Asymptotic Tracking via Funnel Control

Jin Gyu Lee* and Stephan Trenn**

***Abstract*— Funnel control is a powerful and simple method to solve the output tracking problem without the need of a good system model, without identification and without knowlegde how the reference signal is produced, but transient behavior as well as arbitrary good accuracy can be guaranteed. Until recently, it was believed that the price to pay for these very nice properties is that only practical tracking and not asymptotic tracking can be achieved. Surprisingly, this is not true! We will prove that funnel control – without any further assumptions – can achieve asymptotic tracking.**

I. Introduction

Funnel control is based on the idea that for a certain system class (relative degree one with positive high frequency gain and stable zero dynamics) a simple proportional output feedback of the form

$$u(t) = -k \cdot y(t)$$

achieves asymptotic stability of the system for *sufficiently large* feedback gain $k > 0$. If the systems parameters are unknown or uncertain, the simple idea of replacing the constant gain by a time-varying gain $k(t)$ and using the adaption rule

$$\dot{k}(t) = y(t)^2$$

yields asymptotic stability and bounded internal variables; many researches have studied variants of this problem, see e.g. the corresponding references in the survey [1].

However, in the presence of measurement noise or if the output is desired to track a (possibly time-varying) reference signal r, see Figure 1, and the control law becomes an error feedback

$$u(t) = -k(t) \cdot e(t), \tag{1}$$

where $e(t) := y(t) - r(t)$, then the simple adaptation rule $\dot{k}(t) = e(t)^2$ will result in an unbounded growth of the gain k in general.

This problem was resolved in [2] by not aiming anymore for asymptotic tracking but for *practical tracking* by introducing the so-called λ-tracker which uses the slightly modified adaption rule

$$\dot{k}(t) = \begin{cases} (|e(t)| - \lambda)|e(t)|, & |e(t)| \geq \lambda \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is the desired tracking accuracy. It can than be shown (under some mild boundedness assumption) that for any desired accuracy, the λ-tracker achieves practical tracking, i.e. there exists a time $T > 0$ such that $|e(t)| \leq \lambda$ for all $t \geq T$. In contrast to most other tracking problems where the reference signal is assumed to be produced by some *known* exosystem, here no assumption is made on the reference signal apart from being bounded. In fact, it seems quite intuitive that in order to achieve asymptotic tracking some knowledge of the reference signal is necessary and that by only requiring practical tracking this knowledge is not needed anymore.

The λ-tracker had still two main drawbacks: 1) There was no direct control on the transient behavior, in particular, how fast the desired accuracy is reached, see Figure 2; and 2) the gain $k(\cdot)$ is monotonically increasing, so that even if the error is small, the gain remains large and unnecessarily amplifies measurement noise.

The first drawback was resolved by Miller & Davison [3] by introducing a piecewise-constant gain function $k(\cdot)$ which doubles every time the error hits certain thresholds. This guaranteed that there was no overshoot at the beginning and also that for a *given* time $T > 0$ the error satisfied $|e(t)| < \lambda$ for all $t \geq T$.

Both drawbacks of the λ-tracker were resolved simultaneously by the funnel controller proposed by Ilchmann, Ryan and Sangwin [4], which still uses the simple proportional error feedback rule (1), but uses now the non-dynamic time-varying gain as

$$k(t) = \frac{1}{\psi(t) - |e(t)|}$$

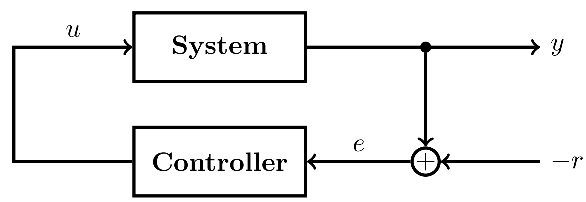

Fig. 1. General feedback structure for the output tracking problem.

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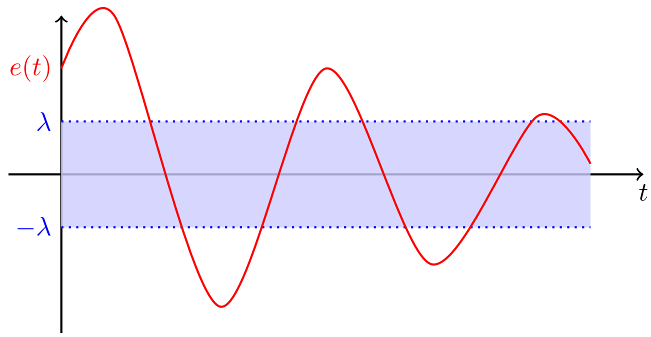

Fig. 2. Problem of λ-tracker: no guarantees for the transient behavior.

where $\psi : [0, \infty) \to \mathbb{R}$ is a *prespecified* (time-varying) error bound, see Figure 3.

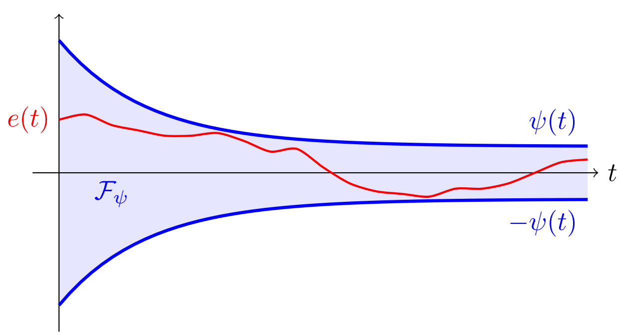

Fig. 3. The funnel: a prespecified time-varying error bound.

The intuition behind this gain function is strikingly simple: If the error $e(t)$ at a given time t is very close to the funnel boundary $\psi(t)$ (or $-\psi(t)$ if $e(t) < 0$), i.e. $\psi(t) - |e(t)|$ is close to zero, then the gain is very large, which in view of the considered system class results in $\dot{e}(t)$ being very negative (or very positive if $e(t) < 0$) and which in turn means that the error is actually moving away from the funnel boundary. On the other hand if the distance between the error and the funnel boundary is not very small, then also the gain is not very large. This approach also works for systems with multiple inputs and outputs (MIMO), in fact, already [4] treated the MIMO case. A key (and obvious) assumption is that the funnel boundary ψ is bounded away from zero, because otherwise the gain would by design grow unbounded which, of course, is undesirable.

The original idea of funnel control was extended in many directions, e.g. more general feedback gain [5], input constraints [6]–[10], higher relative degree [11]–[15], for differential-algebraic equations [16], [17] and is also included in the engineering textbook [18]. However, all of these references exclude a funnel boundary which converges to zero. The only remarkable exception is the reference [19] which try to achieve asymptotic tracking with funnel control, but the authors have to rely on an *internal model principle* to prove their result; interestingly, they show however existence of $\varepsilon > 0$ such that

$$|e(t)| \le (1 - \varepsilon)\psi(t) \quad t \ge 0$$

instead of the usual funnel error bound

$$|e(t)| \le \psi(t) - \varepsilon.$$

Prescribed performance control (PPC) proposed by Bechlioulis & Rovithakis [20] is similar to funnel control and also aims at ensuring that the error evolves within a prespecified time-varying error bound, but also in this approach it is assumed that the error-bound is not approaching zero. Very recently, asymptotic tracking was also investigated for PPC [21] where convergence to zero is achieved indirectly via a backstepping approach and not directly via a converging funnel boundary as proposed here; in particular, the rate of convergence cannot be prespecified via the choice of the funnel shape.

Altogether it seems to be a common assumption in the community, that asymptotic tracking of an arbitrary reference signal (not produced by a known exo-system) with prescribed performance is not possible. We will show here that this is a misconception and with a simple trick of rewriting the funnel control law it is indeed possible to show that asymptotic tracking is possible!

II. Problem setting

A. System class

We consider nonlinear system of the following input-affine form:

$$\begin{aligned} \dot{y} &= f(p_f, y, z) + g(p_g, y, z) \cdot u, \\ \dot{z} &= h(p_h, y, z), \end{aligned} \tag{2}$$

where $y : \mathbb{R}_{\ge 0} \to \mathbb{R}$ is the (scalar) output, $u : \mathbb{R}_{\ge 0} \to \mathbb{R}$ is the (scalar) input, $z : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n-1}$ are the internal dynamics of order $n - 1 \in \mathbb{N}$, $p_f, p_g, p_h : \mathbb{R}_{\ge 0} \to \mathbb{R}^d$ are locally integrable pertubations, $f, g : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and $h : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are locally Lipschitz. We make the following structural assumptions on the system class.

A1 $g(p_g, y, z) > 0$ for all p_g, y, z.

A2 BIBO zero dynamics, i.e. all solutions of $\dot{z} = h(p_h, y, z)$ satisfy the following inequality

$$\|z(t)\| \le b_z(\|p_{h[0,t)}\|_\infty, \|y_{[0,t)}\|_\infty, \|z(0)\|),$$

for some continuous function b_z and for any bounded p_h and y.

A3 The perturbations p_f, p_g, p_h are bounded.

Remark 1: Assuming that the nonlinear system is already in the form (2) may seem rather restrictive; however, the control law does not depend on the knowledge of the specific form, in fact, it is sufficient to know that the actual model is equivalent to a model of the form (2). In particular, if the original model has the form

$$\dot{x} = G(p_G, x, u), \quad y = H(p_H, x)$$

with $G : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $H : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$, then the funnel control law can be applied without change as long as it is reasonable to assume that a diffeomorphic coordinate transformation $x \mapsto (y, z)$ *exists* which results in the form (2). Roughly speaking, this is the case, when the nonlinear system has relative degree one. //

B. Tracking problem and funnel

The control objective is to find an output feedback rule such that the output y of the system tracks a given reference signal $r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with prespecified error performance. The latter is given via a time varying strict error bound $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, i.e. it is required that

$$e(t) := y(t) - r(t) \in \big(\text{-}\psi(t), \psi(t)\big) \quad \forall t \geq 0.$$

The time-varying region where the error is allowed to evolve in is given by

$$\mathcal{F}_\psi := \{(t, e) \mid |e| < \psi(t)\}$$

and is called *funnel* and ψ is called *funnel boundary*, cf. Figure 3. Note that the funnel boundary is chosen according to the requirements of the control application, in particular, the convergence rate and the final accuracy are reflected by the choice of ψ. A typical choice is for example

$$\psi(t) = (\overline{\psi} - \underline{\psi})e^{-\lambda t} + \underline{\psi},$$

where $\overline{\psi} > 0$ is a (known) bound on the initial error, $\underline{\psi} \geq 0$ is the finally desired tracking accuracy and $\lambda > 0$ is the desired convergence rate. By choosing a non-monotonic funnel boundary it is also possible to temporarily allow for larger errors in case of known (periodic) disturbances (e.g. sensor callibrations).

The following assumptions are made on the tracking signal r and the funnel boundary ψ:

A4 $r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuously differentiable, bounded and with bounded derivative

A5 $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuously differentiable, bounded and with bounded derivative

A6 $|e(0)| < \psi(0)$.

C. Classical funnel control

The classical funnel control takes the form

$$u(t) = -K_{\mathcal{F}}(t, e(t)) \cdot e(t) \tag{3}$$

where $K_{\mathcal{F}}(t, e(t))$ is a positive gain function which approaches infinity when the error variable $e(t)$ approaches the funnel boundary, a possible choice is

$$K_{\mathcal{F}}(t, e) = \frac{1}{\psi(t) - |e|}. \tag{4}$$

Since by design the gain grows unbounded when the distance $\psi(t) - |e|$ tends to zero, it follows that asymptotic tracking is impossible with bounded internal variables (including the gain), because asymptotic tracking means that $\psi(t) \to 0$ as $t \to \infty$ and this implies $\psi(t) - |e(t)|$ also tends to zero for $t \to \infty$ (or even earlier, when the error leaves the funnel).

D. An alternative funnel control design

Inspired by the proof technique used in [22] we introduce first the *ratio* between the error and the funnel boundary

$$\eta(t) := \frac{e(t)}{\psi(t)}$$

and suggest the following funnel control:

$$u(t) = -\alpha\left(\eta(t)\right) \cdot \beta\left(\eta(t)\right) \tag{5}$$

where α and β satisfy the following conditions

A7 $\alpha : (-1, 1) \to [0, \infty)$ is continuous and satisfies $\alpha(\eta) \to \infty$ as $|\eta| \to 1$.

A8 $\beta : (-1, 1) \to \mathbb{R}$ is continuous, $\beta(\eta) \not\to 0$ as $|\eta| \to 1$ and $\operatorname{sgn}(\beta(\eta)) = \operatorname{sgn}(\eta)$ for all $\eta \neq 0$.

In contrast to the classical funnel controller where $\psi(t) \to 0$ as $t \to \infty$ lead to an unbounded gain, this is not the case any more with the alternative formulation because the gain $\alpha(e(t)/\psi(t))$ can remain bounded even if $1/\psi(t)$ tends to infinity. It is important to note that the new control law (5) actually results in *exactly the same* control action as the classical funnel control law (3) with gain (4) by choosing

$$\alpha(\eta) = \frac{1}{1 - |\eta|} \quad \text{and} \quad \beta(\eta) = \eta. \tag{6}$$

Before the main proof for asymptotic tracking will be presented, a comparison between the new and the old approach will be illustrated with a simple example.

Example 1: We consider the system

$$\dot{y} = 2 + \sin(t) + u$$

with initial condition $y(0) = 0$ and we want to track the zero trajectory. Note that $(y, u) = (0, 0)$ is *not* an equilibrium and if all the system parameters would be known, one could choose the control $u(t) = -(2 + \sin(t))$ to remain at zero. We consider first a funnel given by $\psi(t) = 3e^{-3t} + 0.5$ which is bounded away from zero. The first simulation is carried out by applying the usual funnel controller

$$u(t) = -k(t)y(t), \quad k(t) := \frac{1}{\psi(t) - |y(t)|}.$$

The result is shown in the left part of Figure 4. We repeat the simulation with the asymptotic funnel given by $\psi(t) = 3.5e^{-3t}$ and the results are shown in the right part of Figure 4. As can be clearly seen, the classical gain $k(t)$ grows very strongly when the funnel boundary approaches zero, while the gain $\alpha(\eta(t)) = \frac{1}{1-|\eta(t)|}$ and the corresponding input $u(t)$ behave very nicely. Note that with the choice of α and β as in (6) it holds that $\alpha = 1 - u$ (for positive errors) and that in the asymptotic case the input automatically adapts to the ideal (but unknown) input $-(2 + \sin(t))$. //

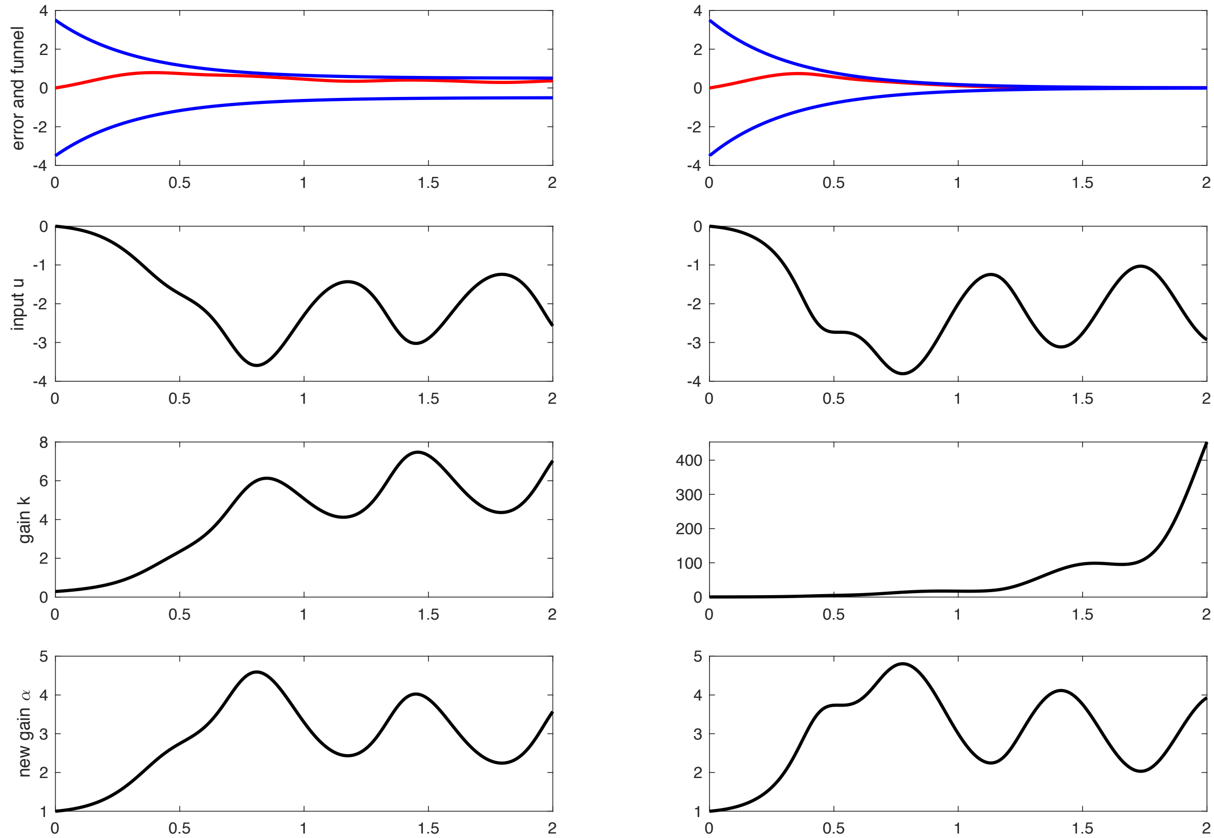

Fig. 4. Simulation of Example 1 with non-asymptotic funnel boundary (left) and asymptotic funnel (right). From top to bottom: the error (red) within the funnel (blue), the control input $u(t)$, the classical gain $k(t)$, the new gain $\alpha(\eta(t))$.

III. Asymptotic tracking result

Theorem 2: Consider the nonlinear system (2) satisfying assumptions **A1**–**A3**. Then for any reference signal r and any funnel boundary ψ satisfying assumptions **A4**–**A6**, the funnel controller (5) with arbitrary α and β satisfying **A7** and **A8** results in a closed loop where all solutions exist on $[0,\infty)$ and remain bounded. In particular, there exists $\varepsilon > 0$ such that

$$|e(t)| \leq (1-\varepsilon)\psi(t) \text{ for all } t \geq 0,$$

i.e. the error between the output and the reference signal remains within the funnel for all times and the distance between the fraction $e(t)/\psi(t)$ and the boundary ± 1 is bounded away from zero. //

Before presenting the proof, the following consequence will be highlighted.

Corollary 3: Under the assumptions of Theorem 2 asymptotic tracking via funnel control can be achieved with arbitrary convergence rates, for example by using $\psi(t) = ce^{-\lambda t}$ for some $c > 0$ and $\lambda > 0$. //

Proof: (of Theorem 2) Standard arguments from ODE theory ensure existence of a (unique) local solution $(y,z) : [0,\omega) \to \mathbb{R} \times \mathbb{R}^{n-1}$ for some $\omega > 0$. In particular, $(t, e(t)) \in \mathcal{F}_\psi$ for all $t \in [0,\omega)$, because only then are α and β well defined. Furthermore,

$$\dot{e} = \dot{y} - \dot{r} = f(p_f, y, z) - \dot{r} + g(p_g, y, z)u.$$

Assume that for some $\varepsilon > 0$ there is a $t_\varepsilon \in [0,\omega)$ with $e(t_\varepsilon) = (1-\varepsilon)\psi(t_\varepsilon)$. Then it holds that $u(t_\varepsilon) = -\alpha(1-\varepsilon) \cdot \beta(1-\varepsilon)$. Furthermore, $y = e + r$ is bounded on $[0,\omega)$ because r is bounded by assumption and e is contained in the bounded funnel. Therefore, BIBO stability of the z-dynamics also guarantee that z is bounded on $[0,\omega)$. Continuity of f and g as well as boundedness of $\dot{r}$, p_f and p_g now guarantee that there exists constants $c_1, c_2 > 0$ (which are independent of ε and ω) such that

$$f(p_f(t), y(t), z(t)) - \dot{r}(t) < c_1$$

and

$$g(p_g(t), y(t), z(t)) > c_2,$$

for all $t \in [0,\omega)$. Hence

$$\dot{e}(t_\varepsilon) < c_1 - c_2\alpha(1-\varepsilon)\beta(1-\varepsilon).$$

By assumption, $\alpha(1-\varepsilon)$ grows unbounded and $\beta(1-\varepsilon) \not\to 0$ for $\varepsilon \to 0$, therefore, also in view of boundedness of $\dot{\psi}$, it holds that $\dot{e}(t_\varepsilon) < \dot{\psi}(t_\varepsilon)$ if $e(t_\varepsilon) = (1-\varepsilon)\psi(t_\varepsilon)$ for sufficiently small $\varepsilon > 0$. Analogoue arguments show that for sufficiently small $\varepsilon > 0$ also $\dot{e}(t_\varepsilon) > -\dot{\psi}(t_\varepsilon)$ if

$e(t_\varepsilon) = -(1-\varepsilon)\psi(t_\varepsilon)$. Altogether, this shows that for sufficiently small $\varepsilon > 0$ the set

$$\mathcal{F}_\varepsilon := \{(t,e) \mid -(1-\varepsilon)\psi(t) \le e \le (1-\varepsilon)\psi(t)\}$$

is positively invariant if $e(0) \in \mathcal{F}_\varepsilon$. In particular, the solution exists globally, the error evolves within the funnel and the fraction $e(t)/\psi(t)$ is bounded away from ± 1 with a distance of at most ε. ■

Remark 4: A careful analysis of the proof shows that it is even possible to achieve *finite time convergence* of the error by simply choosing a funnel boundary which approach zero in finite time. However, in that case the solution stops to exists once it reaches the end-point and an alternative control approach is necessary to continue tracking of the reference signal (e.g. a sliding mode controller). //

IV. Generalization to the MIMO case

The original paper on funnel controller [4] already treated the case of multiple inputs and multiple outputs (MIMO). The goal of this section is to consider MIMO systems and additionally removing the assumption that the systems dynamics are affine in the input. Therefore, consider the following class of systems

$$\dot{y}(t) = F(p_F(t), y(t), z(t), u(t)) \tag{7a}$$

$$\dot{z} = h(p_h(t), y(t), z(t)) \tag{7b}$$

where now $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ and $y : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ are m-dimensional input and output signals for some $m > 1$; $F : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ and $h : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ are locally Lipschitz. As before $p_F, p_h : \mathbb{R} \to \mathbb{R}^d$, $d \in \mathbb{N}$, are locally integrable pertubations. Then the definition of the funnel slightly changes to

$$\mathcal{F}_\psi := \{(t,e) \mid \|e\|_2 < \psi(t)\},$$

where $\|e\|_2 = \sqrt{e^\top e}$ is the usual Euclidian norm. Assumption **A1** is replaced by

A1' F is differentiable with respect to u and there exist $M \ge 1$ and a continuous function $\gamma : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}$ such that the following implication holds for all $\eta, \tilde{\eta} \in \mathbb{R}^m, p \in \mathbb{R}^d, \nu \in \mathbb{R}^m, \zeta \in \mathbb{R}^{n-m}, \mu \in \mathbb{R}^m$:

$$\|\eta\|_2 < 1 \ \wedge\ \|\tilde{\eta}\|_2 \le M \ \wedge\ \eta^\top \tilde{\eta} \ge \|\eta\|_2^2 > 0$$
$$\implies \eta^\top \left[\frac{\partial}{\partial u} F(p,\nu,\zeta,\mu)\right] \tilde{\eta} \ge \gamma(p,\nu,\zeta)\|\eta\|_2^2 > 0.$$

Remark 5: If F in (7) is affine in the input, i.e. $F = f + g \cdot u$, then Assumption **A1'** just means that g must be *positive definitive*. //

Assumption **A2** remains exactly the same for the MIMO case and in Assumption **A3** p_f and p_g are replaced by p_F. Also the assumptions **A4**, **A5**, **A6** remain the same apart from r being now a map to $\mathbb{R}^m$ and $|\cdot|$ is replaced by $\|\cdot\|_2$. For reference purposes we will denote the corresponding MIMO assumptions by **A2'**, **A3'**, **A4'**, **A5'**, **A6'**, respectively. The alternative funnel control law (5) remains formally identical in the MIMO case, however, the domain of α and both the domain and range of β change. Let $\mathbb{B}^m_{<1}$ denote the open unit ball in $\mathbb{R}^m$, i.e. $\mathbb{B}^m_{<1} := \{\eta \in \mathbb{R}^m \mid \|\eta\|_2 < 1\}$ then α and β are assumed to satisfy the following properties.

A7' $\alpha : \mathbb{B}^m_{<1} \to [0,\infty)$ is continuous and satisfies $\alpha(\eta) \to \infty$ as $\|\eta\|_2 \to 1$.

A8' $\beta : \mathbb{B}^m_{<1} \to \mathbb{R}^m$ is bounded by $\delta M > 0$, where $M \ge 1$ is from Assumption **A1'** and $\delta > 0$ is assumed to exist such that

$$\eta^\top \beta(\eta) \ge \delta \|\eta\|_2^2 \quad \forall \eta \in \mathbb{B}^m_{<1}.$$

A possible choice for α and β which satisfies assumptions **A7'** and **A8'** is

$$\alpha(\eta) = \frac{1}{1-\|\eta\|_2}, \quad \beta(\eta) = \eta \quad (\text{choose } \delta = 1).$$

There may however exist applications where β cannot be chosen as the identity map, for example when the input direction can only be chosen from a finite set (but with arbitrary gain).

We can now formulate the MIMO-version of Theorem 2 as follows.

Theorem 6: Consider the nonlinear system (7) satisfying Assumptions **A1'**–**A3'** with reference signal r and prespecified error funnel $\mathcal{F}_\psi$ satisfying Assumptions **A4'**–**A6'**. The closed loop with the funnel control (5) satisfying Assumptions **A7'** and **A8'** ensures global existence of solutions and the tracking error $e = y - r$ evolves within the funnel. In particular, there exists $\varepsilon > 0$ such that

$$\|e(t)\|_2 \le (1-\varepsilon)\psi(t) \quad \forall t \ge 0.$$

Proof: Similar as in the SISO case, existence of a solution $(y,z) : [0,\omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ is guaranteed by standard arguments from ODE theory. Let $e = y - r$ and $\eta = e/\psi$ then (omitting the time dependence)

$$\begin{aligned}\frac{d}{dt}(\|e\|_2^2) &= 2e^\top \dot{e}\\ &= 2\psi\eta^\top F(p_F, y, z, u) - 2\psi\eta^\top \dot{r}\end{aligned}$$

The mean value theorem ensures existence of $\lambda \in (0,1)$ such that

$$F(p_F, y, z, u) = F(p_F, y, z, 0) + \frac{\partial}{\partial u} F(p_F, y, z, \lambda u) \cdot u.$$

By assumption $\|e(t)\|_2 < \psi(t)$ for all $t \in [0,\omega)$ and together with boundedness of r the output y is also bounded on $[0,\omega)$. Due to the BIBO-stability of the zero-dynamics this also implies that z is bounded on $[0,\omega)$. The perturbation is bounded by assumption, hence continuity of F implies existence of $C_F > 0$ such that $\|F(p_F(t), y(t), z(t), 0)\| \le C_F$ for all $t \in [0,\omega)$. By assumption there is also $C_{\dot{r}} > 0$ such that $\|\dot{r}(t)\| \le C_{\dot{r}}$ for all $t \in [0,\omega)$.

Invoking the Cauchy-Schwartz-Inequality together with $\|\eta(t)\|_2 < 1$ we obtain:

$$\frac{d}{dt}(\|e\|_2^2) \le 2\psi\Big(C_F + C_{\dot{r}} - \alpha(\eta)\eta^\top \frac{\partial}{\partial u} F(p_F, y, z, \lambda u)\beta(\eta)\Big).$$

Assume now that for $\varepsilon > 0$ there is $t_\varepsilon \in [0,\omega)$ such that $\|e(t_\varepsilon)\|_2 = (1-\varepsilon)\psi(t_\varepsilon)$. With $\tilde{\eta} = \beta(\eta(t_\varepsilon))/\delta$ the conditions of the implications in Assumption **A1'** are satisfied due to Assumption **A8'**, hence we can conclude

$$\eta(t_\varepsilon)^\top \frac{\partial}{\partial u} F(p_F(t_\varepsilon), y(t_\varepsilon), z(t_\varepsilon), \lambda u(t_\varepsilon))\beta(\eta(t_\varepsilon)) \geq \delta\gamma(p_F(t_\varepsilon), y(t_\varepsilon), z(t_\varepsilon))\|\eta(t_\varepsilon)\|_2^2.$$

By boundedness of p_F, y, z on $[0,\omega)$ and continuity of γ there is $C_\gamma > 0$ such that

$$\gamma(p_F(t), y(t), z(t)) \geq C_\gamma \quad \forall t \in [0,\omega).$$

Therefore, taking into account that $\|\eta(t_\varepsilon)\|_2^2 = (1-\varepsilon)^2$,

$$\frac{d}{dt}(\|e(t_\varepsilon)\|_2^2) \leq 2\psi(t_\varepsilon)\Big(C_F + C_{\dot{r}} - \alpha(\eta(t_\varepsilon))\delta C_\gamma (1-\varepsilon)^2\Big).$$

Note that the constants C_F, $C_{\dot{r}}$, C_γ are independent of ε and $t_\varepsilon \in [0,\omega)$, hence for sufficiently small ε the term $\alpha(\eta(t_\varepsilon))$ grows large enough so that ($\dot{\psi}$ is bounded)

$$C_F + C_{\dot{r}} - \alpha(\eta(t_\varepsilon))C_\gamma(1-\varepsilon)^2 < \dot{\psi}(t_\varepsilon)$$

and hence

$$\frac{d}{dt}(\|e(t_\varepsilon)\|_2^2) \leq 2\psi(t_\varepsilon)\dot{\psi}(t_\varepsilon) = \frac{d}{dt}(\psi^2)(t_\varepsilon).$$

Analogously as in the SISO-case this now shows that the region

$$\mathcal{F}_\varepsilon := \{(t,e) \in \mathcal{F}_\psi \mid \|e\|_2 \leq (1-\varepsilon)\psi(t)\}$$

is a positively invariant set for the error e. Hence $\omega = \infty$ and everything is shown. ∎

V. Conclusion

We have shown that asymptotic tracking with prescribed transient behavior can be achieved without any additional assumptions on the system class and the reference signal compared to the usual funnel control approach. The proof technique even allows to achieve finite time convergence with the additional benefit of having full control of the transient behavior. We believe that almost all existing funnel control results (e.g. input saturations, higher relative) can be extended to also achieve asymptotic tracking by simply rewriting the funnel rule.

Nevertheless we would like to stress that in practical applications the property of asymptotical tracking/convergence is mostly only of minor interest, the main concern is avoiding overshoots and respecting the input constraints; in view of measurement inaccuracy the difference between asymptotic tracking and practical tracking is only academic.

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