i &= x_{(-\infty, t_{i-1})}^{i-1} \\ (E\dot{x}^i)_{[t_{i-1}, \infty)} &= (Ax^i + \tilde{f}^i)_{[t_{i-1}, \infty)} \end{aligned}$$

where  $\tilde{f} := D\sigma_\tau\{x^{i-1}\} + f$ .

First note that  $x$  is a well defined distribution because  $x(\varphi)$  reduces to a finite sum for each test function  $\varphi \in \mathcal{C}_0^\infty$  and it is also easily seen that it is again a piecewise-smooth distribution. Furthermore, by construction  $x_{(-\infty, 0)} = x_{(-\infty, 0)}^0$ . The claim now follows from (for any  $i \geq 1$ )

$$\begin{aligned} &(E\dot{x})_{[t_{i-1}, t_i)} \\ &= E \left( (x_{(-\infty, 0)}^0)' + \sum_{k=1}^{\infty} (x_{[t_{k-1}, t_k)}^k)' \right)_{[t_{i-1}, t_i)} \\ &\stackrel{*}{=} E(-x^{i-1}(t_{i-1}^-)\delta_{t_{i-1}} + (\dot{x}^i)_{[t_{i-1}, t_i)} + x^i(t_{i-1}^-)\delta_{t_{i-1}}) \\ &\stackrel{**}{=} (E\dot{x}^i)_{[t_{i-1}, t_i)} = (Ax^i + \tilde{f}^i)_{[t_{i-1}, t_i)} \\ &= Ax_{[t_{i-1}, t_i)} + D(\sigma_\tau\{x^{i-1}\})_{[t_{i-1}, t_i)} + f_{[t_{i-1}, t_i)} \\ &\stackrel{**}{=} (Ax + D\sigma_\tau\{x\} + f)_{[t_{i-1}, t_i)}, \end{aligned}$$

where  $\star$  follows from  $(\mathcal{F}_{[s,t]})' = (\mathcal{F}')_{[s,t]} + \mathcal{F}(s^-)\delta_s - \mathcal{F}(t^-)\delta_t$  (see [9, Prop. 13]) and  $\star\star$  follows from  $x_{(-\infty, t_i)}^i = x_{(-\infty, t_i)}^{i-1}$ . ■

Note that in contrast to [10], there is no constraint on the initial trajectory because we do not require the existence of classical (in particular, continuous) solutions. However, inconsistencies may amplify in the sense that an initial inconsistency at  $t = 0$  may lead to a Dirac impulse or even derivatives of a Dirac impulse at  $t = \tau$  which in turn results in further derivatives of Dirac impulses at integer multiples of  $\tau$ . However, this amplifying behavior can only occur when the DDAE is of *de-smoothing* type in the sense of [10].

*Remark 5:* The existence and uniqueness of distributional solutions for DDAEs was already hinted in [11] and [12]. To the best of our knowledge, we are the first to present such a result. Results for stronger solution concepts are presented for instance in [13], [14], [15], although under much stronger assumptions on the history function and additional properties of the matrix pair  $(E, A)$ .

#### IV. REGULARIZATION BY DELAY

We are now coming back to the question posed in the introduction, whether it is possible to regularize a DAE by introducing delays. As a motivational example consider the DDAE (2) with  $(E, A, D) = (0, 0, 1)$ , i.e.

$$0 = \sigma_\tau\{x\} + f. \quad (5)$$

Clearly, the matrix pair  $(E, A) = (0, 0)$  is not regular, so Theorem 4 cannot be used to say something about existence and uniqueness of solutions. However, (5) obviously has the unique (acausal) solution  $x = \sigma_{-\tau}\{f\}$  for any inhomogeneity  $f$ . Note however, that it is not possible to freely prescribe the initial trajectory for  $x$  on  $[-\tau, 0)$  because this is already fully specified by  $f$  given on  $[0, \tau)$ .

The example showed that by introducing a time-delay term to a nonregular DAE, we arrive at a DDAE which is “regular” in a certain sense. We formalize the notion of “regularity” to DDAEs as follows:

*Definition 6:* The DDAE (2) is called *delay-regular* if, and only if, for all  $f \in \mathbb{D}_{\text{pw}C^\infty}^n$  there exists  $x \in \mathbb{D}_{\text{pw}C^\infty}^n$  such that (2) holds within the space of piecewise-smooth distributions and for any two solutions  $x, \tilde{x}$  for the same  $f$  it holds that

$$x_{[-\infty, 0)} = \tilde{x}_{[-\infty, 0)} \implies x = \tilde{x}.$$

The matrix triple  $(E, A, D)$  is called *delay-regular* if, and only if, the corresponding DDAE is delay-regular.

It is important to note that for delay-regularity we do *not* require

- 1) causality of the solutions with respect to the inhomogeneity  $f$  and
- 2) the existence of a solution for all initial trajectories.

In fact, the second point is a consequence of the first point because of the possible acausality the current inhomogeneity may determine the past (initial) state.

Of course, in reality, a dependence on the future is not possible, and therefore, one may question the utility of the notion of delay-regularity. However, besides its mathematical relevance, this notion may also be useful in practice if the future value of the inhomogeneity can be interpreted as a *prediction* of that future value.

We would like to discuss the above toy example (5) again and define a notion of *delay-equivalence*: We started with a DDAE given by  $(E, A, D) = (0, 0, 1)$  where the matrix pair  $(E, A)$  is not regular. By inspection we saw that the solutions of the DDAE are equivalently obtained by the DDAE  $(\hat{E}, \hat{A}, \hat{D}) = (0, 1, 0)$  with a shifted inhomogeneity  $\hat{f} := f(\cdot + \tau)$ , where now  $(\hat{E}, \hat{A})$  is regular. This trick of shifting the time-delay from the state to the inhomogeneity is the key idea of our forthcoming delay-regularization algorithm. Unfortunately, it is however not sufficient to consider matrix triplets only, as the following example suggest (compare also [11] and [16]).

*Example 7:* Consider the DDAE (2) with

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Clearly,  $(E, A)$  is not regular, however differentiating the last equations twice and plugging in the first two equations yields

$$0 = \sigma_\tau\{\dot{x}_1\} + f_1 + \dot{f}_2 + \ddot{f}_3.$$

The same trick as in the toy-example (5) can now be used to shift the time-delay into the inhomogeneity, but the resulting equation cannot be written as a DDAE again (due to the presence of a second derivative).

Example 7 motivates to study the more general problem

$$\left( \sum_{j=0}^p P_j \left( \frac{d}{dt} \right)^j \right) x = \left( \sum_{j=0}^q Q_j \left( \frac{d}{dt} \right)^j \right) \sigma_\tau\{x\} + f \quad (6)$$

with  $P_j, Q_j \in \mathbb{R}^{n \times n}$  and the following definition:

*Definition 8:* The two pairs of matrix polynomials  $(\mathcal{P}(s), \mathcal{Q}(s)), (\hat{\mathcal{P}}(s), \hat{\mathcal{Q}}(s)) \in (\mathbb{R}[s]^{n \times n})^2$  are called *delay-equivalent* if, and only if there exists a bijective map  $\mathcal{T} : \mathbb{D}_{\text{pw}C^\infty}^n \rightarrow \mathbb{D}_{\text{pw}C^\infty}^n$  such that for all  $(x, f) \in \mathbb{D}_{\text{pw}C^\infty}^n \times \mathbb{D}_{\text{pw}C^\infty}^n$  and  $\hat{f} := \mathcal{T}f$  the following equivalence holds

$$\mathcal{P} \left( \frac{d}{dt} \right) x = \mathcal{Q} \left( \frac{d}{dt} \right) \sigma_\tau\{x\} + f$$

$$\iff \hat{\mathcal{P}} \left( \frac{d}{dt} \right) x = \hat{\mathcal{Q}} \left( \frac{d}{dt} \right) \sigma_\tau\{x\} + \hat{f}.$$

In this case we write  $(\mathcal{P}(s), \mathcal{Q}(s)) \stackrel{d}{\sim} (\hat{\mathcal{P}}(s), \hat{\mathcal{Q}}(s))$ .

Defining delay-regularity for (6) analogously as in Definition 6, we immediately see<sup>1</sup> that for  $(\mathcal{P}(s), \mathcal{Q}(s)) \stackrel{d}{\sim} (\widehat{\mathcal{P}}(s), \widehat{\mathcal{Q}}(s))$  delay-regularity of  $(\mathcal{P}(s), \mathcal{Q}(s))$  is equivalent to delay-regularity of  $(\widehat{\mathcal{P}}(s), \widehat{\mathcal{Q}}(s))$ . Furthermore, as a consequence of Theorem 4, we have the following sufficient condition for delay-regularity of the general DDAE (6).

*Corollary 9:* Consider the DDAE (6) with  $\det(\mathcal{P}(s)) \neq 0$  and corresponding DDAE-ITP

$$x_{(-\infty, 0)} = x_{(-\infty, 0)}^0, \quad (7a)$$

$$\left(\mathcal{P}\left(\frac{d}{dt}\right)x\right)_{[0, \infty)} = \left(\mathcal{Q}\left(\frac{d}{dt}\right)\sigma_\tau\{x\} + f\right)_{[0, \infty)}, \quad (7b)$$

where  $x^0 \in \mathbb{D}^n$ ,  $f \in \mathbb{D}^n$  and distributional shift operator  $\sigma_\tau : \mathbb{D} \rightarrow \mathbb{D}$  (cf. Theorem 4). Then for any past trajectory  $x^0 \in \mathbb{D}^n$  and any disturbance  $f \in \mathbb{D}^n$  the delay ITP (7) has a unique solution, in particular, (6) is delay-regular.

*Proof:* A standard companion form linearization of (6) yields the DDAE

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{D}\sigma_\tau\{z\} + \mathcal{F} \quad (8)$$

with  $\mathcal{E}, \mathcal{A}, \mathcal{D} \in \mathbb{R}^{pn \times pn}$ , given by

$$\mathcal{E} = \begin{bmatrix} P_p & 0 & \cdots & 0 \\ 0 & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_n \end{bmatrix}, \quad z = \begin{bmatrix} \left(\frac{d}{dt}\right)^{p-1}x \\ \vdots \\ \frac{d}{dt}x \\ x \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} -P_{p-1} & -P_{p-2} & \cdots & -P_0 \\ I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} Q_{p-1} & \cdots & Q_0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The proof now follows by the observation that there exists a constant  $c \neq 0$  (cf. [18]) with

$$0 \neq \det(\mathcal{P}(s)) = c \det(s\mathcal{E} - \mathcal{A})$$

and application of Theorem 4 to (8). ■

*Remark 10:* The initial trajectory  $x_{(-\infty, 0)}^0$  in (7a) not only specifies the state  $x_{(-\infty, 0)}$  but also its (distributional) derivatives and thus providing the initial trajectories for the higher-order differential operator  $\mathcal{P}(d/dt)$  in (7b).

We will now present a *time-delay regularization algorithm* which allows us to decide whether a DDAE is delay-regular or not. It is outlined in Algorithm 1.

Since Algorithm 1 is defined recursively, we first need to establish that the update in Line 8 does not affect delay-regularity.

<sup>1</sup>In fact, any solution of the DDAE-ITP (7) is also a solution of the DDAE  $P\left(\frac{d}{dt}\right)\dot{x} = Q\left(\frac{d}{dt}\right)\sigma_\tau\{x\} + f_{\text{ITP}}$ , where  $f_{\text{ITP}} := f_{[0, \infty)} + \left(P\left(\frac{d}{dt}\right)x^0 - Q\left(\frac{d}{dt}\right)\sigma_\tau\{x^0\}\right)_{(-\infty, 0)}$ , cf. [17] and the corresponding discussion in [5, pp. 167–168].

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### Algorithm 1 Time-delay regularization

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**Input:**  $\mathcal{P}(s), \mathcal{Q}(s) \in \mathbb{R}[s]^{n \times n}$

**Output:** yes/no

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1: if  $\text{rank}_{\mathbb{R}[s]} \mathcal{P}(s) = n$  then
2:   return yes
3: else
4:   Choose unimodular  $U(s) \in \mathbb{R}[s]^{n \times n}$  such that
      $U(s)\mathcal{P}(s) = \begin{bmatrix} \mathcal{P}_1(s) \\ 0 \end{bmatrix}$  and  $U(s)\mathcal{Q}(s) = \begin{bmatrix} \mathcal{Q}_1(s) \\ \mathcal{Q}_2(s) \end{bmatrix}$ ,
     where  $\mathcal{P}_1(s) \in \mathbb{R}[s]^{n_1 \times n}$  has full row rank
5:   if  $\text{rank}_{\mathbb{R}[s]} \mathcal{Q}_2(s) < n - n_1$  then
6:     return no
7:   else
8:     Set  $\mathcal{P}(s) \leftarrow \begin{bmatrix} \mathcal{P}_1(s) \\ -\mathcal{Q}_2(s) \end{bmatrix}$  and  $\mathcal{Q}(s) \leftarrow \begin{bmatrix} \mathcal{Q}_1(s) \\ 0 \end{bmatrix}$ .
9:     Go to line 1
10:  end if
11: end if

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*Proposition 11:* Consider the notation as in Algorithm 1. Then the two pairs of matrix polynomials  $(\mathcal{P}(s), \mathcal{Q}(s))$  and  $\left(\begin{bmatrix} \mathcal{P}_1(s) \\ -\mathcal{Q}_2(s) \end{bmatrix}, \begin{bmatrix} \mathcal{Q}_1(s) \\ 0 \end{bmatrix}\right)$  are delay-equivalent.

*Proof:* Let us first note that shifting and differentiation commutes in  $\mathbb{D}_{\text{pwc}^\infty}$ , i.e., for any  $g \in \mathbb{D}_{\text{pwc}^\infty}^n$  we have  $U\left(\frac{d}{dt}\right)\sigma_\tau\{g\} = \sigma_\tau\{U\left(\frac{d}{dt}\right)g\}$ . This allows us to define the bijection

$$\mathcal{T} : \mathbb{D}_{\text{pwc}^\infty}^n \rightarrow \mathbb{D}_{\text{pwc}^\infty}^n, \quad f \mapsto \begin{bmatrix} U_1\left(\frac{d}{dt}\right)f \\ U_2\left(\frac{d}{dt}\right)\sigma_{-\tau}\{f\} \end{bmatrix},$$

where  $U(s) = \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$ . For  $f \in \mathbb{D}_{\text{pwc}^\infty}^n$  let  $x \in \mathbb{D}_{\text{pwc}^\infty}^n$  satisfy

$$\mathcal{P}\left(\frac{d}{dt}\right)x = \mathcal{Q}\left(\frac{d}{dt}\right)\sigma_\tau\{x\} + f.$$

Since  $U(s)$  from Algorithm 1 is unimodular this is true if and only if  $x$  satisfies

$$\begin{bmatrix} \mathcal{P}_1\left(\frac{d}{dt}\right) \\ 0 \end{bmatrix} x = \begin{bmatrix} \mathcal{Q}_1\left(\frac{d}{dt}\right) \\ \mathcal{Q}_2\left(\frac{d}{dt}\right) \end{bmatrix} \sigma_\tau\{x\} + \begin{bmatrix} U_1\left(\frac{d}{dt}\right)f \\ U_2\left(\frac{d}{dt}\right)f \end{bmatrix}. \quad (9)$$

The lower part of this equation is equivalent to the time-shifted equation

$$0 = \sigma_{-\tau} \left\{ \mathcal{Q}_2\left(\frac{d}{dt}\right)\sigma_\tau\{x\} + U_2\left(\frac{d}{dt}\right)f \right\}.$$

Since shifting and differentiation commutes, this is equivalent to

$$\begin{bmatrix} \mathcal{P}_1\left(\frac{d}{dt}\right) \\ -\mathcal{Q}_2\left(\frac{d}{dt}\right) \end{bmatrix} x = \begin{bmatrix} \mathcal{Q}_1\left(\frac{d}{dt}\right) \\ 0 \end{bmatrix} \sigma_\tau\{x\} + \mathcal{T}f,$$

which completes the result. ■

*Theorem 12:* Assume that Algorithm 1 terminates after a finite number of iterations for the polynomial matrix pair  $(\mathcal{P}(s), \mathcal{Q}(s)) \in \mathbb{R}[s]^{n \times n} \times \mathbb{R}[s]^{n \times n}$ . Then  $(\mathcal{P}(s), \mathcal{Q}(s))$  is delay-regular if and only if Algorithm 1 results in **yes**.

*Proof:* Corollary 9 together with Proposition 11 shows that if Algorithm 1 terminates in line 2 with the

output **yes** the original DDAE (6) was delay-regular. On the other hand, if Algorithm 1 terminates with output **no** then there exists a unimodular matrix  $\mathcal{U}_2(s)$  such that  $\mathcal{U}_2(s)\mathcal{Q}_2(s) = \begin{bmatrix} \tilde{\mathcal{Q}}_2(s) \\ 0 \end{bmatrix}$ . Hence the original DDAE (6) is equivalent to the DDAE

$$\begin{bmatrix} \mathcal{P}_1(\frac{d}{dt}) \\ 0 \\ 0 \end{bmatrix} x = \begin{bmatrix} \mathcal{Q}_1(\frac{d}{dt}) \\ \tilde{\mathcal{Q}}_2(\frac{d}{dt}) \\ 0 \end{bmatrix} x + \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{pmatrix},$$

which clearly is not solvable when  $\hat{f}_3 \neq 0$ . Hence, in view of Proposition 11 the original DDAE was not delay-regular. This completes the proof. ■

*Remark 13:* In the toy example (5), the non-regularity of the matrix pair  $(E, A)$  results in a restriction of the set of initial trajectories that lead to a solution. Note that this restriction is a result of the shifting step in line 8 in Algorithm 1, cf. [11].

Algorithm 1 terminates without a recursive call for both the toy example given by (5) and the slightly more complicated Example 7 and before we discuss the problem of non-termination we would like to consider first some examples, where the algorithm needs to run several rounds until it terminates.

*Example 14:* Consider the matrix polynomials

$$\mathcal{P}(s) = \begin{bmatrix} s^2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}(s) = \begin{bmatrix} 0 & s-1 \\ s & 0 \end{bmatrix}.$$

Applying Algorithm 1 to  $(\mathcal{P}(s), \mathcal{Q}(s))$  yields  $\mathcal{U}(s) = I$  with  $\mathcal{Q}_2(s) = [s \ 0]$  in line 4. A recursive call of Algorithm 1 yields  $\text{rank}_{\mathbb{R}[s]} \begin{bmatrix} s^2 & 0 \\ s & 0 \end{bmatrix} = 1$  and thus we need to perform another row compression of (the updated)  $\mathcal{P}(s)$  and obtain the unimodular matrix polynomial  $\mathcal{U}(s) = \begin{bmatrix} 0 & 1 \\ 1 & -s \end{bmatrix}$  with  $\mathcal{U}(s) \begin{bmatrix} s^2 & 0 \\ s & 0 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$  and  $\mathcal{U}(s) \begin{bmatrix} 0 & s-1 \\ s & 0 \end{bmatrix} = \begin{bmatrix} 0 & s-1 \\ 0 & s-1 \end{bmatrix}$ . A second recursive call terminates the algorithm with **yes**, since

$$\text{rank}_{\mathbb{R}[s]} \begin{bmatrix} s & 0 \\ 0 & s-1 \end{bmatrix} = 2,$$

i.e., Theorem 12 ensures that the polynomial matrix pair  $(\mathcal{P}(s), \mathcal{Q}(s))$  is delay-regular.

*Example 15:* Applying Algorithm 1 to the matrices

$$\mathcal{P}(s) = \begin{bmatrix} s & 1 & 0 \\ s & 0 & s^2 \\ s^2 & s & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}(s) = \begin{bmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ -s & 0 & 0 \end{bmatrix}$$

yields the unimodular matrix polynomial

$$\mathcal{U}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s & 0 & 1 \end{bmatrix}.$$

The update in line 8 yields

$$\mathcal{P}(s) = \begin{bmatrix} s & 1 & 0 \\ s & 0 & s^2 \\ s & 0 & s^2 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}(s) = \begin{bmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

such that the algorithm terminates after another iteration with output **no**, because with the unimodular

transformation  $\mathcal{U}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$  we produce a common zero row, i.e., the pair  $(\mathcal{P}(s), \mathcal{Q}(s))$  is not delay-regular.

## V. (NON-)TERMINATION OF ALGORITHM 1

Regarding the (non-)termination of Algorithm 1 we observe that

$$\text{rank}_{\mathbb{R}[s]} \left( \begin{bmatrix} \mathcal{P}_1(s) \\ -\mathcal{Q}_2(s) \end{bmatrix} \right) \geq \text{rank}_{\mathbb{R}[s]}(\mathcal{P}(s)), \quad (10a)$$

$$\text{rank}_{\mathbb{R}[s]} \left( \begin{bmatrix} \mathcal{Q}_1(s) \\ 0 \end{bmatrix} \right) \leq \text{rank}_{\mathbb{R}[s]}(\mathcal{Q}(s)). \quad (10b)$$

If in each iteration of Algorithm 1 one of these inequalities is strict, then Algorithm 1 terminates after a finite number of iterations. Or equivalently, Algorithm 1, does not terminate if, and only if, after finitely many iterations the ranks in (10) remain constant in all further iterations of the algorithm. The following example shows that this indeed can happen.

*Example 16:* Consider the input data

$$\mathcal{P}(s) = \begin{bmatrix} s & 1 \\ s^2 & s \end{bmatrix}, \quad \mathcal{Q}(s) = \begin{bmatrix} s & 1 \\ 0 & 0 \end{bmatrix},$$

for Algorithm 1. Then  $\mathcal{U}(s) = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$  leads to the desired structure with  $\mathcal{Q}_2(s) = [-s^2, -s]$ . Hence the algorithm is called recursively with the same pair of matrix polynomials.

From this example one may conjecture that once the ranks in (10) remain constant, they also will remain constant in future iterations so that at least the algorithm can be terminated with the output **unknown**. Unfortunately, this is not true as the following example shows:

*Example 17:* Applying Algorithm 1 to the matrices

$$\mathcal{P}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

yields in the first iteration  $\mathcal{U}(s) = I$  and the updated matrices

$$\mathcal{P}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that neither of the rank inequalities in (10) is strict. However, choosing

$$\mathcal{U}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

yields in the second iteration the updated matrix polynomials

$$\mathcal{P}(s) = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{Q}(s) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we conclude that  $(\mathcal{P}(s), \mathcal{Q}(s))$  is delay-regular.

Note that in Example 17 we actually had the following situation after line 4 in Algorithm 1:

$$\text{rank} \begin{bmatrix} \mathcal{Q}_1(s) \\ \mathcal{Q}_2(s) \end{bmatrix} = \text{rank } \mathcal{Q}_1(s),$$

but there exist unimodular matrices  $\mathcal{M}_{11}(s)$  and  $\mathcal{M}_{22}(s)$ , and a polynomial matrix  $\mathcal{M}_{12}(s)$  of appropriate sizes such that

$$\begin{bmatrix} \widetilde{\mathcal{Q}}_1(s) \\ \widetilde{\mathcal{Q}}_2(s) \end{bmatrix} := \begin{bmatrix} \mathcal{M}_{11}(s) & \mathcal{M}_{12}(s) \\ 0 & \mathcal{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \mathcal{Q}_1(s) \\ \mathcal{Q}_2(s) \end{bmatrix} \quad (11)$$

satisfies

$$\text{rank} \begin{bmatrix} \widetilde{\mathcal{Q}}_1(s) \\ \widetilde{\mathcal{Q}}_2(s) \end{bmatrix} > \text{rank } \widetilde{\mathcal{Q}}_1(s).$$

Hence one could add a test after Step 4 in Algorithm 1 whether a rank drop in (10a) can be enforced via a suitable unimodular transformation as in (11). If this is not possible (and therefore the non-termination of Algorithm 1 cannot be excluded) one may test whether there is a polynomial non-singular matrix  $\mathcal{M}(s)$  such that the Algorithm 1 with input  $(\mathcal{M}(s)\mathcal{P}(s), \mathcal{M}(s)\mathcal{Q}(s))$  terminates with output no. In that case we can conclude that also the original pair  $(\mathcal{P}(s), \mathcal{Q}(s))$  was not delay-regular, because every solution of  $(\mathcal{P}(s), \mathcal{Q}(s))$  is also a solution of  $(\mathcal{M}(s)\mathcal{P}(s), \mathcal{M}(s)\mathcal{Q}(s))$ , hence if the latter is not solvable, then the former will also not be solvable.

Nevertheless, as of now, a simple extension of Algorithm 1 to give a definitive answer in all possible case is not available yet and is ongoing research.

## VI. CONCLUSIONS

We have introduced the novel notion of delay-regularity for DDAEs and have shown that regularity of the matrix pair  $(E, A)$  is sufficient for delay-regularity. However, for DDAEs regularity of the matrix pair  $(E, A)$  is not necessary to guarantee existence and uniqueness of solution; the delay can regularize the originally non-regular DAE. We propose a recursive algorithm to test whether a given DDAE is delay-regular or not. We prove that the algorithm returns the correct result if it terminates. We discuss the situation under which non-termination can occur and how this may be prevented. An extension of the algorithm which ensures termination for all possible inputs is still ongoing research.

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