# Asymptotic stability of piecewise affine systems with Filippov solutions via discontinuous piecewise Lyapunov functions

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***Abstract*— Asymptotic stability of continuous-time piecewise affine systems defined over a polyhedral partition of the state space, with possible discontinuous vector field on the boundaries, is considered. In the first part of the paper the feasible Filippov solution concept is introduced by characterizing single-mode Caratheodory, sliding mode and forward Zeno behaviors. Then, a global asymptotic stability result through a (possibly discontinuous) piecewise Lyapunov function is presented. The sufficient conditions are based on pointwise classifications of the trajectories which allow the identification of crossing, unreachable and Caratheodory boundaries. It is shown that the sign and jump conditions of the stability theorem can be expressed in terms of linear matrix inequalities by particularizing to piecewise quadratic Lyapunov functions and using the cone-copositivity approach. Several examples illustrate the theoretical arguments and the effectiveness of the stability result.**

## I. Introduction

Lyapunov theory has been widely used for the asymptotic stability analysis of continuous-time piecewise affine (PWA) systems defined over a polyhedral partition of the state space [1], [2]. When the vector fields are not continuous on the boundaries, which is the case considered in this paper, the stability problem becomes more challenging due to the possible occurrence of sliding mode and Zeno behaviors [3], [4]. For this class of discontinuous systems, to find a globally quadratic Lyapunov function is a nontrivial issue [5], [6] and the existence of such a function is not ensured either [7], [8].

A possible direction for overcoming limitations of global quadratic functions, consists of considering continuous piecewise Lyapunov functions [9]. In particular, piecewise quadratic (PWQ) Lyapunov functions and the $S$-procedure lead to stability conditions for classical solutions which can be expressed in terms of linear matrix inequalities (LMIs), see [10], [11], [12]. In general, to have a continuous PWQ function which is positive definite and decreasing in time in each region it is not sufficient for concluding the asymptotic stability of Filippov solutions of a discontinuous PWA system. Indeed, further conditions for dealing with sliding modes must be added [1], [13]. Other classes of continuous piecewise Lyapunov functions have been considered [14]. For instance, a backstepping procedure for the construction of a continuous Lyapunov function in the form of sum of squares for piecewise polynomial systems with discontinuous vector field is proposed in [15]. In [16] continuous functions given by convex combinations of quadratic forms allow to conclude the stability of some discontinuous PWA systems with sliding modes. The more general class of piecewise smooth Lyapunov functions is considered in [17] but the continuity on the boundaries is assumed therein too.

In this paper we consider the more general case of possibly discontinuous piecewise Lyapunov functions for discontinuous PWA systems. Discontinuous PWQ Lyapunov functions have been considered in [18], [19] for the asymptotic stability of planar PWA systems, but the analysis was restricted to the case of continuous vector fields. The stability conditions proposed in [20] allows discontinuities but the apriori knowledge of the sequence of modes is required. In [21] a discontinuous Lyapunov function designed by looking at the system structure has been proposed for a point mass subject to Coulomb friction in feedback with a PID controller.

The analysis proposed in this paper originates from the preliminary arguments presented in [22] where more restrictive classes of PWA systems and PWQ Lyapunov functions were considered. Herein the possibly discontinuous Lyapunov function does not require to have a PWQ form, although can be particularized to that class thus allowing the formulation of the stability conditions in terms of LMIs through the copositive programming approach [23].

The rest of the paper is organized as follows. The class of continuous-time discontinuous PWA systems with the relevant solution concepts are presented in Sec. II. The classification of the system modes depending on the trajectory behavior on the boundaries is discussed in Sec. III. The main stability theorem with the conditions for the existence of a possibly discontinuous piecewise Lyapunov function is proved in Sec. IV. Conditions for the characterization of boundaries in terms of inequalities to be satisfied on their relative interior is discussed in Sec. V. The analysis is then particularized to the case of PWQ Lyapunov functions in Sec. VI where numerical results confirm the effectiveness of the approach. Sec. VII concludes the paper.

## II. PWA system and solution concept

We consider the PWA system

$$\dot{x} = A_s x + b_s, \quad x \in X_s, \quad s \in \Sigma \tag{1}$$

where $A_s \in \mathbb{R}^{n\times n}$, $b_s \in \mathbb{R}^n$ and $\{X_s\}_{s=1}^S$ is a polyhedral partition of $\mathbb{R}^n$ with $S \in \mathbb{N}$ being the *finite* size of the partition; let $\Sigma := \{1,\ldots,S\}$. In particular, every $X_s$ is a closed convex set resulting from the finite intersection of (closed) half-spaces. Furthermore, we assume that the

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intersection $X_i \cap X_j$ is empty or a common face of the polyhedra $X_i$ and $X_j$ for all $i, j \in \Sigma$. By assumption each $X_s$ is a closed set, hence neighbouring polyhedra have a nonempty intersection and there is some ambiguity in the system definition on these intersections. This ambiguity needs to be handled carefully when defining solutions and also is crucial in the forthcoming stability analysis. A first step for dealing with the nonuniqueness of the dynamics on the boundaries is the introduction of the (dynamic-independent) index set of *current modes* at $x \in \mathbb{R}^n$ as $\Sigma^x := \{ s \in \Sigma \mid x \in X_s \}$. Note that for those $x \in \mathbb{R}^n$ which are not on a boundary, $\Sigma^x$ just contains one index. Rewriting (1) now as a differential inclusion

$$\dot{x} \in \{ A_s x + b_s \mid s \in \Sigma^x \}, \qquad (2)$$

we introduce the following solution concept for (2).

*Definition 1 (Caratheodory solution):* We call $\xi : [t_0, T) \to \mathbb{R}^n$, $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, a *Caratheodory solution* of the PWA system (2) iff

1) $\xi$ is absolutely continuous and
2) for almost all $t \in [t_0, T)$:

$$\dot{\xi}(t) \in \left\{ A_s \xi(t) + b_s \;\middle|\; s \in \Sigma^{\xi(t)} \right\}. \qquad (3)$$

The set of all Caratheodory solutions $\xi$ defined on $[t_0, T)$ with initial condition $\xi(t_0) = x_0$ is denoted by $\mathcal{CS}(x_0)\big|_{[t_0,T)}$. In particular, a Caratheodory solution $\xi : [t_0, T) \to \mathbb{R}^n$ is called *single-mode Caratheodory solution* iff there exists an $s \in \Sigma$ such that $\xi(t) \in X_s$ and $\dot{\xi}(t) = A_s \xi(t) + b_s$ for all $t \in (t_0, T)$.

Since every absolutely continuous function is differentiable almost everywhere, (3) indeed makes sense. Furthermore, away from the boundaries the set $\Sigma^x$ only consists of one index; hence, only on the boundaries of the partition we really have a differential inclusion.

For the (asymptotic) stability analysis it is necessary to consider global solutions (i.e. where $T = \infty$ in the above definition); however, for PWA systems (in constrast to usual linear systems) existence of global solutions is not guaranteed. In order to formalize the notion of solutions for which there is a maximal time until when they exist, we recall the *maximal solution* concept as follows (due to the time-invariant nature of (2) we can restrict ourselves to the case $t_0 = 0$): A Caratheodory solution $\xi : [0, \omega) \to \mathbb{R}^n$ is called maximal, if there is no Caratheodory solution $\xi' : [0, \omega') \to \mathbb{R}^n$ with $\omega' > \omega$ and $\xi = \xi'$ on $[0, \omega)$. The set of all (maximal) Caratheodory solutions starting at $x_0 \in \mathbb{R}^n$ is denoted by

$$\mathcal{CS}(x_0) := \left\{ \xi : [0, \omega) \to \mathbb{R}^n \;\middle|\; \begin{array}{l} \xi \text{ is a Caratheodory sol.} \\ \text{with } \xi(0) = x_0 \text{ and} \\ \text{maximal } \omega > 0 \end{array} \right\}.$$

Note that different solutions in $\mathcal{CS}(x_0)$ may have different time-intervals on which they are defined, i.e. in general there is no common $\omega > 0$ for all solutions in $\mathcal{CS}(x_0)$. Let

$$\omega_{\min}^{\mathcal{CS}}(x_0) := \inf \{ \omega > 0 \mid \xi : [0, \omega) \to \mathbb{R}^n \in \mathcal{CS}(x_0) \}$$

be the minimal length of (maximal) solution-existence for initial value $x_0$.

In general, there may be initial values for which a Caratheodory solution does not exist (i.e. $\mathcal{CS}(x_0) = \emptyset$ for some $x_0 \in \mathbb{R}^n$). Consider for example the scalar PWA system (2) with $A_1 = A_2 = 0$, $b_1 = -1$, $b_2 = 1$, $X_1 = \{ x \in \mathbb{R} \mid x \geq 0 \}$, $X_2 = -X_1$ for which there is a maximal single-mode Carathodory solution for all $x_0 \neq 0$ but there is no Caratheodory solution with initial value $\xi(0) = 0$, see Figure 1a.


(a) Example for non-existing Caratheodory solutions.

(b) Example to illustrate non-feasible Filippov solutions.

Fig. 1: Illustration for different solution behaviors.

This example also has the property that all trajectories starting away from zero reach the origin in finite time, in particular, although the trajectories remain bounded the maximal solution-interval is finite. This is in contrast to *continuous* nonlinear differential equations, where a maximal solution has a finite solution interval only if finite escape time occurs (i.e. the solution grows unbounded in finite time).

The following example shows that non-existence of Caratheorody solutions for some initial values can also occur in PWA systems which exhibit maximal non single-mode Caratheodory solutions.

*Example 2:* Consider the following PWA system on $\mathbb{R}^2$ (see also Figure 2): $\dot{x} = \binom{-1}{1}$ in $X_1 = \{x_1 \geq 0, x_2 \geq 0\}$, $\dot{x} = \binom{-1}{-1}$ in $X_2 = \{x_1 \leq 0, x_2 \geq 0\}$, $\dot{x} = \binom{1}{-1}$ in $X_3 = \{x_1 \leq 0, x_2 \leq 0\}$, $\dot{x} = \binom{1/2}{1}$ in $X_4 = \{x_1 \geq 0, x_2 \leq 0\}$.

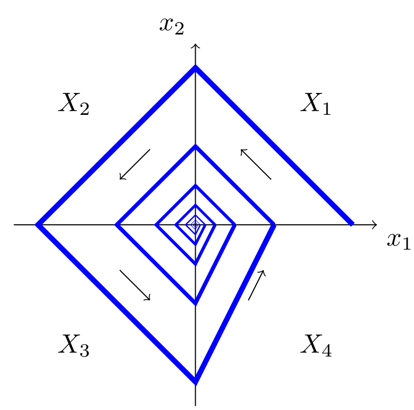


Fig. 2: Planar *backward* Zeno (Caratheodory) solution (also called left-Zeno solution) reaching the origin. There is no Caratheodory solution starting from the origin. If the flow direction is reversed, the system exhibits a *forward* Zeno (Caratheodory) solution (also called right-Zeno solution) starting from the origin.

The trajectories of this example move around the origin with constant speed and since the length halves after each round, the origin is reached in finite time where the Caratheodory solutions stops (i.e. there is no Caratheodory solution starting in the origin). Furthermore, there are infinitely many switches between the different modes in a finite time interval, i.e. a *Zeno behavior*, which leads to problems when attempting to numerically solve the PWA system. ☡

The problem of non-existence of Caratheodory solutions can neatly be circumvented by convexifying the differential inclusion (2), i.e.

$$\dot{x} \in \operatorname{conv}\{\ A_s x + b_s \ \mid s \in \Sigma^x\ \}, \tag{4}$$

where "conv" indicates the convex hull and passing to so called Filippov solutions (in particular, sliding solutions):

*Definition 3 (Filippov solution):* We call $\xi : [t_0, T) \to \mathbb{R}^n$, $t_0, T \in \mathbb{R} \cup \{\infty\}$ with $t_0 < T$, a *Filippov solution* of the PWA system (4) iff

1) $\xi$ is absolutely continuous and
2) for almost all $t \in [t_0, T)$:

$$\dot{\xi}(t) \in \operatorname{conv}\left\{\ A_s \xi(t) + b_s \ \middle|\ s \in \Sigma^{\xi(t)}\ \right\}. \tag{5}$$

*Definition 4 (Sliding solution):* A Filippov solution $\xi : [t_0, T) \to \mathbb{R}^n$ is called *sliding solution* iff it is not a Caratheodory solution on any subinterval of $[t_0, T)$ and there exists an index set $\Sigma^{\xi(\cdot)}_{\text{slide}} \subseteq \Sigma$ such that $\Sigma^{\xi(\cdot)}_{\text{slide}} = \Sigma^{\xi(t)}$ for all $t \in (t_0, T)$ and $\dot{\xi}(t) \in \operatorname{conv}\left\{\ A_s\xi(t) + b_s \ \middle|\ s \in \Sigma^{\xi(\cdot)}_{\text{slide}}\ \right\}$ for almost all $t \in [t_0, T)$.

Clearly, a Caratheodory solution is a Filippov solution, but a sliding solution is not a Caratheodory solution.

Maximality of a Filippov solution is defined analogously as for Caratheodory solutions and the set of all (maximal) Filippov solutions with initial value $x_0 \in \mathbb{R}^n$ is

$$\mathcal{FS}(x_0) := \left\{\ \xi : [0, \omega) \to \mathbb{R}^n \ \middle|\ \begin{array}{l} \xi \text{ is a Filippov sol.} \\ \text{with } \xi(0) = x_0 \text{ and} \\ \text{with maximal } \omega > 0 \end{array} \right\}$$

and any $\xi \in \mathcal{FS}(x_0)$ with $\omega = \infty$ is called *global*. By definition it holds that $\mathcal{FS}(x_0) \supseteq \mathcal{CS}(x_0), \forall x_0 \in \mathbb{R}^n$.

The more general class of Filippov solutions allows $\mathcal{FS}(x_0) \neq \emptyset$ *for all* initial values $x_0 \in \mathbb{R}^n$. In order to show that, we first recall the classical existence result for general differential inclusions by Filippov:

*Theorem 5 ([3, Thm. 2.7.1]):* Let $F : \mathbb{R}^n \quad \mathbb{R}^n$ be a upper-semicontinuous[1] set-valued map for which $F(x)$ is nonempty, bounded, closed and convex for all $x \in \mathbb{R}^n$. Then the differential inclusion $\dot{x} \in F(x)$ has for any initial value $x_0 \in \mathbb{R}^n$ a solution $\xi : [0, \omega) \to \mathbb{R}^n$, i.e. $\xi$ is an absolutely continuous functions with $\xi(0) = x_0$ and which satisfies the differential inclusion for almost all $t \in [0, \omega)$.

Building on this (local) existence result, we can now prove existence of global solutions of the PWA system (4) for arbitrary initial values:

*Theorem 6:* The PWA system (4) has global Filippov solutions for all initial values.

*Proof:* Clearly, $F(x) = \operatorname{conv}\{\ A_s x + b_s \ \mid x \in X_s\ \}$ is nonempty, bounded, closed and convex for all $x \in \mathbb{R}^n$. To show upper-semicontinuity, consider an aribitray $x \in \mathbb{R}^n$ and let $\varepsilon > 0$. Let $M := \max_{s\in\Sigma} \|A_s\|$ and choose $\delta := \varepsilon / M$, then, for any $y \in F(x + \mathbb{B}_\delta)$ there exists some $\lambda_s \in [0, 1]$ with $\sum_{s\in\Sigma} \lambda_s = 1$ such that

$$y \in \sum_{s\in\Sigma} \lambda_s (A_s(x + \mathbb{B}_\delta) + b_s) \subseteq F(x) + \sum_{s\in\Sigma} \lambda_s A_s \mathbb{B}_\delta$$
$$\subseteq F(x) + \sum_{s\in\Sigma} \lambda_s M \mathbb{B}_\delta \subseteq F(x) + \mathbb{B}_\varepsilon.$$

This shows that for any $x \in \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $\delta > 0$ such $F(x + \mathbb{B}_\delta) \subseteq F(x) + \mathbb{B}_\varepsilon$, i.e. $F$ is upper-semicontinuous. Hence all assumptions of Theorem 5 are satisfied and existence of an absolutely continuous $\xi : [0, \omega) \to \mathbb{R}^n$ satisfying (5) follows.

The ability to extend each local solution to a global solution follows from the fact that $\|F(x)\| \le M\|x\| + B$ where $B = \max_{s\in\Sigma} \|b_s\|$, i.e. the right-hand side of the differential inclusion is affinely bounded and finite escape time cannot occur (cf. [24, Prop. 4.12]). ■

Indeed, we have now shown the initial claim that passing from Caratheodory solutions to Filippov solutions resolves the problem of nonexistence of solutions for certain initial values (and as a bonus we actually get that all solutions are global). For instance, the global Filippov solutions of Example 2 consist of a Caratheodory backward Zeno till the origin is reached (in finite time) and then a sliding mode in the origin. However, it is well possible that for some $x_0 \in \mathbb{R}^n$ we have $\emptyset \subsetneq \mathcal{CS}(x_0) \subsetneq \mathcal{FS}(x_0)$, i.e. we have obtained additional Filippov solutions starting in $x_0$ although there already existed Caratheodory solutions starting in $x_0$, see the following example.

*Example 7:* Consider a second order PWA system in the form (4) with a partition in the following three regions $X_1 = \{x_1 \ge 0, x_2 \ge -x_1\}$, $X_2 = \{x_1 \le 0, x_2 \ge x_1\}$, $X_3 = \{x_2 \le -|x_1|\}$ and dynamics $A_1 = A_2 = A_3 = 0$, $b_1 = (-2, 1)^\top$, $b_2 = (2, 1)^\top$, $b_3 = (0, -1)^\top$, see Figure 3. It is

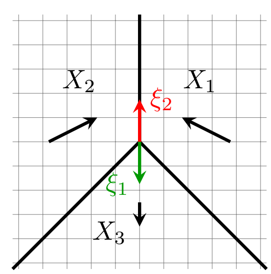


Fig. 3: Example with $\emptyset \subsetneq \mathcal{CS}(x_0) \subsetneq \mathcal{FS}(x_0)$.

easily seen that for any initial value not on the boundary $X_1 \cap X_2$ there is a unique (local) single-mode Caratheodory solution and for any initial value in the relative interior of the boundary $X_1 \cap X_2$ there is a unique sliding solution. There are, however, two Filippov solutions leaving the origin, one single mode Caratheodory solution leaving via region $X_3$ and one sliding solution leaving along the boundary $X_1 \cap X_2$. ☡

While for Example 7 it seems reasonable to allow the situation that for some initial values it is possible to leave via a Caratheodory and a sliding solution both, in other situation this may not be desirable.

[1] A set valued map $F : \mathbb{R}^n \quad \mathbb{R}^n$ is upper-semicontinuous if for every $x \in \mathbb{R}^n$ and for all $\varepsilon > 0$ there is $\delta > 0$ such that $F(x + \mathbb{B}_\delta) \subseteq F(x) + \mathbb{B}_\varepsilon$.

As an example consider the scalar PWA system (4) with $A_1 = A_2 = 0$, $b_1 = 1 = -b_2$ and $X_1 = -X_2 = \{x \geq 0\}$, see Figure 1b, where $\xi(t) \equiv 0$ is a Filippov solution starting in $x_0 = 0$. However, this is an "unnecessary"' sliding solution because there are already two (global) Caratheodory solutions leaving the origin. These unnecessary sliding solutions are not physically feasible, because they cannot be obtained as a limit of a chattering solution and they also lead to unnessarily conservative stability conditions. Therefore, we want to restrict our attention to *feasible Filippov solutions* defined as follows.

*Definition 8 (Feasible Filippov solutions):* A sliding solution $\xi : [t_0, T) \to \mathbb{R}^n$ of (4) is said to exhibit *unnecessary sliding* iff $\mathcal{CS}(\xi(t)) \neq \emptyset$ for some $t \in (t_0, T)$, i.e. iff somewhere along the trajectory it is possible to continue the trajectory with a Caratheodory solution instead of a sliding solution. We now call a Filippov solution $\xi : [t_0, T) \to \mathbb{R}^n$ *feasible* iff there is no subinterval on which $\xi$ is unnecessarily sliding. Or, in other words, a Filippov solution is called *infeasible* iff it contains unnecessary sliding.

Let the set of all (maximal) feasible solutions starting in $x_0 \in \mathbb{R}^n$ be denoted by:

$$\mathcal{FS}^f(x_0) := \{\ \xi \in \mathcal{FS}(x_0)\ \mid \xi \text{ is feasible }\ \}.$$

A natural question rising at this point is whether any global Filippov solution of a PWA system is composed of only Caratheodory (possibly Zeno) and sliding behaviours. Example 2 seems to confirm this claim: for any nonzero initial condition there is a (local) single-mode Caratheodory solution (whose sequence generates the backward Zeno behaviour) and in the origin there is a sliding solution. A simple generalization of this example shows that a global Filippov solution can also exhibit a non-Caratheodory backward Zeno behavior. For instance think at the picture in Figure 2 as a trajectory in $\mathbb{R}^3$ (of a different PWA system) which is constrained to evolve on the plane by the fact that each piece of the trajectory in a quadrant is a sliding motion involving different modes. Then, each piece of the trajectory is a (local) sliding solution but the global Filippov solution cannot be classified as a sliding mode solution since it is not possible to find a common $\Sigma_{\text{slide}}^{\xi(\cdot)}$ for the whole trajectory. This would be a backward Zeno behavior composed by pieces of sliding solutions. Clearly one could also have global Filippov solutions with backward Zeno behaviour generated by the sequence of (local) single-mode Caratheodory and sliding solutions. On the contrary, forward Zeno behavior cannot be locally classified neither as a single-mode Caratheodory nor as a sliding mode.

We will now make certain assumptions on the (Filippov) solution behavior of the PWA system (4). We believe that *all* PWA systems of the form (4) satisfy these assumptions, however, as of now, we are not able to formally prove these properties.

*Assumptions:*

(A1) The PWA system (4) has for all initial values global *feasible* Filippov solutions.

(A2) Let $\xi : [0, \infty) \to \mathbb{R}^n$ be any Filippov solution of the PWA system (4). Then for all $t \geq 0$ there is an $\varepsilon > 0$ such that exactly one of the three cases holds:

1) $\xi\big|_{[t,t+\varepsilon)}$ is a *single-mode Caratheodory solution*.
2) $\xi\big|_{[t,t+\varepsilon)}$ is a *sliding solution*.
3) $\xi\big|_{[t,t+\varepsilon)}$ is a *forward Zeno solution*, i.e. it is neither a single-mode Caratheodory nor a sliding solution and there exists a sequence of positive and strictly decreasing numbers $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ and for each $k \in \mathbb{N}$ the piece $\xi|_{[t+\varepsilon_{k+1},t+\varepsilon_k)}$ is either a single-mode Caratheodory or sliding solution.

Assumption (A1) almost looks like the property already shown in Theorem 6; however, although we know that for any initial value there is a *global* Filippov solution starting in this point, it is not clear, whether this statement is also true when we restrict ourselves to *feasible* Filippov solutions. In particular, we do not know whether (A1) actually rules out certain PWA systems or not.

The first two cases in Assumption (A2) we have already seen in the simple Examples illustrated in Figure 1 and in Example 2 (Figure 2); the third case in Assumption (A2) is illustrated by the PWA system which has forward Zeno solutions in Example 2 with a reverted vector field (i.e. where all solutions are the ones of the original system running backward in time).

An important consequence of Assumption (A2) is the following technical result about the nature of Filippov solutions.

*Lemma 9:* Consider the PWA system (4) satisfying Assumption (A2). Then for every Filippov solution $\xi : [0, \infty) \to \mathbb{R}^n$ there exists a family of open intervals $(\mathcal{I}_k)_{k\in K}$, $K$ some index set, such that $[0, \infty) \setminus \bigcup_{k\in K} \mathcal{I}_k$ is at most *countable* and $\xi$ is on each interval $\mathcal{I}_k$ either a single-mode Caratheodory or a sliding solution.

*Proof:* We will construct the desired family of intervals as follows. Let $t_0 := 0$ and choose $t_{\ell+1} > t_\ell$ inductively by the condition that $\xi$ is either a single-mode Caratheordory, a sliding or a forward Zeno solution on $[t_\ell, t_{\ell+1})$. If $\xi$ is a single-mode Caratheodory or a sliding solution we add the open interval $(t_\ell, t_{\ell+1})$ to our family of intervals; for a forward Zeno solution we add the corresponding countable family of open subintervals $(t_\ell + \varepsilon_{k+1}, t_\ell + \varepsilon_k)$, $k \in \mathbb{N}$ to the family of intervals. If $t_{\ell+1} = \infty$ for some $\ell$ or $t_\ell \to \infty$ the claim of the lemma is shown. Otherwise repeat the procedure with the new initial time $t_0 := \lim_{\ell\to\infty} t_\ell$. By adding the countably many end-points of the open intervals we completely cover the interval $[0, \infty)$ and on each open interval $\xi$ is either a single-mode Caratheodory or a sliding solution. ∎

*Remark 10 (A "counterexample" to Lemma 9):* One may be tempted to argue that the statement of Lemma 9 is a simple corollary from the much more general statement about the membership property of an absolutely continuous trajectory with respect to a compact set in $\mathbb{R}^n$:

For any absolutely continuous function $\xi : [0, \infty) \to \mathbb{R}^n$ and any compact set $X \subseteq \mathbb{R}^n$ there exists a family of open intervals $(\mathcal{I})_{k\in K}$ for some index set $K$ such that either $\xi(t) \in X$ for all $t \in \mathcal{I}_k$ or $\xi(t) \notin X$ for all $t \in \mathcal{I}_k$ and $[0,\infty) \setminus \bigcup_{k\in K} \mathcal{I}$ is countable.

However, this statement is not correct! A counterexample can be constructed already on the interval $[0,1]$ and in $\mathbb{R}^1$ as follows:

Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, \ldots\}$ be the (countable) set of rational numbers in the interval $[0,1]$ and let $r_i := 2^{-(i+1)}$ (then $\sum_{i=1}^{\infty} r_i = 1/2$) and choose $\phi_i : [0,1] \to \mathbb{R}$ such that

- $\phi_i$ is smooth
- $\phi_i(q_i) = r_i$
- $\phi_i(t) = 0$ for all $t \in [0,1]$ with $|t - q_i| \geq r_i/2$
- $0 < \phi_i(t) \leq r_i$ for all $t \in [0,1]$ with $|t - q_i| < r_i/2$.

Then $\phi := \sum_{i=1}^{\infty} \phi_i$ is well defined (because $\sum_{i=1}^{\infty} \|\phi_i\|_\infty = 1/2 < \infty$) and smooth. Let $\lambda$ denote the Lesbesgue measure, then

$$\begin{aligned}&\lambda(\{\ t \in [0,1] \ \mid \ \phi(t) \neq 0\ \}) \\ &= \lambda\left(\bigcup_{i\in\mathsf{N}} \{\ t \in [0,1] \ \mid \ |t - q_i| < r_i/2\ \}\right) \\ &\leq \sum_{i=1}^{\infty} \underbrace{\lambda(\{\ t \in [0,1] \ \mid \ |t - q_i| < r_i/2\ \})}_{=r_i} = 1/2.\end{aligned}$$

Hence the measure of all points $t$ where $\phi(t) = 0$ is positive, in particular, there are uncountably many such points. Furthermore, each $t \in [0,1]$ with $\phi(t) = 0$ cannot be contained in an interval $(a,b)$ with $a < b$ and $\phi$ being identically zero on $(a,b)$, because there exists a rational number $q \in (a,b)$ and $\phi(q) \neq 0$ by construction of $\phi$. Hence each of the uncountable many points $t \in [0,1]$ with $\phi(t) = 0$ is not contained in any open interval where $\phi$ is identically zero. ☡

## III. Pointwise mode classifications

### A. (Strict) forward and backward modes

In addition to the current modes $\Sigma^x$ of a point $x \in \mathbb{R}^n$ it is useful for the forthcoming stability analysis to introduce also backward and forward modes. Towards this end we first introduce the set of forward and backward feasible Filippov solutions as follows:

$$\mathcal{FS}_+^f(x_0) := \left\{\ \xi : [0,\infty) \to \mathbb{R}^n \ \middle|\ \begin{array}{l}\xi \text{ is a feasible} \\ \text{Filippov sol. of (4)} \\ \text{with } \xi(0) = x_0\end{array}\right\},$$

$$\mathcal{FS}_-^f(x_0) := \left\{ \xi : [-\omega, 0) \to \mathbb{R}^n \ \middle|\ \begin{array}{l}\xi \text{ is a feasible} \\ \text{Filippov sol. of (4)} \\ \text{with } \xi(0^-) = x_0 \\ \text{and maximal } \omega > 0\end{array}\right\}.$$

*Remarks 11:* Consider the PWA system (4) and the set of feasible forward and backward solutions as above.

1) Assumption (A1) yields that $\mathcal{FS}_+^f(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{R}^n$.
2) If general Filippov solutions would be considered in the definition of $\mathcal{FS}_-^f(x_0)$ then, by time-reversibility, it follows that $\mathcal{FS}_-^f(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{R}^n$ (and the corresponding $\omega$ would be infinity); however, by restricting ourself to feasible Filippov solutions, there may be initial values $x_0$ for which $\mathcal{FS}_-^f(x_0) = \emptyset$, i.e. these initial values cannot be reached via a feasible Filippov solution (cf. the example illustrated in Figure 1b, where the origin is not reachable via a feasible Filippov solution).
3) The possibility to have $\mathcal{FS}_-^f(x_0) = \emptyset$ is another motivation to consider only feasible Filippov solutions: Having $\mathcal{FS}_-^f(x_0) = \emptyset$ will significantly reduce the number of jump-conditions for the forthcoming stability result (in fact, for points which are not reachable the corresponding Lyapunov-functions do not need to satisfy any additional "crossing-condition" in those points). ☡

*Definition 12 (Forward and strict forward mode):* For $x \in \mathbb{R}^n$ we call $s \in \Sigma$ a *forward mode* for $x$ with respect to the PWA system (4) if there exists a solution $\xi \in \mathcal{FS}_+^f(x)$ such that $\xi(t) \in X_s$ for infinitely many small $t > 0$, or, more formally, the set of all forward modes for $x$ is

$$\Sigma_+^x := \bigcup_{\xi \in \mathcal{FS}_+^f(x)} \bigcap_{\varepsilon > 0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(\tau)}.$$

We call $s \in \Sigma$ a *strict forward mode* for $x \in \mathbb{R}^n$ if there exists a single-mode Caratheodory solution $\xi : [0,\varepsilon) \to \mathbb{R}^n$, $\varepsilon > 0$, such that $\xi(t) \in \operatorname{int} X_s$ for all $t \in (0,\varepsilon)$; the set of all strict forward modes for $x$ is denoted by $\Sigma_{++}^x$.

*Definition 13 (Backward and strict backward mode):* For $x \in \mathbb{R}^n$ we call $s \in \Sigma$ a *backward mode* of $x$ with respect to the PWA system (4) if there exists a solution $\xi \in \mathcal{FS}_-^f(x)$ such that $\xi(-t) \in X_s$ for infinitely many small $t > 0$, or, more formally, the set of all backwards modes for $x$ is[2]

$$\Sigma_-^x := \bigcup_{\xi \in \mathcal{FS}^f(x)} \bigcap_{\varepsilon > 0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(-\tau)}.$$

A mode $s \in \Sigma$ is a *strict backward mode* for $x$ if it is a strict forward mode for the time-reversed PWA system (4), i.e. if there exists a single-mode Caratheodory solution $\xi : (-\varepsilon, 0] \to \mathbb{R}^n$, $\varepsilon > 0$ with $\xi(t) \in \operatorname{int} X_s$ for all $t \in (-\varepsilon, 0)$; the set of all strict backward modes for $x$ is denoted by $\Sigma_{--}^x$.

It is clear, that strict forward/backward modes are always forward/backward modes, i.e. $\Sigma_{++}^x \subseteq \Sigma_+^x$ and $\Sigma_{--}^x \subseteq \Sigma_-^x$. Furthermore, if some point $x \in \mathbb{R}^n$ is not reachable via a feasible Filippov solution (i.e. $\mathcal{FS}_-^f(x) = \emptyset$) then there are no backwards mode for $x$, i.e. $\Sigma_-^x = \emptyset$.

Concerning some typical solution behaviors around a point $x$ on a $n-1$-dimensional boundary $X_i \cap X_j$ we can formulate the following (informal) "classifications" (cf. a similar classification in [25, Sec. 3.1]):

[2]We use the convention that $^{\xi(-\tau)} = \emptyset$, whenever $\tau > \omega$ and $\xi \in \mathcal{FS}_-^f(x)$ is only defined on $[-\omega, 0)$. Furthermore, if $\mathcal{FS}_-^f(x) = \emptyset$ then we use the convention that a union over an empty index-set is the empty set.

- $x$ is a $(i,j)$-"crossing" point $\Leftrightarrow \Sigma_-^x = \{i\}$, $\Sigma_+^x = \{j\}$.
- $x$ is a "splitting" point $\Leftrightarrow \Sigma_-^x = \emptyset$ and $\Sigma_+^x = \{i,j\}$.
- $x$ is a "sliding" point $\Leftrightarrow \Sigma_-^x = \Sigma_+^x = \{i,j\}$.

### B. Sliding modes

The situation $\Sigma_-^x = \Sigma_+^x = \{i,j\}$ for some $x \in X_i \cap X_j$ which indicates possible sliding behavior along the boundary, can also occur for Caratheodory solutions passing through $x$ (when at least one vector field is tangential to the boundary). In order to distinguish genuine sliding behavior from "classical" solution behavior, we introduce the following index set.

*Definition 14 (Sliding mode):* We call $s \in \Sigma$ a sliding mode for $x \in \mathbb{R}^n$ with respect to the PWA system (4) if there is a (feasible) sliding solution $\xi : [t_0, T) \to \mathbb{R}^n$ with $\xi(t_0) = x$ and $s \in \Sigma_{\text{slide}}^{\xi(\cdot)}$, with $\Sigma_{\text{slide}}^{\xi(\cdot)}$ as in Definition 4.

Even in the planar case there are much more complicated solution behaviors possible, in particular, for points $x$ which are located at the boundary of a boundary (i.e. on intersections of boundaries). While in the planar case these boundaries of boundaries have dimension zero (i.e. are isolated points), in higher dimension these boundaries can have positive dimension without being $n-1$-dimensional faces.

*Example 15 (Examples 2 and 7 revisited):* Consider the PWA system from Example 2 exhibiting backward Zeno behavior. After the trajectory has reached the origin in finite time only a sliding Filippov solution exists (which remains in the origin). For any point $x \neq 0$ there is exactly one forward and backward mode, so the solution behavior is rather standard away from the origin. However, for $x = 0$ we have $\Sigma_+^0 = \Sigma_-^0 = \Sigma_{\text{slide}}^0 = \{1,2,3,4\}$, $\Sigma_{++}^0 = \Sigma_{--}^0 = \emptyset$ and for any solution $\xi : [-\omega, \infty) \to \mathbb{R}^2$ with $\xi(0) = 0$ and $\xi(-t) \neq 0$ for all $t \in (0, \omega)$ we have that $\Sigma_\pm^{\xi(-t)}$ only contains *one* mode each.

It is also possible to revert the direction of the vector fields, then there will be (many) non-single-mode Caratheodory solutions starting at the origin (and there is no feasible Filippov solution reaching the origin), i.e. for this different planar system we have $\Sigma_+^0 = \{1,2,3,4\}$, $\Sigma_-^0 = \Sigma_{++}^0 = \Sigma_{--}^0 = \emptyset$ and for any solution $\xi : [0,\infty) \to \mathbb{R}^2$ we have that $\Sigma_\pm^{\xi(t)}$ only contains one mode each for any $t > 0$. Moreover there exists an unnecessary sliding solution starting and remaining in the origin, i.e. an infeasible Filippov solution.

We also discuss the mode sets for Example 7: the sets $\Sigma_+^x$ and $\Sigma_-^x$ contain exactly one element for all $x \notin X_1 \cap X_2$ and for these $x$ also $\Sigma_{\text{slide}}^x = \emptyset$, $\Sigma_{--}^x = \Sigma_-^x$ and $\Sigma_{++}^x = \Sigma_+^x$. For $x \in \operatorname{ri}(X_1 \cap X_2)$ we have $\Sigma_+^x = \Sigma_-^x = \Sigma_{\text{slide}}^x = \Sigma_{--}^x = \{1,2\}$ and $\Sigma_{++}^x = \emptyset$. Finally, for $x = 0$ the situation is quite interesting: $\emptyset = \Sigma_-^0 \subsetneq \Sigma_{\text{slide}}^0 = \{1,2\} \subsetneq \Sigma_+^0 = \{1,2,3\}$ and $\Sigma_{--}^0 = \Sigma_-^0$, but $\Sigma_{++}^0 = \{3\} \neq \Sigma_+^0$. ∎

In higher dimension it is also possible (especially on boundaries with dimensions less then $n-1$) to have that there are multiple forward modes, multiple backwards modes and for example the following situation is possible:

$$\emptyset \neq \Sigma_-^x \cap \Sigma_+^x \subsetneq \Sigma_\pm^x \subsetneq \Sigma_+^x \cup \Sigma_-^x.$$

While there is no general subspace-relationship between $\Sigma_+^x$ and $\Sigma_-^x$ the following properties of the backward, current and forward modes are always true:

*Lemma 16:* Consider the PWA system (4) with corresponding mode sets $\Sigma^x$, $\Sigma_+^x$ and $\Sigma_-^x$ for $x \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$ and any $\xi \in \mathcal{FS}_+^f(x)$ the following holds.

(i) $\Sigma_+^x \subseteq \Sigma^x$ and $\Sigma_-^x \subseteq \Sigma^x$.

(ii) If $\Sigma^{\xi(\varepsilon)} = \Sigma^x$ for all sufficiently small $\varepsilon > 0$ then $\Sigma^x = \Sigma_+^x$.

(iii) $\forall t_0 > 0 \ \exists \varepsilon^* > 0 \ \forall \tau^* \in (0, \varepsilon^*)$ :

$$\Sigma_+^{\xi(t_0 - \tau^*)} \cap \Sigma_-^{\xi(t_0)} \neq \emptyset, \tag{6}$$

(iv) $\forall t_0 \geq 0 \ \exists \varepsilon^* > 0 \ \forall \tau^* \in (0, \varepsilon^*)$ :

$$\Sigma_+^{\xi(t_0 + \tau^*)} \subseteq \Sigma_+^{\xi(t_0)}. \tag{7}$$

Before proving the above Lemma, we would like to give some remarks about the subspace relationships.

*Remarks 17:* Concerning the four statements of Lemma 16 we want to highlight the following:

(i) This statement means that trajectories can reach or leave some value $x \in \mathbb{R}^n$ only through regions in which $x$ is currently contained in; this is in fact a consequence of continuity of trajectories and closedness of the regions $X_s$.

(ii) This statement clarifies when equality may hold in the subspace relation $\Sigma_+^x \subseteq \Sigma^x$, apart from the trivial case when $x \in \operatorname{int} X_s$.

(iii) In general the subsets $\Sigma_+^x$ and $\Sigma_-^x$ don't have a specific relationship to each other (apart from being both subsets of $\Sigma^x$); in particular, they can be disjoint non-empty sets. However, for points on a trajectory reaching some $x \in \mathbb{R}^n$ sufficiently close to $x$ there is always at least one forward mode which is also a backward mode for $x$. This common mode, however, may depend on the point along the trajectory, cf. Example 2.

(iv) The final subspace relationships means that no additional forward modes can occur for points on a trajectory starting at $x$ and which are sufficiently close to $x$. ∎

*Proof of Lemma 16.* (i) Let $s \in \Sigma_+^x$. Then there exists $\overline{\xi} \in \mathcal{FS}_+^f(x)$ for which for all $\varepsilon > 0$ there is a $\tau \in (0, \varepsilon)$ such that $\overline{\xi}(\tau) \in X_s$. In particular, there is a sequence $(t_k)_{k \in \mathbb{N}}$ of positive numbers with $t_k \to 0$ as $k \to \infty$ and $\overline{\xi}(t_k) \in X_s$. By continuity of $\overline{\xi}$ and closeness of $X_s$ it therefore follows that $x = \overline{\xi}(0) \in X_s$, and hence $s \in \Sigma^x$. The analogous argument shows that $\Sigma_-^x \subseteq \Sigma^x$ (unless $\Sigma_-^x = \emptyset$, but in this case the subspace inclusion holds trivially).

(ii) By assumption, for all $\varepsilon > 0$ exists $\tau \in (0, \varepsilon)$ such that $\Sigma^{\xi(\tau)} = \Sigma^x$, consequently $\xi(\tau) \in \bigcap_{\overline{s} \in \Sigma^x} X_{\overline{s}} \subseteq X_s$ for any $s \in \Sigma^x$. Therefore, by definition, $s \in \Sigma_+^x$ for any $s \in \Sigma^x$. This shows $\Sigma^x \subseteq \Sigma_+^x$ and together with (i) the claim is shown.

(iii) First note that $\mathcal{FS}_-^f(\xi(t_0))$ is nonempty because $\xi(\cdot + t_0)$ is a solution defined on $[-t_0, \infty)$ with $t_0 > 0$ and passing through $\xi(t_0)$ at $t = 0$. For any $\overline{\xi} \in \mathcal{FS}_-^f(\xi(t_0))$ and $\varepsilon > 0$

let

$$\Sigma_-^{\bar{\xi},\varepsilon} := \bigcup_{\tau\in(0,\varepsilon)} \Sigma^{\bar{\xi}(-\tau)}.$$

Clearly, for $0 < \varepsilon_1 < \varepsilon_2$, $\Sigma_-^{\bar{\xi},\varepsilon_1} \subseteq \Sigma_-^{\bar{\xi},\varepsilon_2} \subseteq \Sigma$. Since $\Sigma$ is finite, the sequence $\Sigma_-^{\bar{\xi},\varepsilon}$ must get stationary as $\varepsilon \to 0$. In other words, there exists an $\bar{\varepsilon} > 0$ (depending on $\bar{\xi}$ and $\xi(t_0)$) such that for all $\varepsilon \in (0, \bar{\varepsilon})$: $\Sigma_-^{\bar{\xi},\varepsilon} = \Sigma_-^{\bar{\xi},\bar{\varepsilon}}$. In particular, for $\bar{\xi} \in \mathcal{FS}_-^f(\xi(t_0))$ with $\bar{\xi}(t) = \xi(t + t_0)$ for $t \in [-t_0, 0)$ let $\varepsilon^* = \bar{\varepsilon}$, then

$$\Sigma_-^{\xi(t_0)} = \bigcup_{\bar{\xi}\in\mathcal{FS}^f(\xi(t_0))} \Sigma_-^{\bar{\xi},\bar{\varepsilon}}$$

$$\supseteq \Sigma_-^{\xi(\cdot+t_0),\varepsilon} = \bigcup_{\tau\in(0,\varepsilon\ )} \Sigma^{\xi(t_0-\tau)}. \quad (8)$$

Now let $\tau^* \in (0, \varepsilon^*)$ and we will show that there is $\bar{\tau} \in (0, \tau^*)$ such that

$$\Sigma^{\xi(t_0-\bar{\tau})} \subseteq \Sigma_+^{\xi(t_0-\tau\ )}. \quad (9)$$

We have, using the same finiteness argument as above,

$$\Sigma_+^{\xi(t_0-\tau\ )} \supseteq \bigcap_{\varepsilon>0} \bigcup_{\tau\in(0,\varepsilon)} \Sigma^{\xi(t_0-\tau\ +\tau)} = \bigcup_{\tau\in(0,\tilde{\varepsilon})} \Sigma^{\xi(t_0-\tau\ +\tau)},$$

where $\tilde{\varepsilon} > 0$ is chosen sufficiently small. For some $\tau \in (0, \min\{\tilde{\varepsilon}, \tau^*\})$ let $\bar{\tau} := \tau^* - \tau > 0$, then (9) holds. Since $\bar{\tau} < \tau^* < \varepsilon^*$, we also have

$$\Sigma^{\xi(t_0-\bar{\tau})} \subseteq \bigcup_{\tau\in(0,\varepsilon\ )} \Sigma^{\xi(t_0-\tau)}$$

and together with (8) and (9) we can conclude that

$$\emptyset \neq \Sigma^{\xi(t_0-\bar{\tau})} \subseteq \Sigma_+^{\xi(t_0-\tau\ )} \cap \Sigma_-^{\xi(t_0)}.$$

(iv) Assume there is $t_0 \geq 0$ such that for all $\varepsilon > 0$ there is $\tau_\varepsilon^* \in (0, \varepsilon)$ such that (7) does not hold, i.e. there is $s_{\tau_\varepsilon} \in \Sigma_+^{\xi(t_0+\tau_\varepsilon)}$ with $s_{\tau_\varepsilon} \notin \Sigma_+^{\xi(t_0)}$. Because $s_{\tau_\varepsilon}$ is contained in the finite set $\Sigma$ for all $\varepsilon > 0$, there is a decreasing sequence $(\varepsilon_k)_{k\in\mathsf{N}}$ of positive numbers converging to zero and an $s^* \in \Sigma$ such that

$$\forall k \in \mathbb{N}: \quad s^* \in \Sigma_+^{\xi(t_0+\tau_{\varepsilon_k})} \setminus \Sigma_+^{\xi(t_0)}.$$

By redefining $\tau_\varepsilon^* := \tau_{\varepsilon_k}^*$ for $\varepsilon \in (\varepsilon_k, \varepsilon_{k+1})$ we therefore have for all $\varepsilon > 0$ that there exists a $\tau_\varepsilon^* \in (0, \varepsilon)$ such that $s^* \in \Sigma_+^{\xi(t_0+\tau_\varepsilon)} \setminus \Sigma_+^{\xi(t_0)}$. Consequently there exists $\bar{\xi}_{\tau_\varepsilon} \in \mathcal{FS}(\xi(t_0 + \tau_\varepsilon^*))$ such that

$$\forall \bar{\varepsilon} > 0\ \exists \bar{\tau} \in (0, \bar{\varepsilon}) : \bar{\xi}_{\tau_\varepsilon}(\bar{\tau}) \in X_s\ .$$

The latter allows us to chose a sequence $(\bar{\tau}_k)_{k\in\mathsf{N}}$ converging to zero with $\bar{\xi}_{\bar{\tau}_\varepsilon}(\tau_k) \in X_s$ . By continuity of $\bar{\xi}_{\tau_\varepsilon}$ and closedness of $X_s$ it follows that

$$\xi(t_0 + \tau_\varepsilon^*) = \bar{\xi}_{\tau_\varepsilon}(0) \in X_s\ .$$

Hence there exists a solution $\bar{\xi}$ starting at $\xi(t_0)$, namely $\bar{\xi} = \xi(\cdot + t_0)$, such that for all $\varepsilon > 0$ there exists $\tau \in (0, \varepsilon)$, namely $\tau = \tau_\varepsilon^*$, such that $\bar{\xi}(\tau) = \xi(t_0 + \tau_\varepsilon^*) \in X_s$ , i.e. $s^* \in \Sigma_+^{\xi(t_0)}$. This contradicts our assumption and we have therefore shown that (7) holds. ■

## IV. Stability with piecewise Lyapunov functions

We will now study stability of the PWA system (4) with (feasible) Filippov solutions.

*Definition 18 (Global asymptotic stability):* The PWA (4) is called *stable* iff

(S1) $\mathcal{FS}_+^f(x_0) \neq \emptyset$ for all $x_0 \in \mathbb{R}^n$ and all (feasible Filippov) solutions are defined on $[0, \infty)$.

(S2) The origin is stable, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all solutions $\xi \in \mathcal{FS}_+^f(x_0)$ the following implication holds:

$$\|\xi(0)\| < \delta \implies \|\xi(t)\| < \varepsilon \quad \forall t \geq 0.$$

It is called *globally asymptotically stable* if additionally the origin is globally attractive, i.e.

(S3) $\xi(t) \to 0$ as $t \to \infty$ for all $\xi \in \mathcal{FS}_+^f(x_0)$ and all $x_0 \in \mathbb{R}^n$.

Assumption (A1) ensures that condition (S1) is satisfied, this would not be the case when considering Caratheodory solutions or when the partition of $\mathbb{R}^n$ has infinitely many elements. For linear systems attractivity already implies stability of the origin, however for PWA systems this is not necessarily the case; as an example consider a planar PWA system qualitatively given in Figure 4.

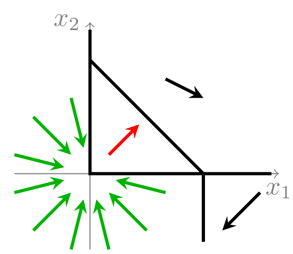


Fig. 4: A PWA system whose origin is attractive but where solution starting close to zero can first go away by a certain minimal amount before coming back.

Our goal is to prove stability of the PWA system (4) via a piecewisely defined Lyapunov function. For this we first define "local" Lyaypunov functions.

*Definition 19 (Local Lyapunov function):* Consider the PWA system (4). We call $V_s : \mathbb{R}^n \to \mathbb{R}$ a *local Lyapunov function for mode* $s \in \Sigma$ iff

(L1) $V_s$ is continuous on $\mathbb{R}^n$ and continuously differentiable on $X_s$.

(L2) $V_s$ is positive definite on $X_s$, i.e. $V_s(x) > 0$ for all $x \in X_s \setminus \{0\}$ and if $0 \in X_s$ then $V_s(0) = 0$.

(L3) $V_s$ is radially unbounded in the following sense:

$$\forall \bar{v} \in V_s(X_s) \subseteq \mathbb{R}^n : V_s^{-1}([0, \bar{v}]) \cap X_s \text{ is compact},$$

(L4) $V_s$ is decreasing along "classical" solutions within $X_s$ in the following sense

$$\nabla V_s(x)(A_s x + b_s) < 0 \quad \forall x \in X_s \setminus \{0\},$$

*Remark 20:* If $X_s$ is bounded (and hence compact) continuity of $V_s$ already implies that (L3) is satisfied. Furthermore, conditions (L2) and (L3) together with continuity of

$V_s$ yields that

$$\forall \varepsilon > 0\ \exists \gamma_s^\varepsilon > 0 : \quad V_s^{-1}([0, \gamma_s^\varepsilon]) \cap X_s \subseteq \mathbb{B}_\varepsilon. \tag{10}$$

Note that (10) is trivially satisfied for all modes $s \in \Sigma$ with $0 \notin X_s$, because from continuity and (L3) it follows that $\min_{x \in X_s} V_s(x) > 0$, hence $V^{-1}([0, \gamma_s^\varepsilon]) \cap X_s = \emptyset$ for sufficiently small $\gamma_s^\varepsilon > 0$. Finally, condition (L4) can slightly be relaxed, because it is not necessary to require a decreasing local Lyapunov function in points where the trajectory leaves $X_s$. ☡

The challenge is to formulate suitable compatibility conditions for this Lyapunov function on the boundaries. The simplest case (but also most restrictive case) is the assumption that there is a common Lyapunov function for all modes, then stability is obviously guaranteed. It is common to assume continuity of the local Lyapunov functions across the boundaries, then asymptotic stability is guaranteed if no sliding and no Zeno-behavior occur. However, requiring continuity is neither necessary nor sufficient for proving stability; for the latter see e.g. [1, Example 4.9].

Our main result will not impose continuity of the local Lyapunov functions across the boundaries, but we will now present weaker suitable compatibility conditions which, if satisfied, ensure stability of the PWA system (4) with feasible Filippov solutions; including sliding and Zeno behaviors as well as non-unique solutions.

*Theorem 21:* Consider the PWA system (4) satisfying Assumptions (A1) and (A2). Assume that for each mode $s \in \Sigma$ there is a local Lyapunov-function $V_s : \mathbb{R}^n \to \mathbb{R}$ as in Definition 19. Furthermore, assume that the different Lyapunov functions are compatible in the following sense:

(B1) $\forall x \in \mathbb{R}^n\ \forall (i,j) \in \Sigma_-^x \times \Sigma_+^x : \quad V_i(x) \geq V_j(x)$.

(B2) $\exists \mu > 0\ \forall x \in \mathbb{R}^n$ with $\Sigma_{\text{slide}}^x \neq \emptyset\ \exists i_x \in \Sigma_{\text{slide}}^x$ :

$$\nabla V_{i_x}(x)(A_j x + b_j) \leq -\mu \|x\| \quad \forall j \in \Sigma_{\text{slide}}^x. \tag{11}$$

Then (4) is globally asymptotically stable

Before proving our main result, we would like give a few remarks.

*Remarks 22:* 1) Conditions (B1) and (B2) are trivially satisfied for all $x$ in the interior of some $X_s$, hence it need only to be checked for points $x$ on the boundaries. Furthermore, (B1) is also trivially satisfied for those $x$ with $\Sigma_-^x = \emptyset$.

2) We do not explicitly require equality of the Lyapunov function values at sliding points. However, for a sliding solution $\xi : [0, \omega) \to \mathbb{R}^n$ it will turn out, that for almost all $t \in [0, \omega)$ the equality $\Sigma_-^{\xi(t)} = \Sigma_+^{\xi(t)}$ holds; consequently, (B1) implicitly implies equality of the Lyapunov function values.

3) Condition (B2) is satisfied if the in general stronger conditions $\nabla V_i(x) = \nabla V_j(x)$ for all $i, j \in \Sigma_{\text{slide}}^x$ holds. Note that, similar as in [1], we are *not* requiring (11) to hold for *all pairs* $(i,j) \in \Sigma_{\text{slide}}^x \times \Sigma_{\text{slide}}^x$, this is in contrast to other recent approaches, see e.g. [13].

4) It is straightforward to extend Definition 18 to PWA systems (4) with general Filippov solutions (i.e. not restricting the solution space to *feasible* Filippov solutions). Then Assumption (A1) can be dropped in the formulation of Theorem 21. However, in that case $\Sigma_-^x$ will never be empty, so that jump condition (B1) have to be satisfied on *all* boundaries; in particular, for "splitting" boundaries an "unnecessary" sliding can occur, which in turn enforces an "unnecessary" continuity requirement of the Lyapunov function on that boundary. ☡

*Proof of Theorem 21.* Let

$$V(x) := \max_{s \in \Sigma_+^x} V_s(x)$$

We will now proof global asymptotic stability of (4) in several steps.

*Step 1: We show that $V$ is decreasing along solutions.* Let $\xi : [0, \infty) \to \mathbb{R}^n$ be a feasible Filippov solution of (4) and let

$$v(t) := V(\xi(t)).$$

Note that by positive definitness of the local Lyapunov-functions $v(t) = 0$ if, and only if, $\xi(t) = 0$.

*Step 1a:* We show that $v$ cannot jump upwards anywhere. Note that at this point it is not clear yet whether $v$ is left- or right-continuous. In particular, $v(t^-)$ and $v(t^+)$ may not be well defined and we therefore have to formulate the property "not jumping upwards at $t \in [0, \infty)$" as follows:

$$\lim_{\varepsilon \searrow 0} \inf_{\tau \in (0,\varepsilon)} v(t - \tau) \geq v(t) \geq \lim_{\varepsilon \searrow 0} \sup_{\tau \in (0,\varepsilon)} v(t + \tau). \tag{12}$$

Note that (12) is trivially satisfied (with equality) at all continuity points of $v$. In order the prove the left inequality of (12) for any $t > 0$ we first observe that for sufficiently small $\tau > 0$

$$V(\xi(t-\tau)) = \max_{i \in \Sigma_+^{\xi(t-\tau)}} V_i(\xi(t-\tau)) \overset{(6)}{\geq} \min_{i \in \Sigma_-^{\xi(t)}} V_i(\xi(t-\tau)). \tag{13}$$

Furthermore, from continuity of $\xi$ and of each $V_s$ together with finiteness of $\Sigma_-^{\xi(t)}$ we can conclude that

$$\lim_{\varepsilon \searrow 0} \inf_{\tau \in (0,\varepsilon)} \min_{i \in \Sigma_-^{\xi(t)}} V_i(\xi(t-\tau)) = \lim_{\varepsilon \searrow 0} \min_{i \in \Sigma_-^{\xi(t)}} V_i(\xi(t-\varepsilon)) = \min_{i \in \Sigma_-^{\xi(t)}} \lim_{\varepsilon \searrow 0} V_i(\xi(t-\varepsilon)) = \min_{i \in \Sigma_-^{\xi(t)}} V_i(\xi(t)). \tag{14}$$

Altogether we have

$$\lim_{\varepsilon \searrow 0} \inf_{\tau \in (0,\varepsilon)} v(t-\tau) \overset{(13)+(14)}{\geq} \min_{i \in \Sigma_-^{\xi(t)}} V_i(\xi(t)) \overset{\text{(B1)}}{\geq} \max_{j \in \Sigma_+^{\xi(t)}} V_j(\xi(t)) = v(t).$$

The right inequality of (12) for $t \geq 0$ is shown as follows:

$$\begin{aligned} v(t) &= \max_{s\in\Sigma_+^{\xi(t)}} V_s(\xi(t) = \max_{s\in\Sigma_+^{\xi(t)}} \lim_{\varepsilon\searrow 0} V_s(\xi(t+\varepsilon)) \\ &= \lim_{\varepsilon\searrow 0} \max_{s\in\Sigma_+^{\xi(t)}} V_s(\xi(t+\varepsilon)) \\ &= \lim_{\varepsilon\searrow 0} \sup_{\tau\in(0,\varepsilon)} \max_{s\in\Sigma_+^{\xi(t)}} V_s(\xi(t+\tau)) \\ &\overset{(7)}{\geq} \lim_{\varepsilon\searrow 0} \sup_{\tau\in(0,\varepsilon)} \max_{s\in\Sigma_+^{\xi(t+\tau)}} V_s(\xi(t+\tau)) \\ &= \lim_{\varepsilon\searrow 0} \sup_{\tau\in(0,\varepsilon)} v(t+\tau). \end{aligned}$$

*Step 1b:* We show that $v$ is decreasing on intervals where $\xi$ is a single-mode Caratheodory solution.
Let $\mathcal{I} \subseteq [0,\omega)$ be an open interval on which $\xi$ is a single-mode Caratheodory solution not passing through the origin, i.e. $\xi(t) \in X_s \setminus \{0\}$ for some $s \in \Sigma$ and all $t \in \mathcal{I}$ and $\dot{\xi}(t) = A_s\xi(t) + b_s$ for almost all $t \in \mathcal{I}$. We first show that then $v(t) = V_s(\xi(t))$. By construction, $v(t) \geq V_s(\xi(t))$. Furthermore, from $\xi(t-\varepsilon) \in X_s$ for all sufficiently small $\varepsilon > 0$ it follows that $s \in \Sigma_-^{\xi(t)}$. Hence, by (B1), $V_s(\xi(t)) \geq \max_{j\in\Sigma_+^{\xi(t)}} V_j(\xi(t)) = v(t)$, which shows $v(t) = V_s(\xi(t))$ for all $t \in \mathcal{I}$. Hence, $v$ is absolutely continuous on $\mathcal{I}$ and for almost all $t \in \mathcal{I}$,

$$\dot{v}(t) = \nabla V_s(\xi(t))(A_s\xi(t) + b_s) \overset{\text{(L4)}}{<} 0.$$

*Step 1c:* We show that $v$ is decreasing on intervals where $\xi$ is a sliding solution.
Let $\mathcal{I} \subseteq [0,\omega)$ be an open interval on which $\xi$ is a sliding solution not passing through the origin. Hence there exists $S \subseteq \Sigma$ with $S = \Sigma^{\xi(t)}$ and $\xi(t) \neq 0$ for all $t \in \mathcal{I}$. From Lemma 16 and by assumption we can conclude that $S = \Sigma_+^{\xi(t)} = \Sigma_{\text{slide}}^{\xi(t)}$ for all $t \in \mathcal{I}$. With an analogous argument as in the proof of Lemma 16(ii) we can also conclude that $\Sigma_-^{\xi(t)} = \Sigma^{\xi(t)} = S$ for all $t \in \mathcal{I}$. Hence by (B1) we have $V_i(\xi(t)) = V_j(\xi(t))$ for all $i, j \in S$ and all $t \in \mathcal{I}$. In particular, $v(t) = V_s(\xi(t))$ for any $s \in S$ and all $t \in \mathcal{I}$ and, therefore, $v$ is absolutely continuous on $\mathcal{I}$ and for almost all $t \in \mathcal{I}$

$$\dot{v}(t) = \nabla V_s(\xi(t)) \sum_{j\in S} \lambda_j(t)(A_j\xi(t) + b_j)$$

for some $\lambda_j(t) \in [0,1]$ with $\sum_{j\in S}\lambda_j(t) = 1$ and any $s \in S$. By Assumption (B2) we can pick for each $t$ an index $i_t \in S = \Sigma_{\text{slide}}^{\xi(t)}$ such that $\nabla V_{i_t}(\xi(t))(A_j\xi(t) + b_j) < 0$ for all $t \in \mathcal{I}$ and all $j \in \Sigma_{\text{slide}}^{\xi(t)} = S$. Consequently,

$$\dot{v}(t) = \sum_{j\in S} \lambda_j(t) \nabla V_{i_t}(\xi(t))(A_j\xi(t) + b_j) < 0.$$

*Step 1d:* We show monotonicity of $v$.
Invoking Lemma 9 we can conclude that $t \mapsto v(t)$ has at most countable many discontinuities and is differentiable almost everywhere. By Step 1a, $v$ is not increasing at the discontinuities and has negative derivate for almost all $t$ where $v(t) > 0$ by Steps 1b and 1c. If $v(t_0) = 0$ for some $t_0 > 0$ then $v(t) = 0$ for all $t \geq t_0$, because assuming the contrary immediately results in a contradiction to Steps 1a, 1b and/or 1c. Altogether this shows that $v$ is strictly decreasing as long as $v(t) > 0$ and remains at zero once it reaches zero.

*Step 2: We show stability of the origin.*
We will show that for all $\varepsilon > 0$ there exists $\gamma, \delta > 0$ such that

$$\mathbb{B}_\delta \subseteq V^{-1}([0,\gamma]) \subseteq \mathbb{B}_\varepsilon.$$

It then follows that for any solution $\xi : [0,\infty) \to \infty$ of (4) with $\|\xi(0)\| < \delta$ we have $V(\xi(t)) \leq V(\xi(0)) \leq \gamma$ and hence $\xi(t) \in V^{-1}([0,\gamma]) \subseteq \mathbb{B}_\varepsilon$, i.e. $\|\xi(t)\| \leq \varepsilon$.

For $s \in \Sigma$ choose $\gamma_s^\varepsilon > 0$ as in (10) and let $\gamma := \min_{s\in\Sigma}\gamma_s^\varepsilon$ then

$$\forall s \in \Sigma : \quad V_s^{-1}([0,\gamma]) \cap X_s \subseteq \mathbb{B}_\varepsilon.$$

Or in other words, for all $x \in \mathbb{R}^n$ and all $s \in \Sigma^x$ it follows from $V_s(x) \leq \gamma$ that $x \in \mathbb{B}_\varepsilon$. The implication remains true if the stronger assumption $\max_{s\in\Sigma_+^x} V_s(x) \leq \gamma$ is used instead (taking into account that $\Sigma_+^x \subseteq \Sigma^x$), hence we have shown that $V^{-1}([0,\gamma]) \subseteq \mathbb{B}_\varepsilon$.

To show $\mathbb{B}_\delta \subseteq V^{-1}([0,\gamma])$ we first observe that for those $s \in \Sigma$ for which $0 \in X_s$ we have by assumption (L2) that $V_s(0) = 0$ and continuity of $V_s$ at $x = 0$ means that there is $\delta_s > 0$ such that $V_s(\mathbb{B}_{\delta_s}) \subseteq [0,\gamma]$, and hence also

$$V_s(\mathbb{B}_{\delta_s} \cap X_s) \subseteq [0,\gamma]. \tag{15}$$

For those $s \in \Sigma$ with $0 \notin X_s$ we chose $\delta_s > 0$ smaller than the (positive) distance of 0 to $X_s$; by this choice (15) is trivially satisfied also for those $s$. Consequently,

$$V(x) = \max_{s\in\Sigma_+^x} V_s(x) \leq \max_{s\in\Sigma^x} V_s(x) \leq \gamma \quad \forall x \in \mathbb{B}_\delta \cap X_s$$

where $\delta := \min_{s\in\Sigma}\delta_s > 0$. Since, by definition, $s \in \Sigma^x$ if, and only if, $x \in X_s$ the latter implies $V(\mathbb{B}_\delta) \subseteq [0,\gamma]$ which in turn implies the desired subset relationship.

*Step 3: We show that $V$ converges towards zero along solutions.*
We first show that any solution $\xi : [0,\infty) \to \mathbb{R}^n$ evolves within a compact set. For that let

$$t_s := \inf\left\{ t \in [0,\infty) \;\middle|\; s \in \Sigma_+^{\xi(t)} \wedge V_s(\xi(t)) = V(\xi(t)) \right\}$$

be the first time, the local Lyapunov function of mode $s$ determines the global value of the Lyapunov function (note however, that in general $V_s(\xi(t_s))$ may be smaller than $V(\xi(t_s))$). Note that $t_s = \infty$ is possible, for example, when $\xi$ is not evolving through $X_s$. It then follows that for any $t \in [0,\infty)$ and for any $s \in \Sigma_+^{\xi(t)}$ with $t_s < t$ we have by monotonicity of $V(\xi(\cdot))$ that

$$V_s(\xi(t)) \leq V(\xi(t)) \leq V(\xi(t_s + \varepsilon_k)) = V_s(\xi(t_s + \varepsilon_k))$$

for a suitable sequence of nonnegative[3] numbers $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ as $k \to \infty$ and therefore, by continuity of $V_s$,

$$V_s(\xi(t)) \leq V_s(\xi(t_s)) =: \overline{v}_s. \tag{16}$$

[3]If $t_s$ is actually a minimum (instead of the infimum) then $\varepsilon_k = 0$ for all $k \in \mathbb{N}$ can be chosen.

Since for every $t \in [0, \infty)$ there is always an $s_{\max} \in \Sigma_+^{\xi(t)}$ with $V_{s_{\max}}(\xi(t)) = V(\xi(t))$ we have $t \geq t_{s_{\max}}$. In conclusion, for *every* $t \in [0, \infty)$ we have an $s \in \Sigma$ such that $\xi(t) \in X_s$ and (16) holds and

$$\xi(t) \in \bigcup_{\substack{s\in\Sigma \\ t_s<\infty}} V_s^-([0, \overline{v}_s]) \cap X_s =: K.$$

By assumption (L3) we have that $K$ is compact.

Seeking a contradiction we now assume that $\lim v(t) := \underline{v} > 0$. As shown in Step 2 there is a $\delta > 0$ such that $V(\mathbb{B}_\delta) \subseteq [0, \underline{v}]$, hence we can conclude that $\xi$ evolves within the compact set $K_\delta := K \setminus \mathbb{B}_\delta$ which does not contain the origin. Hence for each $s$ where $t_s < \infty$ the continuous functions $x \mapsto |\nabla V_s(x)(A_s x + b_s)|$ attain a minimum on $K_\delta \cap X_s$, say $d_s$. Because of (L4) it holds that $d_s > 0$, hence $\dot{v}(t) \leq -\min_{s\in\Sigma} d_s =: -d < 0$ on intervals where $\xi$ is a single-mode Caratheodory solution (with the convention that $d_s = \infty$ if $t_s = \infty$). On intervals where $\xi$ is a sliding solution it follows from Step 1c and $\|\xi(t)\| \geq \delta$ that

$$\dot{v}(t) = \sum_{s\in S} \lambda_s(t) \nabla V_{i_t}(\xi(t))(A_s\xi(t) + b_s)$$
$$\overset{\text{(B2)}}{\leq} -\sum_{s\in S} \lambda_s(t)\mu\|\xi(t)\| \leq -\mu\delta,$$

where $S = \Sigma_{\text{slide}}^{\xi(t)}$, which, as shown in Step 1c, is independent of $t$ within a given interval on which $\xi$ is a sliding solution. Altogether we have for almost all $t \in [0, \infty)$ that

$$\dot{v}(t) \leq -\min\{d, \mu\delta\} < 0.$$

However, this contradicts $v(t) \geq 0$ and we have shown that $0 = \underline{v} = \lim_{t\to\infty} v(t)$.

*Step 4: We show that all solution converge to zero.* We have already shown in Step 2 that for all $\varepsilon > 0$ there is $\gamma > 0$ such that $V(x) \leq \gamma$ implies $\|x\| < \varepsilon$, hence $V(\xi(t)) \to 0$ as $t \to \infty$ implies $\xi(t) \to 0$ as $t \to \infty$. ∎

*Remark 23:* In the proof of Theorem 21 we did not explicitly utilize *linearity* of the individual modes, the *polyhedral nature* of the partition, nor the assumption that the relative interior of the intersection $X_i \cap X_j$ is empty. Therefore, we believe that Theorem 21 can be significantly generalized. However, the formal extension to the most general case is outside the scope of this paper as we would like to also present a constructive method to prove stability. ☡

## V. BOUNDARIES CHARACTERIZATIONS

The implementation of Theorem 21 requires tools for: i) checking the sign of candidate local Lyapunov functions in the corresponding polyhedra and their derivatives along the trajectories, and ii) verifying the pointwise conditions (B1) and (B2). The former issue will be tackle in Sec. VI by using the cone-copositive approach with quadratic functions. The pointwise conditions (B1) and (B2) can be recast in the same framework for verifying them on all points in some boundaries. This approach heavily relies on the assumption that the partition is chosen suitable in the sense that most of the boundaries have a uniform behavior with respect to pointwise conditions (B1) and (B2). Towards this goal we first use the short hand notation $\Sigma_+^{\text{ri}(X_\mathcal{B})}, \Sigma_-^{\text{ri}(X_\mathcal{B})}, \ldots$ to implicitly assume that $\Sigma_+^x, \Sigma_-^x, \ldots$ are the same for all $x \in \text{ri}(X_\mathcal{B})$. Now we can introduce the following boundary classification:

*Definition 24 (Boundary classification):* A (non-empty) boundary $X_\mathcal{B}$ for $\mathcal{B} \subseteq \Sigma$ is called

(i) *unreachable boundary* iff $\Sigma_-^{\text{ri}(X_\mathcal{B})} = \emptyset$;
(ii) *crossing boundary* iff it is not unreachable and $\Sigma_-^{\text{ri}(X_\mathcal{B})} \cap \Sigma_+^{\text{ri}(X_\mathcal{B})} = \emptyset$;
(iii) *Caratheodory boundary* iff $\Sigma_{\text{slide}}^{\text{ri}(X_\mathcal{B})} = \emptyset$ and $\mathcal{FS}_+^f(x) = \mathcal{CS}(x)$ for all $x \in \text{ri}(X_\mathcal{B})$;
(iv) *sliding boundary* iff $\Sigma_{\text{slide}}^{\text{ri}(X_\mathcal{B})} = \mathcal{B}$;
(v) *unclassified boundary* otherwise.

Definition 24(i) means that no solutions can reach that type of boundary. Definition 24(ii) means that for a crossing boundary there exists at least one backward mode and one (different) forward mode, although the type of backward and forward solutions could be different and each of them can be single-mode Caratheodory, sliding or Zeno. Definition 24(iii) means that all forward solutions (possibly Zeno) starting from the relative interior of that boundary are Caratheodory solutions. Therefore from Caratheodory boundaries cannot start sliding solutions neither forward Zeno solutions with pieces of sliding. Definition 24(iv) means that all solutions lying on that boundary are characterized by sliding.

The remainder of the section will present results which may assist the classification of the boundaries. However, there is no general method available yet to fully characterize a given boundary, so far we can only provide sufficient conditions; in particular some boundaries may remain "unclassified". However, this doesn't prevent our method to work in the sense that this will just impose stricter (possibly unnecessary) continuity assumptions on the sought PWQ Lyapunov function. Nevertheless, if the stability conditions return a solution it will result in a PWQ Lyapunov function proving asymptotic stability of the PWA system, even if too many boundaries were "unclassified".

Consider a boundary $X_\mathcal{B}$ of the partition with $\mathcal{B} \subseteq \Sigma$ the set of indices of all polyhedra sharing the relative interior of that boundary, say $\text{ri}(X_\mathcal{B})$, i.e. $\mathcal{B} = \Sigma^x$ for all $x \in \text{ri}(X_\mathcal{B})$. Consider a generic $s \in \mathcal{B}$. By definition, $X_\mathcal{B}$ is a face of $X_s$. In particular, it can be written as a finite intersection of some facets of $X_s$. Each of these facets is itself an intersection of $X_s$ with another polyhedron $X_\ell$, say $X_{\ell s} = X_\ell \cap X_s$, for some $\ell \in \mathcal{B}$. More specifically, for each boundary $X_\mathcal{B}$ and for each $s \in \mathcal{B}$, consider the set of indices $\mathcal{L}_s = \{\ell_1, \ell_2, \ldots, \ell_{\alpha_s}\} \subseteq \mathcal{B}$ such that

$$X_\mathcal{B} = \bigcap_{\ell\in\mathcal{L}_s} X_{\ell s}, \tag{17}$$

where $X_{\ell s}$, $\ell \in \mathcal{L}_s$, are facets of $X_s$. As an example, consider $x \in \mathbb{R}^3$ and the semiaxis $X_\mathcal{B} = \{x_1 \geq 0, x_2 = x_3 = 0\}$ as a boundary of the polyhedron $X_s = \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. Then (17) holds with $\mathcal{L}_s = \{\ell_1, \ell_2\}$, $X_{\ell_1} = \{x_1 \geq 0, x_2 \leq 0, x_3 \geq 0\}$, $X_{\ell_2} = \{x_1 \geq 0, x_2 \geq 0, x_3 \leq 0\}$.

Consider now the affine hull of each facet $X_{\ell s}$ which is an affine hyperplane

$$\mathcal{H}_{\ell s} = \left\{ \ x \in \mathbb{R}^n \ \middle| \ h_{\ell s}^\top x + g_{\ell s} = 0 \ \right\}$$

for some normal vector $h_{\ell s} \in \mathbb{R}^n$ and offset $g_{\ell s} \in \mathbb{R}$. For any normal vector $h_{\ell s}$ of $\mathcal{H}_{\ell s}$ also $\lambda h_{\ell s}$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ is a normal vector of $\mathcal{H}_{\ell s}$ (with offset $\lambda g_{\ell s}$). Hence it is no restriction of generality to assume that $h_{\ell s}$ is chosen such that it points from $X_\ell$ to $X_s$, i.e. we can assume that

$$h_{\ell s}^\top x + g_{\ell s} > 0, \quad x \in X_s \setminus X_{\ell s}, \tag{18a}$$
$$h_{\ell s}^\top x + g_{\ell s} < 0, \quad x \in X_\ell \setminus X_{\ell s}. \tag{18b}$$

Note that with this convention the normal vectors $h_{\ell s}$ and $h_{s\ell}$ will have opposite directions.

For the pointwise case, in the Appendix we report some iff conditions to determine whether for a given $x \in X_\mathcal{B}$ it is $s \in \Sigma_{++}^x$, $s \in \Sigma_{--}^x$ or neither of the two, see Lemma 33 and Lemma 34. In particular, if Lemma 33 is not satisfied for all $s \in \mathcal{B}$ then $\Sigma_{++}^x = \emptyset$. Analogously, if Lemma 34 is not satisfied for all $s \in \mathcal{B}$ then $\Sigma_{--}^x = \emptyset$. Even if we have $\Sigma_{++}^x = \Sigma_{--}^x = \emptyset$, i.e. the point $x$ can be classified to be "non single-mode Caratheodory", there are still three quite different cases possible:

- It is possible to leave $x$ via a forward Zeno solution.
- There is a sliding solution from $x$ along the boundary $X_\mathcal{B}$.
- There is sliding solution from $x$ leaving $X_\mathcal{B}$ and evolving along $X_{\mathcal{B}'}$ for some proper $\mathcal{B}' \subset \mathcal{B}$.

We are now ready to characterize the boundaries where a single-mode (forward and/or backward) Caratheodory solution exists, i.e. $\Sigma_{++}^x \neq \emptyset$ and/or $\Sigma_{--}^x \neq \emptyset$ for the points $x$ belonging to the boundary. For a boundary $X_\mathcal{B}$ such that the sets $\Sigma_{++}^x$ and $\Sigma_{--}^x$ do not depend on $x$ for all $x \in \mathrm{ri}(X_\mathcal{B})$, it is possible to define the sets $\Sigma_{++}^{\mathrm{ri}(X_\mathcal{B})} \subseteq \mathcal{B}$ and $\Sigma_{--}^{\mathrm{ri}(X_\mathcal{B})} \subseteq \mathcal{B}$. In the following we provide sufficient conditions for $s \in \Sigma_{++}^{\mathrm{ri}(X_\mathcal{B})}$.

*Lemma 25:* Consider the PWA system (4), a boundary of the partition $X_\mathcal{B}$, with $\mathcal{B} \subseteq \Sigma$, a generic $s \in \mathcal{B}$, $\ell \in \mathcal{L}_s$ as in (17), $\{h_{\ell s}\}_{\ell \in \mathcal{L}_s}$ the normal vectors according to the convention in (18). If for each $\ell \in \mathcal{L}_s$ there exists an integer $k_{\ell s} \in \{1, 2, \ldots, n\}$ such that $\forall k \in \{1, \ldots, k_{\ell s} - 1\}$, for all vertices $\{v_i\}_{i=1}^\lambda$ of $X_\mathcal{B}$ and for all rays $\{r_j\}_{j=1}^\rho$ of $X_\mathcal{B}$ it is

$$h_{\ell s}^\top A_s^{k-1}(A_s v_i + b_s) = 0 \tag{19a}$$
$$h_{\ell s}^\top A_s^k r_j = 0 \tag{19b}$$
$$\forall x \in \mathrm{ri}(X_\mathcal{B}) : h_{\ell s}^\top A_s^{k_{\ell s}-1}(A_s x + b_s) > 0, \tag{19c}$$

$i = 1, \ldots, \lambda$, $j = 1 \ldots, \rho$, then $s \in \Sigma_{++}^{\mathrm{ri}(X_\mathcal{B})}$.

*Proof:* Recall that the boundary can be written as

$$X_\mathcal{B} = \mathrm{conv}\{v_i\}_{i=1}^\lambda + \mathrm{cone}\{r_j\}_{j=1}^\rho, \tag{20}$$

i.e. any $x \in X_\mathcal{B}$ can be expressed as the sum of a convex combination of the vertices of $X_\mathcal{B}$ and the conical combination of its rays. Then conditions (19a)–(19b) imply that $h_{\ell s}^\top A_s^{k-1}(A_s x + b_s) = 0$ for any $x \in X_\mathcal{B}$ and the proof follows by applying Lemma 33. ■

Sufficient conditions for $s \in \Sigma_{--}^{\mathrm{ri}(X_\mathcal{B})}$ can be obtained analogously.

*Lemma 26:* Consider the PWA system (4), a boundary of the partition $X_\mathcal{B}$, with $\mathcal{B} \subseteq \Sigma$, a generic $s \in \mathcal{B}$, $\ell \in \mathcal{L}_s$ as in (17), $\{h_{\ell s}\}_{\ell \in \mathcal{L}_s}$ the normal vectors according to the convention in (18). If for each $\ell \in \mathcal{L}_s$ there exists an integer $k_{\ell s} \in \{1, 2, \ldots, n\}$ such that $\forall k \in \{1, \ldots, k_{\ell s} - 1\}$, for all vertices $\{v_i\}_{i=1}^\lambda$ of $X_\mathcal{B}$ and for all rays $\{r_j\}_{j=1}^\rho$ of $X_\mathcal{B}$ it is

$$h_{\ell s}^\top A_s^{k-1}(A_s v_i + b_s) = 0 \tag{21a}$$
$$h_{\ell s}^\top A_s^k r_j = 0 \tag{21b}$$
$$\forall x \in \mathrm{ri}(X_\mathcal{B}) : h_{\ell s}^\top (-A_s)^{k_{\ell s}-1}(A_s x + b_s) < 0, \tag{21c}$$

$i = 1, \ldots, \lambda$, $j = 1 \ldots, \rho$, then $s \in \Sigma_{--}^{\mathrm{ri}(X_\mathcal{B})}$.

*Proof:* The proof follows through steps similar to Lemma 25 by considering the time-reversed variant of the PWA system (4), i.e. $\dot{x}(\tau) = -A_s x(\tau) - b_s$ with $x(\tau) \in X_s$. Indeed, $s$ being a strict backward mode for $x$ is equivalent to $s$ being a strict forward mode for the same $x$ in the time-reversed system. Then the proof directly follows by applying Lemma 34. ■

*Remark 27 (Relative interior):* The verification of (19c) (condition (21c), respectively) also on the boundary of the boundary of $X_\mathcal{B}$ it is not sufficient for concluding that $s \in \Sigma_{++}^{X_\mathcal{B}}$ ($s \in \Sigma_{--}^{X_\mathcal{B}}$), i.e. to extend Lemma 25 (Lemma 26) to the boundary of the boundary of $X_\mathcal{B}$. As an example consider the planar PWA given by $\dot{x} = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$, $x \in \bigcup_{s=1}^4 X_s$ where $X_1$, $X_2$, $X_3$, $X_4$ are the four canonical quadrants, see Figure 5. Clearly, for all $x \in \mathrm{ri}(X_{41})$ it is $\{1\} \in \Sigma_{++}^{\mathrm{ri}(X_{41})}$ and the inequalities (19c) hold for $k_{41}^x = 1$. However, such inequalities also hold for $x = 0$, but there does not exist a solution starting in the origin and evolving for some positive time in $X_1$, i.e. $\{1\} \notin \Sigma_{++}^0$. Moreover, the origin can be written as the intersection of the facets $X_{12}$ and $X_{32}$. Then it is easy to verify that conditions of Lemma 25 are satisfied for $s = 2$, i.e. $\{2\} \subseteq \Sigma_{++}^0$. ⧫

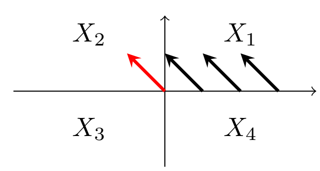


Fig. 5: Illustration that (19c) is not sufficient for points on the boundary of a boundary.

Lemmas 25 and 26 are sufficient conditions which could be used for searching boundaries which are both crossing and Caratheodory. The sets satisfying the conditions in the lemmas, say $\widetilde{\Sigma}_{++}^{\mathrm{ri}(X_\mathcal{B})}$ and $\widetilde{\Sigma}_{--}^{\mathrm{ri}(X_\mathcal{B})}$, are subsets of $\Sigma_{++}^{\mathrm{ri}(X_\mathcal{B})}$ and $\Sigma_{--}^{\mathrm{ri}(X_\mathcal{B})}$, respectively. In the case that $\widetilde{\Sigma}_{++}^{\mathrm{ri}(X_\mathcal{B})}$ and $\widetilde{\Sigma}_{--}^{\mathrm{ri}(X_\mathcal{B})}$ are nonempty, disjoint, and $\widetilde{\Sigma}_{++}^{\mathrm{ri}(X_\mathcal{B})} \cup \widetilde{\Sigma}_{--}^{\mathrm{ri}(X_\mathcal{B})} = \mathcal{B}$, the boundary can be classified as crossing and Caratheodory type, see Definition 24. Clearly, in this case it will be $\Sigma_{+}^{\mathrm{ri}(X_\mathcal{B})} = \widetilde{\Sigma}_{++}^{\mathrm{ri}(X_\mathcal{B})}$ and $\Sigma_{-}^{\mathrm{ri}(X_\mathcal{B})} = \widetilde{\Sigma}_{--}^{\mathrm{ri}(X_\mathcal{B})}$. More in general, it could be that $\widetilde{\Sigma}_{--}^{\mathrm{ri}(X_\mathcal{B})} \cup \widetilde{\Sigma}_{++}^{\mathrm{ri}(X_\mathcal{B})} \subset \mathcal{B}$. In this case, if the sets are nonempty, one can define the Caratheodory trajectories moving from any mode of $\widetilde{\Sigma}_{--}^{\mathrm{ri}(X_\mathcal{B})}$ to any other mode of $\widetilde{\Sigma}_{++}^{\mathrm{ri}(X_\mathcal{B})}$, although it is not possible to state that the boundary is of crossing type for all trajectories. Another interesting case is when the condition $h_{\ell s}^\top A_s^{k_{\ell s}-1}(A_s x + b_s) = 0$ holds $\forall x \in \mathrm{ri}(X_\mathcal{B})$ and for $k_{\ell s} = 1, \dots, n$. This corresponds to Caratheodory trajectories lying on the boundary for mode $s$.

## VI. PWQ Lyapunov function

The conditions in Theorem 21 on the local-Lyapunov functions are pointwise and then not easily implementable. In order to formulate practical conditions, we look for guaranteeing conditions (B1) and (B2) in Theorem 21 on the whole boundaries by exploiting the classification from Definition 24 and local quadratic Lyapunov functions.

### A. Positivity test for quadratic functions on polyhedral sets

Consider a general quadratic function

$$q(x) = x^\top P x + 2\nu^\top x + \omega \tag{22}$$

and a polyhedral set $X \subseteq \mathbb{R}^n$. In the following we assume $\omega = 0$ if $0 \in X$. We want to find a sufficient conditions in terms of $P, \nu, \omega$ and the vertices and rays of $X$ which guarantees that $q(x) > 0 \; \forall x \in X \setminus \{0\}$ or $q(x) \geq 0 \; \forall x \in X$.

Sufficient conditions for the positivity of $q(x)$ in $X$ can be obtained by using the cone-copositive approach. Each polyhedron $X$ can be represented in the form (20) which identifies the so-called *$\mathcal{V}$-representation* of the polyhedron. If the origin is the only vertex, then the polyhedron is a pointed polyhedral cone, say $\mathcal{C}$. The matrix $R \in \mathbb{R}^{n\times\rho}$ whose columns are the rays in an arbitrary order, is called ray matrix of the cone. Any $v \in \mathcal{C}$ can be written as $v = R\theta$ where $\theta \in \mathbb{R}_+^\rho$.

The conical hull $\mathcal{C}_X$ of $X$ is obtained by interpreting the vertices also as rays, i.e. $\mathcal{C}_X = \mathrm{cone}\{\{v_\ell\}_{\ell=1}^\lambda, \{r_\ell\}_{\ell=1}^\rho\}$, and the corresponding ray matrix is

$$R = \begin{pmatrix} v_1 & \cdots & v_\lambda & r_1 & \dots & r_\rho \end{pmatrix}. \tag{23}$$

In the following for simplicity we assume that all possible vertices and rays redundancies in the ray matrices have been eliminated. The conic homogenization of a polyhedron $X$ is defined as $\mathcal{C}_{\widehat{X}} = \mathrm{cone}\{\{\widehat{v}_i\}_{i=1}^\lambda, \{\widehat{r}_j\}_{j=1}^\rho\}$, and the corresponding ray matrix is

$$\widehat{R} = \begin{pmatrix} \widehat{v}_1 & \cdots & \widehat{v}_\lambda & \widehat{r}_1 & \dots & \widehat{r}_\rho \end{pmatrix} \tag{24}$$

where $\widehat{v}_i = \mathrm{col}(v_i, 1)$ for all $i$ and $\widehat{r}_j = \mathrm{col}(r_j, 0)$ for all $j$. A sufficient condition for the sign of a quadratic function on a polyhedron can be written in terms of LMIs, so as shown by the following lemmas whose proofs can be easily derived from [11].

*Lemma 28:* Consider (22), $x \in X$, $\omega = 0$ if $0 \in X$, $\widehat{R}$ the ray matrix of the cone $\mathcal{C}_{\widehat{X}}$, the symmetric matrix $\widehat{P} \in \mathbb{R}^{(n+1)\times(n+1)}$ defined as

$$\widehat{P} = \begin{pmatrix} P & \nu \\ \nu^\top & \omega \end{pmatrix}. \tag{25}$$

If there exists a symmetric (entrywise) nonnegative matrix $\overline{N}$ such that

$$\widehat{R}^\top \widehat{P} \widehat{R} - \overline{N} < 0 \tag{26}$$

holds, then $q(x) \geq 0$, $x \in X$.

If $0 \notin X$, the implication in Lemma 28 is valid for strict inequalities if $\overline{N}$ is replaced by a matrix $N$ with (strictly) positive entries. To obtain a strict inequality also for the case that $0 \in X$ an additional condition is required:

*Lemma 29:* Consider (22), $x \in X$, $0 \in X$, $\omega = 0$, $R$ the ray matrix of the cone $\mathcal{C}_X$. Let $e_i \in \mathbb{R}^{\lambda+\rho}$ be the $i$-th unit vector. If there exists a symmetric matrix $N$ with (strictly) positive entries such that the following conditions

$$\left.\begin{array}{l} R^\top P R - N < 0 \quad \wedge \\ 2\nu^\top R e_i \geq 0, \; i = 1, \dots, \lambda + \rho \end{array}\right\} \tag{27}$$

hold, then $q(x) > 0$, $x \in X \setminus \{0\}$.

### B. Local quadratic Lyapunov function for a mode

Let us associate to the mode $s \in \Sigma$ the quadratic function

$$V_s(x) = x^\top P_s x + 2\nu_s^\top x + \omega_s \tag{28}$$

with $P_s \in \mathbb{R}^{n\times n}$ symmetric matrix, $\nu_s \in \mathbb{R}^n$, $\omega_s \in \mathbb{R}$. In the following we verify that (28) satisfy the conditions in Definition 19 which are a prerequisite for Theorem 21. In particular, we distinguish the cases when the origin belongs to $X_s$, i.e. $s \in \Sigma^0$, and when it does not, i.e. $s \in \overline{\Sigma^0} = \Sigma \setminus \Sigma^0$.

*Lemma 30:* Consider the PWA system (4) and for each mode $s \in \Sigma$ a quadratic function $V_s : \mathbb{R}^n \to \mathbb{R}$ as in (28). Furthermore, define the following conditions:

(C1) for all $s \in \overline{\Sigma^0}$

$$\widehat{R}_s^\top \widehat{P}_s \widehat{R}_s - N_s < 0, \tag{29a}$$

$$-\widehat{R}_s^\top (\widehat{A}_s^\top \widehat{P}_s + \widehat{P}_s \widehat{A}_s) \widehat{R}_s - M_s < 0, \tag{29b}$$

with $\widehat{R}_s$ defined according to (24), $N_s$ and $M_s$ unknown symmetric entrywise positive matrices,

$$\widehat{A}_s = \begin{pmatrix} A_s & b_s \\ 0 & 0 \end{pmatrix}, \quad \widehat{P}_s = \begin{pmatrix} P_s & \nu_s \\ \nu_s^\top & \omega_s \end{pmatrix};$$

(C2) for all $s \in \Sigma^0$

$$\left.\begin{array}{l} R_s^\top P_s R_s - N_s < 0 \quad \wedge \\ 2\nu_s^\top R_s e_i \geq 0, \; i = 1, \dots, \lambda_s + \rho_s \end{array}\right\} \tag{30a}$$

$$\left.\begin{array}{l} -R_s^\top (A_s^\top P_s + P_s A_s) R_s - M_s < 0, \quad \wedge \\ -2\nu_s^\top A_s R_s e_i \geq 0, \; i = 1, \dots, \lambda_s + \rho_s \end{array}\right\} \tag{30b}$$

with $R_s$ defined through (23), $e_i$ are the unit vectors, $N_s$ and $M_s$ unknown symmetric entrywise positive matrices.

If the set of LMIs (29)–(30) have a solution $\{P_s, \nu_s, \omega_s, N_s, M_s\}_{s\in\Sigma}$ then the quadratic functions (28) are local Lyapunov functions for the PWA system (4).

*Proof:* The proof consists of verifying that conditions in Definition 19 are satisfied.

Condition (L1) in Definition 19 is trivially satisfied by (28) in $\mathbb{R}^n$.

As regards (L2) we get $V_s(0) = 0$ by imposing $\omega_s = 0$ if $s \in \Sigma^0$. It is trivial to verify that it is always possible to find a quadratic function which is positive in a polyhedron. In particular, the LMIs (29a) and (30a) allows one to "construct" such a positive $V_s(x)$ in $X_s$ by using Lemma 28 and Lemma 29, respectively, with $X = X_s$, $R = R_s$, $P = P_s$, $\nu = \nu_s$, $\omega = \omega_s$, $N = N_s$.

The radial unboundedness condition (L3) in Definition 19 has to be verified for all unbounded $X_s$ of the (finite) polyhedral partition of the state space. The quadratic nature of $V_s$, its continuity and positive definiteness implied by (29a) and (30a) allows us to prove the radially unboundedness property. First consider the case $s \in \Sigma^0$ with $X_s$ unbounded. Clearly the radially unboudedness on $\mathcal{C}_{X_s}$ implies that on $X_s \subseteq \mathcal{C}_{X_s}$. For any $\tilde{x} \in \mathcal{C}_{X_s}$ then also $\tau\tilde{x} \in \mathcal{C}_{X_s}$ with $\tau$ any positive real number. Therefore for all $x = \tau\tilde{x}$ it is

$$\lim_{\|x\|\to+\infty} V_s(x) = \lim_{\tau\to+\infty} (\tau^2 \tilde{x}^\top P_s \tilde{x} + 2\tau\nu_s^\top \tilde{x}) = +\infty$$

where we used the conditions in Lemma 29. In the case $s \in \overline{\Sigma^0}$ and $X_s$ unbounded consider $\lim_{\|x\|\to+\infty} V_s(x) = \lim_{\|\bar{x}\|\to+\infty} \bar{x}^\top \hat{P}_s \bar{x}$ where $\bar{x} = \mathrm{col}(x, 1)$ and $x \in X_s$. Since $\bar{x} \in \mathcal{C}_{\widehat{X}_s}$, by using (26), with $\overline{N}_s$ replaced by a matrix $N_s$ with (strictly) positive entries, we can conclude that $V_s$ is radially unbounded for any unbouded $X_s$.

As regards condition (L4) in Definition 19, by using (28) one can write

$$\begin{aligned}\nabla V_s(x)(A_s x + b_s) =& x^\top (A_s^\top P_s + P_s A_s) x \\ &+ 2(b_s^\top P_s + \nu_s^\top A_s) x + 2\nu_s^\top b_s\end{aligned}$$

which is a quadratic function. The conditions (29b) and (30b) imply the sign condition of $\nabla V_s(x)(A_s x + b_s)$ on $X_s$ by using Lemma 28 and Lemma 29 with $P = A_s^\top P_s + P_s A_s$, $\nu^\top = b_s^\top P_s + \nu_s^\top A_s$ and $\omega = 2\nu_s^\top b_s$. ∎

The existence of $\{P_s, \nu_s, \omega_s\}$ such that (L2) and (L4) are *both* satisfied is not ensured for any polyhedron $X_s$ and pairs $\{A_s, b_s\}$. In the case of quadratic forms, i.e. $\nu_s = 0$ and $\omega_s = 0$, it is easy to verify that if $A_s$ has some unstable eigenvector whose eigenspace has a nontrivial intersection with $X_s$, then it is not possible to find any $P_s$ which satisfies (L2) and (L4). The same holds for unbounded polyhedra containing the origin and quadratic functions, if the eigenspace is contained in $X_s$.

On the contrary there are cases when the existence of a positive $V_s$ with negative derivative in a polyhedron is guaranteed. For instance, if $X_s$ is bounded and it does not contain the origin, it is enough to choose a sufficiently large $\omega_s > 0$ for having a positive $V_s$. Moreover from $\nabla V_s(x)(A_s x + b_s) \le \lambda_P^{\max}\|x\|^2 + 2\|b_s^\top P_s\|\|x\| + \nu_s^\top(A_s x + b_s)$ with $\lambda_P^{\max}$ the maximum eigenvalue of $P$, $P = A_s^\top P_s + P_s A_s$, $\nu^\top = b_s^\top P_s + \nu_s^\top A_s$ and $\omega = 2\nu_s^\top b_s$, one can choose $P_s = 0$ and $\nu_s$ such that $\nu_s^\top \max\{A_s x + b_s\}_{x\in X_s} < 0$, where $\max$ must be intended componentwise. This result is not dependent on the eigenvalues of $A_s$.

### C. PWQ stability with jump conditions

In order to apply Theorem 21 we need to guarantee the compatibility conditions (B1) and (B2) for all local quadratic Lyapunov functions. From Remark 22 it is enough to consider these conditions on the polyhedra boundaries. The characterization of the boundaries allows one to obtain operative conditions in terms of LMIs.

*Corollary 31:* Consider the PWA system (4) satisfying Assumptions (A1) and (A2) and consider for each mode $s \in \Sigma$ a quadratic function $V_s : \mathbb{R}^n \to \mathbb{R}$ as in (28). Furthermore, define the following conditions:

(D1) for each $s \in \Sigma$ the LMIs (29)–(30);

(D2) for each boundary $X_{\mathcal{B}}$ which can be classified as a unreachable boundary according to Definition 24(i), there are no additional conditions;

(D3) for each boundary $X_{\mathcal{B}}$ which can be classified as a crossing boundary according to Definition 24(ii),

$$\widehat{R}_{\mathcal{B}}^\top (\widehat{P}_i - \widehat{P}_j) \widehat{R}_{\mathcal{B}} - N_{ij} < 0, \qquad (32)$$

with $N_{ij}$ unknown symmetric entrywise nonnegative matrix, for all pairs $(i, j) \in \Sigma_-^{\mathrm{ri}(X_{\mathcal{B}})} \times \Sigma_+^{\mathrm{ri}(X_{\mathcal{B}})}$ with $\widehat{R}_{\mathcal{B}}$ being the ray matrix of the conic homogenization of $X_{\mathcal{B}}$;

(D4) for each remaining boundary $X_{\mathcal{B}}$ which can be classified as a Caratheodory boundary according to Definition 24(iii), the continuity conditions

$$\widehat{R}_{\mathcal{B}}^\top (\widehat{P}_i - \widehat{P}_j) \widehat{R}_{\mathcal{B}} = 0, \qquad (33)$$

for all pairs $i, j \in \mathcal{B}$;

(D5) for all other boundaries $X_{\mathcal{B}}$, including those which can be classified as a sliding boundaries according to Definition 24(iv), the pairwise continuity conditions (33) for all pairs $i, j \in \mathcal{B}$, together with

$$-\widehat{R}_{\mathcal{B}}^\top (\widehat{A}_j^\top \widehat{P}_i + \widehat{P}_i \widehat{A}_j + \mu I) \widehat{R}_{\mathcal{B}} - M_{ij} < 0, \qquad (34)$$

for an arbitrary $i \in \mathcal{B}$ and for all $j \in \mathcal{B}$, with unknowns symmetric entrywise positive matrices $M_{ij}$ and $\mu > 0$.

If the set of LMIs with equality constraints in (D1)–(D5) have a solution, then all solutions of the PWA system (4) converge asymptotically to zero.

*Proof:* From (D1) and Lemma 30 it follows that $V_s$ for all $s \in \Sigma$ are local Lyapunov functions for the system. The rest of the proof consists of verifying that the local Lyapunov functions are compatible on the polyhedra boundaries, i.e. (D2)–(D5) imply (B1) and (B2) of Theorem 21 for all boundaries. The verification of (32) implies that (B1) is satisfied for all crossing boundaries. Indeed, by using (28) the inequality $V_i(x) \ge V_j(x)$ can be rewritten as

$$x^\top (P_i - P_j) x + 2(\nu_i^\top - \nu_j^\top) x + \omega_i - \omega_j \ge 0$$

for all $x \in X_{\mathcal{B}}$, which follows from (32) by using Lemma 28 with $P = P_i - P_j$, $\nu^\top = \nu_i^\top - \nu_j^\top$, $\omega = \omega_i - \omega_j$ and the $\mathcal{V}$-representation of the boundary. For all the other boundaries which are not unreachable, the continuity conditions in (D3) imply that (B1) is satisfied.

Assume that (D3)–(D5) are satisfied. Then (B2) holds for all remaining boundaries. By picking an arbitrary $i \in \mathcal{B}$ the inequalities (11) can be rewritten as

$$x^\top (A_j^\top P_i + P_i A_j + \mu I)x + 2(b_j^\top P_i + \nu_i^\top A_j)x + 2\nu_i^\top b_j \leq 0$$

for all $x \in X_\mathcal{B}$ and for all $j \in \mathcal{B}$, which follows from (34) by using Lemma 28 with $P = -(A_j^\top P_i + P_i A_j + \mu I)$, $\nu^\top = -(b_j^\top P_i + \nu_i^\top A_j)$, $\omega = -2\nu_i^\top b_j$ for all $j \in \mathcal{B}$, and the $\mathcal{V}$-representation of the boundary. ■

The LMIs for the asymptotic stability are (29), (30), (32) and (34). These LMIs are coupled through the ray matrices because some polyhedra could share some vertices and/or rays, see (24). Moreover the constraints (33) must be considered too. The unknown variables are $\{P_s, \nu_s, \omega_s, N_s, M_s\}_{s\in\Sigma}$ together with the matrices $N_{ij}$, $M_{ij}$ and the scalar $\mu$ in (D3)–(D5). If the LMIs have no solution, one could try with a refined partition by considering the same dynamics for each refined polyhedron, thus increasing the degrees of freedom of the candidate PWQ Lyapunov function. Moreover, the refinement could be obtained by partitioning boundaries which cannot be classified as crossing or unreachable but allow a classification for the derived subsets.

### *D. Example*

The following example is shown to not have a *continuous* PWQ Lyapunov function for the given state-space decomposition, whereas our approach provides a positive answer for the asymptotic stability. The PWA system has the $\mathbb{R}^2$ state space partitioned as in Figure 6: $X_0 := \{|x_1| \leq 1, |x_2| \leq 1\}$, $X_1 := \{x_2 \geq \max\{|x_1|, 1\}\}$, $X_2 := \{x_1 \geq \max\{|x_2|, 1\}\}$, $X_3 := -X_1$, $X_4 := -X_2$. The dynamics are given by (1), where $s \in \Sigma = \{0, 1, 2, 3, 4\}$ and $A_0 = -I$, $A_1 = A_2 = A_3 = A_4 = 0$, $b_0 = 0$, $b_1 = \left(\begin{smallmatrix}1\\-0.1\end{smallmatrix}\right)$, $b_2 = \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)$, $b_3 = \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)$, $b_4 = -b_2$.

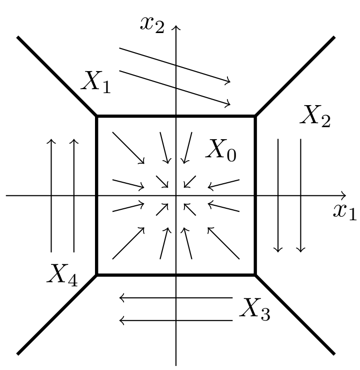


Fig. 6: PWA system for which only a discontinuous PWQ Lyapunov function exists.

The following Lemma excludes the existence of a continuous PWQ Lyapunov function.

*Lemma 32:* The above example does not allow a *continuous* PWQ Lyapunov function (including linear and constant terms) defined on the given state space partition.

*Proof:* Let the candidate Lyapunov function on region $i \in \{0, 1, 2, 3, 4\}$ be given by

$$V_i(x) = \alpha_i x_1^2 + \beta_i x_2^2 + 2\gamma_i x_1 x_2 + \delta_i x_1 + \eta_i x_2 + r_i.$$

| $X_\mathcal{B}$ | $\sim\mathrm{ri}(X_\mathcal{B})_{++}$ | $\mathrm{ri}(X_\mathcal{B})_{+}$ | $\sim\mathrm{ri}(X_\mathcal{B})_{--}$ | $\mathrm{ri}(X_\mathcal{B})_{-}$ |
|---|---|---|---|---|
| $X_0 \cap X_1$ | $\{0\}$ | $\{0\}$ | $\{1\}$ | $\{1\}$ |
| $X_0 \cap X_2$ | $\{0\}$ | $\{0, 2\}$ | $\emptyset$ | $\{2\}$ |
| $X_0 \cap X_3$ | $\{0\}$ | $\{0, 3\}$ | $\emptyset$ | $\{3\}$ |
| $X_0 \cap X_4$ | $\{0\}$ | $\{0, 4\}$ | $\emptyset$ | $\{4\}$ |
| $X_1 \cap X_2$ | $\{2\}$ | $\{2\}$ | $\{1\}$ | $\{1\}$ |
| $X_1 \cap X_4$ | $\{1\}$ | $\{1\}$ | $\{4\}$ | $\{4\}$ |
| $X_2 \cap X_3$ | $\{3\}$ | $\{3\}$ | $\{2\}$ | $\{2\}$ |
| $X_3 \cap X_4$ | $\{4\}$ | $\{4\}$ | $\{3\}$ | $\{3\}$ |

TABLE I: Characterization of facets for the example in Fig. 6.

| $X_\mathcal{B}$ | $\sim\mathrm{ri}(X_\mathcal{B})_{++}$ | $\mathrm{ri}(X_\mathcal{B})_{+}$ | $\sim\mathrm{ri}(X_\mathcal{B})_{--}$ | $\mathrm{ri}(X_\mathcal{B})_{-}$ |
|---|---|---|---|---|
| $X_0 \cap X_1 \cap X_2$ | $\{0\}$ | $\{0, 2\}$ | $\{1\}$ | $\{1\}$ |
| $X_0 \cap X_2 \cap X_3$ | $\{0\}$ | $\{0, 3\}$ | $\emptyset$ | $\{2\}$ |
| $X_0 \cap X_3 \cap X_4$ | $\{0\}$ | $\{0, 4\}$ | $\emptyset$ | $\{3\}$ |
| $X_0 \cap X_1 \cap X_4$ | $\{0\}$ | $\{0\}$ | $\emptyset$ | $\{4\}$ |

TABLE II: Characterization of points which are boundaries for the example in Fig. 6.

Note that $V_0(0) = 0$ requires $r_0 = 0$ and $V_0(x) \geq 0$ in a ball around the origin requires $\delta_0 = 0$ and $\eta_0 = 0$; it is also clear that positive definiteness of $V_0$ implies $\alpha_0 > 0$ and $\beta_0 > 0$. Furthermore, continuity on the intersection $X_0 \cap X_2$ and $X_0 \cap X_3$ means that for $h_i^2(\lambda) := V_i((1, \lambda)^\top)$, $i = 0, 2$, and $h_j^3(\lambda) := V_j((\lambda, -1)^\top)$, $j = 0, 3$, $h_0^2(\lambda) = h_2^2(\lambda)$ and $h_0^3(\lambda) = h_3^3(\lambda)$ $\forall \lambda \in [-1, 1]$. Consequently, also the derivatives of the corresponding function have to be equal on $[-1, 1]$:

$$2\lambda\beta_0 + 2\gamma_0 = h_0^{2\prime}(\lambda) = h_2^{2\prime}(\lambda) = 2\lambda\beta_2 + 2\gamma_2 + \eta_2,$$
$$2\lambda\alpha_0 - 2\gamma_0 = h_0^{3\prime}(\lambda) = h_3^{3\prime}(\lambda) = 2\lambda\alpha_3 - 2\gamma_3 + \delta_3.$$

Therefore, continuity yields $\alpha_0 = \alpha_3$, $\beta_0 = \beta_2$, $\gamma_0 = \gamma_2 + \frac{1}{2}\eta_2$, $\gamma_0 = \gamma_3 - \frac{1}{2}\delta_3$. Evaluating the decreasing condition of $V_2$ and $V_3$ along solutions yields

$$0 > \dot{V}_2(x) = \nabla V_2(x) \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -2\beta_2 x_2 - 2\gamma_2 x_1 - \eta_2,$$
$$0 > \dot{V}_3(x) = \nabla V_3(x) \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -2\alpha_3 x_1 - 2\gamma_3 x_2 - \delta_3,$$

which has to hold for all $x \in X_2$ or $x \in X_3$, respectively. In particular, it has to hold for $x = (1, -1)^\top \in X_2$ and $x = (-1, -1) \in X_3$, resulting in the following constraints $\gamma_2 + \frac{1}{2}\eta_2 > \beta_2$ and $\gamma_3 - \frac{1}{2}\delta_3 < -\alpha_3$. By invoking the equalities above and positivity of $\alpha_0$ and $\beta_0$ we arrive at the contradiction $\gamma_0 > \beta_0 > 0$ and $\gamma_0 < -\alpha_0 < 0$. ■

Lemmas 25 and 26 together with the following considerations allow the boundaries characterization reported in Table I and Table II. As an example, for each point on the segments which are the boundaries given by the intersections between $X_0$ and $X_2$, $X_3$ and $X_4$, there are two Caratheodory solutions, one which remains on the boundary and another that converges to the origin inside $X_0$. Moreover, for the vertex $X_0 \cap X_1 \cap X_4$, the vector fields analysis allows one to deduce that $\{1\} \notin \Sigma_{++}^x$ and $\{1\} \notin \Sigma_{--}^x$. In particular, by definition it is also $\{1\} \notin \Sigma_{+}^x$ and $\{1\} \notin \Sigma_{-}^x$.

By using Corollary 31 we obtained the asymptotic stability

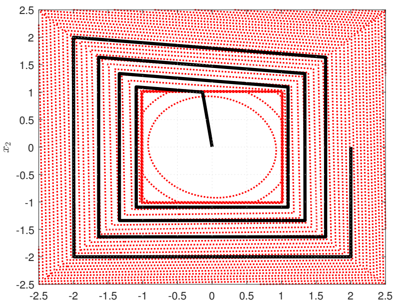


Fig. 7: State space with a trajectory (black line) and the PWQ Lyapunov function level curves (red line).

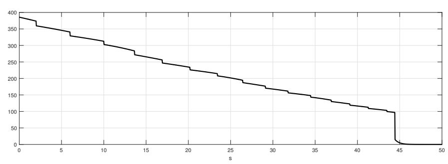

Fig. 8: Time evolution of the PWQ Lyapunov function along the trajectory in Figure 7.

of the origin with the following matrices:

$$P_0 = \begin{pmatrix} 15.3020 & 1.0085 \\ 1.0085 & 15.2899 \end{pmatrix}, \; \hat{P}_1 = \begin{pmatrix} 0.2193 & 6.7605 & 0.4975 \\ 6.7605 & 81.6087 & 7.1308 \\ 0.4975 & 7.1308 & 3.0925 \end{pmatrix}$$

$$\hat{P}_2 = \begin{pmatrix} 88.1542 & 1.3678 & 6.8188 \\ 1.3678 & -0.0901 & 0.1591 \\ 6.8188 & 0.1591 & 5.7816 \end{pmatrix},$$

$$\hat{P}_3 = \begin{pmatrix} -0.1254 & -0.9845 & 0.3704 \\ -0.9845 & 79.6640 & -6.4175 \\ 0.3704 & -6.4175 & 6.2516 \end{pmatrix},$$

$$\hat{P}_4 = \begin{pmatrix} 72.9043 & 0.7773 & -6.1179 \\ 0.7773 & -0.1354 & -0.4988 \\ -6.1179 & -0.4988 & 5.3389 \end{pmatrix}.$$

Figure 7 shows the state space with level curves and a trajectory for the example. The corresponding time evolution of the PWQ Lyapunov function along the trajectory is represented in Figure 8: discontinuities occur when the trajectory crosses the polyhedra boundaries.

## VII. Conclusion

The effectiveness of discontinuous Lyapunov functions for proving the asymptotic stability of the origin in continuous-time piecewise affine systems with discontinuous vector field has been demonstrated. Sliding mode and Zeno behaviours have been included in the feasible Filippov solution concept for which a stability theorem has been proved. The classification of the boundaries in the polyhedral partition of the state space has been carried out by proposing operative conditions. The particularization of the stability conditions to Lyapunov functions in piecewise quadratic form has conducted to the formulation of linear matrix inequalities whose solution directly provides the parameters of the Lyapunov function.

Directions for future research could be to consider weaker conditions for the boundaries characterization and corresponding refinement strategies of the polyhedral partition. The exploitation of the stability theorem for control design is a further interesting direction for future studies.

## Appendix

*Lemma 33:* Consider the PWA system (4) and a point $x$ in some boundary $X_{\mathcal{B}}$. Choose $s \in \mathcal{B}$, the set of indices $\mathcal{L}_s = \{\ell_1, \ell_2, \dots, \ell_{\alpha_s}\}$ as in (17). Then $s$ is a strict forward mode for $x$, i.e. $s \in \Sigma^x_{++}$, if, and only if, for each $\ell \in \mathcal{L}_s$ there exists an integer $k^x_{\ell s} \in \{1, 2, \dots, n\}$ such that

$$\forall k \in \{1, \dots, k^x_{\ell s} - 1\} : h^\top_{\ell s} A^{k-1}_s (A_s x + b_s) = 0 \tag{35}$$

and

$$h^\top_{\ell s} A^{k^x_{\ell s}-1}_s (A_s x + b_s) > 0. \tag{36}$$

where $h_{\ell s}$ with $\ell \in \mathcal{L}_s$ are normal vectors according to the convention in (18).

*Proof: Necessity.* Let $\xi : [0, \varepsilon) \to \mathbb{R}^n$ be a single-mode Caratheodory solution with $\xi(t) \in \operatorname{int} X_s$ for all $t \in (0, \varepsilon)$ for some $s \in \mathcal{B}$. Since $X_s$ is locally an intersection of halfspaces defined by the normal vectors $h_{\ell s}$ with $\ell \in \mathcal{L}_s$ it follows that $0 < h^\top_{\ell s}(\xi(t) - x)$ $\forall \ell \in \mathcal{L}_s$ and, hence

$$0 \le \lim_{t \searrow 0} h^\top_{\ell s} \frac{\xi(t) - \xi(0)}{t} = h^\top_{\ell s} \dot{\xi}(0^+) = h^\top_{\ell s}(A_s x + b_s)$$

with $\xi(0) = x$. If $h^\top_{\ell s}\dot{\xi}(0^+) > 0$, then the claim is shown for $k^x_{\ell s} = 1$. If on the other hand $h^\top_{\ell s}\dot{\xi}(0^+) = 0$ we proceed inductively and show that if $h^\top_{\ell s}\xi^{(k)}(0^+) = 0$ for $k = 1, 2, \dots, k^x_{\ell s} - 1$ and $h^\top_{\ell s}\xi^{(k^x_{\ell s})}(0^+) \neq 0$ for some $k^x_{\ell s} \le n$, then (36) holds. From $h^\top_{\ell s}\xi^{(k)}(0^+) = 0$ for $k = 1, 2, \dots, k^x_{\ell s} - 1$ it follows that

$$\begin{aligned} 0 &\le \lim_{t \searrow 0} h^\top_{\ell s} (\xi(t) - \xi(0)) \frac{k^x_{\ell s}!}{t^{k^x_{\ell s}}} = h^\top_{\ell s}\xi^{(k^x_{\ell s})}(0^+) \\ &= h^\top_{\ell s} A^{k^x_{\ell s}-1}_s (A_s x + b_s) \end{aligned}$$

and consequently (as it was assumed that $h^\top_{\ell s}\xi^{(k^x_{\ell s})}(0^+) \neq 0$) the desired inequality (36) is shown.

*Sufficiency.* Let $\xi(t) := e^{A_s t}x + \int_0^t e^{A_s(t-\tau)} b_s d\tau$ for $t \ge 0$, then $\dot{\xi} = A_s\xi + b_s$ and by considering the Taylor-expansion, we have for all $t > 0$

$$\begin{aligned} h^\top_{\ell s}(\xi(t) - x) &= \overbrace{h^\top_{\ell s}(A_s x + b_s)}^{=0} t + \overbrace{h^\top_{\ell s} A_s (A_s x + b_s)}^{=0} \frac{t^2}{2} \\ &\quad + \dots + \underbrace{h^\top_{\ell s} A^{k^x_{\ell s}-1}_s (A_s x + b_s) \frac{t^{k^x_{\ell s}}}{k^x_{\ell s}!}}_{>0} + o(t^{k^x_{\ell s}}). \end{aligned}$$

Then for all sufficiently small $t$ we have $h^\top_{\ell s}(\xi(t) - x) > 0$. Since $x$ is in the relative interior of $X_{\ell s}$ this implies $\xi(t) \in \operatorname{int} X_s$. ■

*Lemma 34:* Consider the PWA system (4) and a point $x$ in some boundary $X_{\mathcal{B}}$. Choose $s \in \mathcal{B}$, the set of indices $\mathcal{L}_s = \{\ell_1, \ell_2, \dots, \ell_{\alpha_s}\}$ as in (17). Then $s$ is a strict backward mode for $x$, , i.e. $s \in \Sigma^x_{--}$, if, and only if, for each $\ell \in \mathcal{L}_s$ there exists an integer $k^x_{\ell s} \in \{1, 2, \dots, n\}$ such that (35) is satisfied and

$$h^\top_{\ell s}(-A_s)^{k^x_{\ell s}-1}(A_s x + b_s) < 0,$$

where $h_{\ell s}$ with $\ell \in \mathcal{L}_s$ are normal vectors according to the convention in (18).

*Proof:* The proof follows through steps similar to Lemma 33 by considering that $s$ being a strict backward mode for $x$ is equivalent to $s$ being a strict forward mode for the same $x$ in the time-reversed system $\dot{x}(\tau) = -A_s x(\tau) - b_s$ with $x(\tau) \in X_s$. ■

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