

The one-step-map for switched singular systems in discrete-time

Pham Ky Anh^a, Pham Thi Linh^a, Do Duc Thuan^b, and Stephan Trenn^c

Abstract—We study switched singular systems in discrete time and first highlight that in contrast to continuous time regularity of the corresponding matrix pairs is not sufficient to ensure a solution behavior which is causal with respect to the switching signal. With a suitable index-1 assumption for the whole switched system, we are able to define a one-step-map which can be used to provide explicit solution formulas for general switching signals.

I. INTRODUCTION

We consider switched singular systems (SwSS) in discrete time of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad k \in \mathbb{N}, \quad (1)$$

where $\sigma : \mathbb{N} \rightarrow \{1, 2, \dots, n\}$ is the switching signal determining at each time $k \in \mathbb{N}$ which of the $n \in \mathbb{N}$ system modes is active, $x(k) \in \mathbb{R}^n$, $n \in \mathbb{N}$, is the state and $u(k) \in \mathbb{R}^m$, $m \in \mathbb{N}$, is the input at time k . The different modes of the switched systems are described by the matrices $E_1, E_2, \dots, E_n, A_1, A_2, \dots, A_n \in \mathbb{R}^{n \times n}$ and $B_1, B_2, \dots, B_n \in \mathbb{R}^{n \times m}$. SwSS of the form (1) are a special case of time-varying singular systems where only finitely many different matrix triples (E_k, A_k, B_k) describe the dynamics. The introduction of a switching signal is motivated by the situation that mode changes are induced by relatively rare events, e.g. faults or event triggered control actions. In case all matrices E_k are invertible, then premultiplying (1) from the left with $E_{\sigma(k)}^{-1}$ leads to an equivalent switched system of the form

$$x(k+1) = \bar{A}_{\sigma(k)}x(k) + \bar{B}_{\sigma(k)}u(k)$$

for which existence and uniqueness of solutions is well established. Singular coefficients E_k naturally occur when modeling dynamical processes subject to algebraic constraints, see e.g. [15].

For continuous time systems the solution theory of switched singular systems is well established [23], in particular, if considered in an appropriate distributional solution space, existence and uniqueness of solutions for all switching signals is guaranteed if, and only if, all matrix pairs (E_k, A_k) are regular (see the forthcoming Definition 2.1). Two important properties in the continuous time case are: 1) causality of the solutions with respect to the switching signal, i.e. a future

change in the switching signal will not change the solution at the current time and 2) uniqueness of solutions for given switching signal and given initial value. The following example shows that this is not the case anymore in the discrete time, which indicates that a straightforward generalization from the continuous time case to the discrete time case is not possible (note that this non-causality and non-uniqueness even occurs for “well-behaved” matrix pairs; indeed, the matrix pairs in the following example are regular and *not* of higher index, see Section II for a formal definition).

Example 1.1: Consider the SwSS (1) with $n = 2$ modes and

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ A_1 = A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = B_2 = 0.$$

For the constant switching signals $\sigma_1 \equiv 1$ and $\sigma_2 \equiv 2$ the solutions are then given by $x(k) = \begin{pmatrix} x_0^1 \\ 0 \end{pmatrix}$ and $x(k) = \begin{pmatrix} 0 \\ x_0^2 \end{pmatrix}$, respectively, for all $k \in \mathbb{N}$ and some $x_0^1, x_0^2 \in \mathbb{R}$; in particular, the initial value uniquely determines the solutions. We now consider the switched system (1) with a switching signal with one switch at time $k_s > 0$, e.g. $\sigma = \sigma_{12}$, where

$$\sigma_{12}(k) = \begin{cases} 1, & k < k_s \\ 2, & k \geq k_s. \end{cases}$$

The corresponding SwSS (1) reads as

$$\begin{array}{ll} k < k_s : & k \geq k_s : \\ x_1(k+1) = x_1(k) & 0 = x_1(k) \\ 0 = x_2(k), & x_2(k+1) = x_2(k). \end{array}$$

We first observe that the value $x_1(k_s)$ is constraint by both modes: For $k = k_s - 1$ we have from mode 1 that $x_1(k_s) = x_1(k_s - 1)$ and for $k = k_s$ we have from mode 2 that $0 = x_1(k_s)$. As a consequence, $0 = x_1(k_s) = x_1(k_s - 1) = \dots = x_1(0)$, hence $x(k) = 0$ for all $k < k_s$, i.e. the presence of the switch at time $k_s > 0$ invalidates all non-zero solutions from the past (loss of causality). The second observation is that $x_2(k_s)$ is neither constraint by the first mode (it does not appear in the equations of the first mode for $k < k_s$), nor is it constraint by the second mode (where it simply plays the role of a free initial value), i.e. the presence of a switch results in the loss of unique solvability. //

The loss of causality and uniqueness due to switching is usually an undesired property and our goal is to give conditions for well-posedness of the switched system (1) in the sense that a family of *one-step-maps* $\Phi_{i,j} \in \mathbb{R}^{n \times n}$, $i, j \in \{1, 2, \dots, n\}$ exists, such that for any switching signal

The third author was supported by NAFOSTED project 101.01-2017.302 and the forth author was supported by the NWO Vidi grant 639.032.733

^aFaculty of Mathematics, Mechanics, and Informatics, Vietnam National University, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam,

^bSchool of Applied Mathematics and Informatics, Hanoi University for Science and Technology, 1 Dai Co Viet, Hanoi, Vietnam

^cBernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands

we have that $x(\cdot)$ is a solution of (1) with $u = 0$ if, and only if,

$$x(k+1) = \Phi_{\sigma(k+1), \sigma(k)} x(k).$$

Although some authors have already studied discrete-time singular switched (or time-varying) systems (e.g. [15], [2], [16], [3], [26], [29], [31], [28], [6], [7], [27], [1]) it seems that the existence of a one-step-map was not investigated so far and we want to close this gap with this contribution.

This note is structured as follows. After some preliminaries on non-switched singular systems, we investigate first the homogeneous case in Section III, where Theorem 3.5 is our main result about the existence of a one-step-map for general switched singular systems. In Section IV we present a constructive way to calculate the one-step-map, some technical results of the section will also play an important role in Section V, where we generalize the notion of the one-step-map to the inhomogeneous case.

II. PRELIMINARIES

We first recall some important notation and properties of non-switched homogeneous singular systems of the form

$$Ex(k+1) = Ax(k), \quad (2)$$

where $E, A \in \mathbb{R}^{n \times n}$ are given.

Definition 2.1: A matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ is called *regular* if, and only if, the polynomial $\det(sE - A)$ is not identically zero. //

Lemma 2.2 ([34], [12]): A matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n \times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \quad (3)$$

where $N \in \mathbb{R}^{n_N \times n_N}$ is nilpotent and $J \in \mathbb{R}^{n_J \times n_J}$ with $n_N + n_J = n$. ■

In view of [4] we call (3) a *quasi Weierstrass form (QWF)* of (E, A) . The QWF is unique up to similarity of the matrices J and N ; in particular, the nilpotency index of N (the smallest number $\nu \in \mathbb{N}$ such that $N^\nu = 0$) is independent of the choices for S and T and we will define the *index* of a regular matrix pair (E, A) as the nilpotency index of N in the QWF. In the index-1 case it is actually easy to see that $T = [T_1, T_2]$ and $S = [ET_1, AT_2]^{-1}$ transform (E, A) into QWF if, and only if, the full column rank matrices T_1, T_2 are chosen so that

$$\begin{aligned} \text{im } T_1 &= \boxed{\mathcal{S} := A^{-1}(\text{im } E)} := \{\xi \in \mathbb{R}^n : A\xi \in \text{im } E\}, \\ \text{im } T_2 &= \ker E; \end{aligned}$$

Note that then $\mathcal{S} \cap \ker E = \{0\}$. In fact, the following stronger results holds.

Lemma 2.3 ([9, Appendix A, Thm. 13], cf. [5, Prop. 9]): The matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ is regular and of index-1 if, and only if,

$$\mathcal{S} \oplus \ker E = \mathbb{R}^n. \quad (4)$$

Remark 2.4: From the dimension formula for the Wong-sequences given in [4, Lem. 2.3], it follows that for regular matrix pairs (E, A) the index-1 condition (4) is in fact equivalent to

$$\mathcal{S} \cap \ker E = \{0\}. \quad (5)$$

The main relevance of regularity and index-1 is the following statement about existence and uniqueness of solutions of the non-switched singular system (2).

Lemma 2.5: Assume (E, A) is regular and of index-1, i.e. it satisfies (5), then (2) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ has a unique solution if, and only if $x_0 \in \mathcal{S}$ and the solution is then given by

$$x(k) = \Phi_{(E,A)}^k x_0, \quad \text{with } \Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where T and J are given by the QWF (3) and $\Phi_{(E,A)}$ is independent from the specific choice of T .

Proof: The proof is straight forward and omitted due to space limitations. ■

Remark 2.6: The matrix $\Phi_{(E,A)}$ corresponds to the matrix A^{diff} in continuous time, see e.g. [25] and can be interpreted as the *one-step map* for (2), i.e. every solution of (2) satisfies

$$x(k+1) = \Phi_{(E,A)} x(k), \quad k \in \mathbb{N}. \quad (6)$$

However, it is important to note that this interpretation is only valid if we assume that (2) holds for *at least two time steps*. In fact, from

$$Ex(1) = Ax(0)$$

we can only conclude that

$$x(1) \in \{\Phi_{(E,A)} x(0)\} + \ker E.$$

In order to conclude that $x(1) = \Phi_{(E,A)} x(0)$ we additionally have to take into account

$$Ex(2) = Ax(1) \quad (\text{which implies } x(1) \in \mathcal{S})$$

together with the index-1 assumption (5). If the matrix pair (E, A) is not index-1, i.e. (5) is not valid, then one has to consider also the equation

$$Ex(3) = Ax(2)$$

which implies that $x(2) \in \mathcal{S}$ and therefore $x(1) \in A^{-1}(ES)$. For index-2 systems $A^{-1}(ES) \cap \ker E = \{0\}$ which now can be used to conclude uniqueness of $x(1)$. In general, for index ν , the one-step-map (6) from $x(k)$ to $x(k+1)$ is only valid if the difference equation (2) is assumed to also hold for the *future times* $k+2, k+3, \dots, k+\nu$. //

Our goal will be to define a suitable one-step-map also for the switched case and to be able to define a state-transition map. Therefore, we conclude this section by recalling the definition of the state-transition map for non-singular (switched) systems.

Definition 2.7: Consider a switched (non-singular) linear system

$$x(k+1) = A_{\sigma(k)} x(k), \quad k \in \mathbb{N}, \quad (7)$$

where $\sigma : \mathbb{N} \rightarrow \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$. The *state transition matrix* $\Phi_\sigma(k, h)$ for system (7) is defined as

$$\Phi_\sigma(k, h) = A_{\sigma(k-1)} A_{\sigma(k-2)} \dots A_{\sigma(h)}.$$

for $k > h$ and $\Phi_\sigma(h, h) = I$. //

It is easily seen that all solutions of (7) satisfy

$$x(k) = \Phi_\sigma(k, h)x(h), \quad \forall k, h \in \mathbb{N} \text{ with } k \geq h,$$

in particular, the initial value problem (7), $x(0) = x_0 \in \mathbb{R}^n$, has the unique solution

$$x(k) = \Phi_\sigma(k, 0)x_0, \quad k \in \mathbb{N}.$$

Note that, in contrast to the continuous time, the transition matrix can in general not be defined backwards in time, i.e. for $k < h$, because the matrices A_i can be singular.

III. HOMOGENEOUS SWITCHED SINGULAR SYSTEMS

In this section we consider the homogeneous case of (1), i.e. the following SwSS:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \quad (8)$$

and first define a desired solvability property:

Definition 3.1: SwSS (8) is called *causal (with respect to the switching signal)* iff for all switching signals σ and all corresponding solutions x the following implication holds for any switching signal $\tilde{\sigma}$ and any $\tilde{k} \in \mathbb{N}$

$$\begin{aligned} \sigma(k) = \tilde{\sigma}(k) \quad \forall k \leq \tilde{k} \\ \implies \exists \text{ sol. } \tilde{x} \text{ of (8) with } \tilde{\sigma} : \tilde{x}(k) = x(k) \quad \forall k \leq \tilde{k}. \end{aligned}$$

In other words, (8) is called causal if changing the switching signal in the future, does not make it necessary to change the solution in the past. //

Note that for the definition of causality it is not required that the switched system (8) has unique solutions.

Example 1.1 already showed that regularity and index-1 of the individual matrix pairs will not be enough to guarantee causality (in contrast to the continuous time case).

We now propose a generalization of the index-1 property from individual matrix pairs to the a whole family of matrix pairs as follows:

Definition 3.2 (cf. [2], [3], [14]): A family of matrix pairs $\{(E_1, A_1), \dots, (E_n, A_n)\}$ or the corresponding SwSS (8) is called *index-1* iff it satisfies the following conditions

- (i) Each matrix pair (E_i, A_i) , $i = 1, \dots, n$, is regular,
- (ii) $\mathcal{S}_i \cap \ker E_j = \{0\}$, $\forall i, j \in \{1, 2, \dots, n\}$, where $\mathcal{S}_i := A_i^{-1}(\text{im } E_i)$. //

In view of Lemma 2.3 Condition (ii) is indeed a generalization of the index-1 property (5) for a single matrix pair; in particular, it implies that each (regular) pair of the family is index-1. Observe that Definition 3.2 does in general not guarantee the regularity and index-1 property of the ‘‘mixed’’ matrix pairs (E_i, A_j) for $i \neq j$.

Revisiting Example 1.1, it is easily seen that

$$\begin{aligned} \mathcal{S}_1 = \text{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \ker E_1 = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \mathcal{S}_2 = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \ker E_2 = \text{im} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned}$$

and Condition (ii) is clearly not satisfied for $i = 1, j = 2$ as well as for $i = 2, j = 1$.

Before providing the main result concerning the solvability of (8), we first highlight an important consequence from the index-1 property.

Lemma 3.3: Let the SwSS (8) be of index-1. Then the following statements hold:

1. $\text{rank } E_i = \text{const} =: r$.
2. Condition (ii) is equivalent to the relation

$$\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n \quad (9)$$

for all $i, j \in \{1, 2, \dots, n\}$.

Proof: 1. Suppose that condition (ii) holds. Due to Remark 2.4 each regular matrix pair (E_i, A_i) is therefore of index-1, in particular (9) holds for $i = j$. Thus, we get $\dim \mathcal{S}_i = n - \dim(\ker E_i)$ for all $i = 1, 2, \dots, n$.

On the other hand, condition (ii) implies that $\dim \mathcal{S}_i \leq n - \dim \ker E_j$, which gives

$$\begin{aligned} \dim \mathcal{S}_i = n - \dim \ker E_i &\leq n - \dim \ker E_j \\ \Leftrightarrow \dim \ker E_i &\geq \dim \ker E_j. \end{aligned}$$

Since the last relation holds for all $i, j = 1, 2, \dots, n$, it follows $\dim \ker E_i = \text{const}$ for all $i = 1, 2, \dots, n$, or $\text{rank } E_i = \text{const} =: r$, $i = 1, 2, \dots, n$.

2. Obviously, condition (9) implies (ii). As shown above, the index-1 property implies $\dim \mathcal{S}_i = n - \dim \ker E_i = r$ for each mode i . This gives $\dim \mathcal{S}_i + \dim \ker E_j = n$ for all $i, j \in \{1, \dots, n\}$, hence, condition (ii) implies (9). ■

Note that the case $r = n$ is not excluded; however, then (ii) is trivially satisfied and the switched system is equivalent to a switched nonsingular system (7) for which a solution theory is already established, hence in the following we will only consider the case $r < n$.

The following lemma (without proof) highlights a simple geometric property of subspaces and will be crucial for deriving the upcoming explicit solution formula for (8).

Lemma 3.4: Consider two subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ such that

$$\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n$$

and let $\Pi_{\mathcal{V}}^{\mathcal{W}}$ be the unique projector onto \mathcal{V} along \mathcal{W} (i.e. $\text{im } \Pi_{\mathcal{V}}^{\mathcal{W}} = \mathcal{V}$ and $\ker \Pi_{\mathcal{V}}^{\mathcal{W}} = \mathcal{W}$). Then for any $x \in \mathbb{R}^n$ the following holds:

$$\mathcal{V} \cap (\{x\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}}x\},$$

in other words, for any $x \in \mathbb{R}^n$ there exists a unique vector $y \in \mathcal{V}$ for which there exists $w \in \mathcal{W}$ with $y = x + w$ and this vector is given by $y = \Pi_{\mathcal{V}}^{\mathcal{W}}x$. ■

We are now ready to present our main result.

Theorem 3.5: The SwSS (8) of index-1 in the sense of Definition 3.2 has for every switching signal σ a solution with $x(0) = x_0 \in \mathbb{R}^n$ if, and only if, $x_0 \in \mathcal{S}_{\sigma(0)}$, where $\mathcal{S}_\ell := A_\ell^{-1}(\text{im } E_\ell)$, $\ell \in \{1, 2, \dots, n\}$. This solution is unique and satisfies

$$x(k+1) = \Phi_{\sigma(k+1), \sigma(k)}x(k) \quad \forall k \in \mathbb{N} \quad (10)$$

where $\Phi_{i,j}$ is the *one-step map* from mode j to mode i given by

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \Phi_{(E_j, A_j)}$$

with $\Pi_{\mathcal{S}_i}^{\ker E_j}$ being the unique projector onto \mathcal{S}_i along $\ker E_j$ and $\Phi_{(E_j, A_j)}$ being the one-step map corresponding to mode j as in Lemma 2.5.

Proof: Clearly, $x_0 \in \mathcal{S}_{\sigma(0)}$ is necessary for existence of a solution. We show sufficiency by induction: Assume that $x(\ell)$ already satisfies (8) for $\ell = 0, 1, \dots, k$ and that $x(k) \in \mathcal{S}_{\sigma(k)}$. In order to extend this solution to $k+1$ it suffices to find $x(k+1)$ such that

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k)$$

and

$$E_{\sigma(k+1)}\xi = A_{\sigma(k+1)}x(k+1) \text{ for some } \xi \in \mathbb{R}^n.$$

In view of Remark 2.6 the first condition is equivalent to

$$x(k+1) \in \{\Phi_{(E_{\sigma(k)}, A_{\sigma(k)})}x(k)\} + \ker E_{\sigma(k)}$$

and the second condition is equivalent to

$$x(k+1) \in A_{\sigma(k+1)}^{-1}(\text{im } E_{\sigma(k+1)}) = \mathcal{S}_{\sigma(k+1)}.$$

By assumption $\ker E_{\sigma(k)} \cap \mathcal{S}_{\sigma(k+1)} = \{0\}$, hence Lemma 3.3 together with Lemma 3.4 yields that $x(k+1)$ is uniquely given by (10). ■

Remark 3.6: In contrast to the nonsingular case (7), the one-step-map (10) from $x(k)$ to $x(k+1)$ depends not only on the mode at time k but also on the mode at time $k+1$. //

The existence of a one-step-map now allows us to define a state-transition map in a similar way as for the non-singular case (cf. Definition 2.7).

Definition 3.7: Consider a family of matrix pairs $\{(E_i, A_i) \mid i = 1, 2, \dots, n\}$ of index-1 and the corresponding SwSS (8). The transition matrix for (8) is given by

$$\Phi_{\sigma}(k, h) = \Phi_{\sigma(k), \sigma(k-1)} \Phi_{\sigma(k-1), \sigma(k-2)} \cdots \Phi_{\sigma(h+1), \sigma(h)}$$

for $k > h$ and $\Phi_{\sigma}(h, h) = \Pi_{\mathcal{S}_{\sigma(h)}}^{\ker E_{\sigma(h)}}. //$

With this definition we arrive at the following corollary of Theorem 3.5.

Corollary 3.8: Consider an index-1 SwSS (1) with corresponding transition matrix Φ_{σ} as in Definition 3.7. Then all solutions are given by

$$x(k) = \Phi_{\sigma}(k, 0)x_0, \quad x_0 \in \mathbb{R}^n. \quad (11)$$

In particular,

$$x(0) = \Pi_{\mathcal{S}_{\sigma(0)}}^{\ker E_{\sigma(0)}}x_0$$

and $x(0) = x_0$ if, and only if, $x_0 \in \mathcal{S}_{\sigma(0)}$.

Remark 3.9: One may wonder how *necessary* the index-1 assumption from Definition 3.2 really is for existence and uniqueness of solutions of the SwSS (1). It is not difficult to see that in the non-switched case only regularity is necessary to ensure existence and uniqueness of solutions. However, in view of Remark 2.6 for higher index systems it is not possible to conclude existence of the one-step-map by just

looking at the current and the next mode. In particular, the switched system would then not be causal w.r.t. the switching signal (Definition 3.1). Furthermore, assuming a one-step-map $\Phi_{i,j}$ exists, then $x(0) = 0$ should imply $x(1) = 0$ for any switching signal σ with $\sigma(0) = j$ and $\sigma(1) = i$ (in particular, independently from the values $\sigma(k)$ for $k > 1$). However $E_jx(1) = A_jx(0) = 0$ and $E_i\xi = A_i x(1)$ for some $\xi \in \mathbb{R}^n$ is satisfied if, and only if, $x(1) \in \ker E_j \cap \mathcal{S}_1 \supseteq \{0\}$, hence $x(1) = 0$ is not the only possible solution of (1) considered for $k = 0, 1$, therefore, a one step map cannot exist. Altogether, the index-1 assumption for (1) is necessary for causality of the switched system as well as for the existence of a one-step-map (which only depends on the current and past mode). //

IV. A CONSTRUCTIVE FORMULA FOR THE ONE-STEP-MAP

In what follows we give a constructive formula for the matrix $\Phi_{i,j}$ as well as for the unique solution of SwSS (8).

Although the following results are partially obtained by similar arguments as in [2], [3], [14], we will give their proofs here to make our presentation self-contained. Furthermore, these properties also play a crucial rule for the treatment of the inhomogeneous case later.

Lemma 4.1: Consider the SwSS (8) and assume that it is of index-1. For $i = 1, \dots, n$, let $V_i := [s_i^1, \dots, s_i^r, h_i^{r+1}, \dots, h_i^n]$ be such that its columns form bases of \mathcal{S}_i and $\ker E_i$, respectively. Let $P := \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$, where I_r is an $r \times r$, identity matrix, $Q := I_n - P$. Finally, let $P_i := V_i P V_i^{-1} = \Pi_{\mathcal{S}_i}^{\ker E_i}$, $Q_i := I - P_i = \Pi_{\ker E_i}^{\mathcal{S}_i}$ and $Q_{i,j} := V_j Q V_i^{-1}$ for $i, j = 1, \dots, n$. Then the following properties hold

- (i) $G_{i,j} := E_i + A_i Q_{i,j}$ is nonsingular for all $i, j \in \{1, 2, \dots, n\}$,
- (ii) $\Pi_{\mathcal{S}_i}^{\ker E_j} = I - Q_{i,j} G_{i,j}^{-1} A_i$,
- (iii) $\Phi_{(E_i, A_i)} = P_i G_{ii}^{-1} A_i$,
- (iv) $\Phi_{i,j} = (I - Q_{i,j} G_{i,j}^{-1} A_i) P_j G_{jj}^{-1} A_j$.

Proof: (i) Assume that $x \in \ker G_{i,j}$, then $A_i Q_{i,j} x = -E_i x \in \text{im } E_i$, hence $Q_{i,j} x \in \mathcal{S}_i$. Further, $Q_{i,j} x = V_j Q V_i^{-1} x \in \text{im } V_j Q = \ker E_j$. Since $\mathcal{S}_i \cap \ker E_j = \{0\}$, we get $Q_{i,j} x = 0$, hence, $E_i x = -A_i Q_{i,j} x = 0$, therefore $x \in \ker E_i = \text{im } Q_i$. Since Q_i is a projector, we have $x = Q_i x$. On the other hand $Q_i x = V_i V_j^{-1} Q_{i,j} x = 0$, thus $x = Q_i x = 0$. This shows that $\ker G_{i,j} = \{0\}$, i.e., the square matrix $G_{i,j}$ is nonsingular.

(ii) We will show that $Q_{i,j} G_{i,j}^{-1} A_i$ is the projection along \mathcal{S}_i onto $\ker E_j$, it then follows that $I - Q_{i,j} G_{i,j}^{-1} A_i = \Pi_{\mathcal{S}_i}^{\ker E_j}$. First observe that $G_{i,j} V_i Q = (E_i + A_i Q_{i,j}) V_i Q = A_i V_j Q$ because $E_i V_i Q = 0$ by definition, hence $(Q_{i,j} G_{i,j}^{-1} A_i)^2 = Q_{i,j} G_{i,j}^{-1} A_i V_j Q V_i^{-1} G_{i,j}^{-1} A_i = Q_{i,j} G_{i,j}^{-1} G_{i,j} V_i Q V_i^{-1} G_{i,j}^{-1} A_i = Q_{i,j} G_{i,j}^{-1} A_i$, i.e. $Q_{i,j} G_{i,j}^{-1} A_i$ is idempotent and therefore a projector.

It remains to be shown that $\text{im } Q_{i,j} G_{i,j}^{-1} A_i = \ker E_j$ and $\ker Q_{i,j} G_{i,j}^{-1} A_i = \mathcal{S}_i$. From $E_j V_j Q = 0$ it follows that immediately that $\text{im } Q_{i,j} G_{i,j}^{-1} A_i \subseteq \ker E_j$. For $x \in \ker E_j \subseteq \text{im } Q_j$ we have $x = Q_j x$ and therefore $\text{im } Q_{i,j} G_{i,j}^{-1} A_i \supseteq$

$Q_{i,j}G_{i,j}^{-1}A_i x = Q_{i,j}G_{i,j}^{-1}A_i V_j Q V_j^{-1} x = Q_{i,j}G_{i,j}^{-1}A_i x = Q_{i,j}G_{i,j}^{-1}G_{i,j}V_i Q V_j^{-1} x = V_j Q V_j^{-1} x = x$, which shows that also $\ker E_j \subseteq \text{im } Q_{i,j}G_{i,j}^{-1}A_i$. Finally, the following equivalences hold:

$$\begin{aligned} x \in \mathcal{S}_i &\iff A_i x = E_i \xi \text{ for some } \xi \\ &\iff G_{i,j}^{-1}A_i x = G_{i,j}^{-1}E_i \xi = P_i \xi \\ &\iff V_i^{-1}G_{i,j}^{-1}A_i x = P V_i^{-1} \xi \\ &\iff Q V_i^{-1}G_{i,j}^{-1}A_i x = 0 \\ &\iff Q_{i,j}G_{i,j}^{-1}A_i x = 0. \end{aligned}$$

This shows $\text{im } \mathcal{S}_i = \ker Q_{i,j}G_{i,j}^{-1}A_i$.

(iii) In view of Lemma 2.2 and the discussing thereafter, it holds that

$$(E_i V_i P + A_i V_i Q)^{-1} A_i V_i = \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix}$$

for some $J_i \in \mathbb{R}^{r \times r}$. Hence,

$$\Phi_{(E_i, A_i)} = V_i P (E_i V_i P + A_i V_i Q)^{-1} A_i.$$

On the other hand

$$P_i G_{i,i}^{-1} A_i = V_i P (E_i V_i + A_i V_i Q)^{-1} A_i$$

and since $E_i V_i P = E_i V_i$ the claim is shown.

(iv) This is a direct consequence from (ii),(iii) and Theorem 3.5. \blacksquare

We illustrate the usefulness of the derived formulas via the following example.

Example 4.2: Let

$$\begin{aligned} (E_1, A_1) &= \left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 5 \\ 3 & 3 & -1 \\ -5 & -1 & 7 \end{bmatrix} \right), \\ (E_2, A_2) &= \left(\begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & -1 \\ -5 & -3 & 1 \end{bmatrix} \right), \\ (E_3, A_3) &= \left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 5 & 1 \end{bmatrix} \right) \end{aligned}$$

A simple computation shows that

$$\begin{aligned} \ker E_1 &= \text{span}\{(-1, 1, 0)^\top\}, \\ \ker E_2 &= \text{span}\{(0, -1, 1)^\top\}, \\ \ker E_3 &= \text{span}\{(1, 0, 1)^\top\}, \\ \mathcal{S}_1 &= \text{span}\{(-2, 1, -1)^\top, (-1, -1, 0)^\top\}, \\ \mathcal{S}_2 &= \text{span}\{(0, 1, 1)^\top, (1, -1, 2)^\top\}, \\ \mathcal{S}_3 &= \text{span}\{(0, 1, 1)^\top, (-1, -1, 0)^\top\}, \end{aligned}$$

hence $\mathcal{S}_i \cap \ker E_j = \{0\}$, $i, j = 1, 2, 3$. It means that system (8) with the above data is of index-1. Furthermore, we can chose V_1, V_2, V_3 such that its columns form bases of \mathcal{S}_i and $\ker E_i$, respectively, then we can calculate $G_{i,j}$ and obtain

$$\begin{aligned} \Phi_{1,1} &= \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} & \frac{1}{2} \\ 2 & 2 & -1 \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, & \Phi_{1,2} &= \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}, \\ \Phi_{1,3} &= \begin{bmatrix} 1 & -4 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}, & \Phi_{2,1} &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \\ \Phi_{2,2} &= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -1 & -1 \end{bmatrix}, & \Phi_{2,3} &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} \end{bmatrix}, \\ \Phi_{3,1} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, & \Phi_{3,2} &= \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}, \\ \Phi_{3,3} &= \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Choosing $\sigma(k) = (k \bmod 3) + 1$, we can compute the corresponding solution as follows

$$\begin{aligned} x(3k) &= (\Phi_{1,3} \Phi_{3,2} \Phi_{2,1})^k x_0 \\ x(3k+1) &= \Phi_{2,1} (\Phi_{1,3} \Phi_{3,2} \Phi_{2,1})^k x_0 \\ x(3k+2) &= \Phi_{3,2} \Phi_{2,1} (\Phi_{1,3} \Phi_{3,2} \Phi_{2,1})^k x_0. \end{aligned}$$

V. INHOMOGENEOUS SWSS

We return our attention to the inhomogeneous SwSS (1).

Theorem 5.1: Suppose that the family of matrix pairs $\{(E_i, A_i)\}_{i=1}^N$ of SwSS (1) is of index-1. Then there exists matrices $\bar{\Phi}_{i,j,\ell} \in \mathbb{R}^{n \times n}$ and $\bar{\Psi}_{i,j,\ell}, \tilde{\Psi}_{i,j} \in \mathbb{R}^{n \times m}$ such that all solutions of (1) satisfy for all $k \in \mathbb{N}$

$$\begin{aligned} x(k+1) &= \bar{\Phi}_{\sigma(k+1), \sigma(k), \sigma(k-1)} x(k) \\ &\quad + \bar{\Psi}_{\sigma(k+1), \sigma(k), \sigma(k-1)} u(k) + \tilde{\Psi}_{\sigma(k+1), \sigma(k)} u(k+1), \end{aligned} \quad (12)$$

where $\sigma(-1) := \sigma(0)$.

In fact, with V_i and $G_{i,j}$ as defined in Lemma 4.1 let

$$V_i^{-1} G_{i,j}^{-1} A_i V_j = \begin{bmatrix} \bar{A}_{i,j}^1 & 0 \\ \bar{A}_{i,j}^2 & I_{n-r} \end{bmatrix}, \quad (13)$$

where $\bar{A}_{i,j}^1 \in \mathbb{R}^{r \times r}$, $\bar{A}_{i,j}^2 \in \mathbb{R}^{(n-r) \times r}$ and $\bar{B}_{i,j} = V_i^{-1} G_{i,j}^{-1} B_i = \begin{bmatrix} \bar{B}_{i,j}^1 \\ \bar{B}_{i,j}^2 \end{bmatrix}$, where $\bar{B}_{i,j}^1 \in \mathbb{R}^{r \times m}$ and $\bar{B}_{i,j}^2 \in \mathbb{R}^{(n-r) \times m}$. Then

$$\begin{aligned} \bar{\Phi}_{i,j,\ell} &= V_j \begin{bmatrix} \bar{A}_{j,\ell}^1 & 0 \\ -\bar{A}_{j,\ell}^2 \bar{A}_{j,\ell}^1 & 0 \end{bmatrix} V_\ell^{-1}, \\ \bar{\Psi}_{i,j,\ell} &= V_j \begin{bmatrix} \bar{B}_{j,\ell}^1 \\ -\bar{A}_{j,\ell}^2 \bar{B}_{j,\ell}^1 \end{bmatrix}, \\ \tilde{\Psi}_{i,j} &= V_j \begin{bmatrix} 0 \\ -\bar{B}_{i,j}^2 \end{bmatrix}. \end{aligned}$$

Furthermore, there exists a solution of (1) with $x(0) = x_0$ if, and only if,

$$x_0 \in \text{im } V_{\sigma(-1)} \begin{bmatrix} I & 0 \\ -\bar{A}_{\sigma(0), \sigma(-1)}^2 & \bar{B}_{\sigma(0), \sigma(-1)}^2 \end{bmatrix}.$$

Proof: Observe that $G_{i,j} P_i = (E_i + A_i V_j Q V_i^{-1}) V_i P V_i^{-1} = E_i P_i + A_i V_j Q P V_i^{-1} = E_i P_i$. Further, since Q_i is the projection onto $\ker E_i$ along \mathcal{S}_i , it follows $E_i Q_i = 0$, therefore, $E_i P_i = E_i (P_i + Q_i) = E_i$. Thus, $G_{i,j} P_i = E_i$, hence $P_i = G_{i,j}^{-1} E_i$. According to the proof of item (ii) of Lemma 4.1, $G_{i,j} V_i Q = A_i V_j Q$, hence, $V_i^{-1} G_{i,j}^{-1} A_i V_j Q = Q$. Therefore, we obtain

$$\begin{aligned} \bar{A}_{i,j} &:= V_i^{-1} G_{i,j}^{-1} A_i V_j = \begin{bmatrix} \bar{A}_{i,j}^1 & O \\ \bar{A}_{i,j}^2 & I_{n-r} \end{bmatrix}, \\ \bar{E}_{i,j} &:= V_i^{-1} G_{i,j}^{-1} E_i V_i = \begin{bmatrix} I_r & O \\ O & O_{n-r} \end{bmatrix}. \end{aligned}$$

Multiplying both sides of system (1) by $V_{\sigma(k)}^{-1} G_{\sigma(k), \sigma(k-1)}^{-1}$, and using the transformation $\bar{x}(k) = V_{\sigma(k-1)}^{-1} x(k)$, we get

$$\bar{E}_{\sigma(k), \sigma(k-1)} \bar{x}(k+1) = \bar{A}_{\sigma(k), \sigma(k-1)} \bar{x}(k) + \bar{B}_{\sigma(k), \sigma(k-1)} u(k). \quad (14)$$

Putting $\bar{x}(k) := (v(k)^\top, w(k)^\top)^\top$, where $v(k) \in \mathbb{R}^r$, $w(k) \in \mathbb{R}^{n-r}$, we can reduce system (14) to system

$$v(k+1) = \bar{A}_{\sigma(k),\sigma(k-1)}^1 v(k) + \bar{B}_{\sigma(k),\sigma(k-1)}^1 u(k)$$

together with the constraint

$$w(k) = -\bar{A}_{\sigma(k),\sigma(k-1)}^2 v(k) - \bar{B}_{\sigma(k),\sigma(k-1)}^2 u(k),$$

which is equivalent to

$$\begin{aligned} w(k+1) &= -\bar{A}_{\sigma(k+1),\sigma(k)}^2 v(k+1) - \bar{B}_{\sigma(k+1),\sigma(k)}^2 u(k+1) \\ &= -\bar{A}_{\sigma(k+1),\sigma(k)}^2 \bar{A}_{\sigma(k),\sigma(k-1)}^1 v(k) \\ &\quad - \bar{A}_{\sigma(k+1),\sigma(k)}^2 \bar{B}_{\sigma(k),\sigma(k-1)}^1 u(k) \\ &\quad - \bar{B}_{\sigma(k+1),\sigma(k)}^2 u(k+1). \end{aligned}$$

By transforming back to the original coordinates via $x(k+1) = V_{\sigma(k)} \begin{bmatrix} v(k+1) \\ w(k+1) \end{bmatrix}$ and $\begin{bmatrix} v(k) \\ w(k) \end{bmatrix} = V_{\sigma(k-1)}^{-1} x(k)$ we arrive at (12). Finally, existence of a solution is guaranteed, if and only if, $x(0)$ is consistent with (1), or in the (v, w) -coordinates, if and only if there exists $u(0) \in \mathbb{R}^m$ such that

$$w(0) = -\bar{A}_{\sigma(0),\sigma(-1)}^2 v(0) - \bar{B}_{\sigma(0),\sigma(-1)}^2 u(0)$$

where $v(0) \in \mathbb{R}^r$ is arbitrary. In other words,

$$\begin{bmatrix} v(0) \\ w(0) \end{bmatrix} \in \text{im} \begin{bmatrix} I & 0 \\ -\bar{A}_{\sigma(0),\sigma(-1)}^2 & \bar{B}_{\sigma(0),\sigma(-1)}^2 \end{bmatrix},$$

which yields the claimed condition by applying the coordinate transformation $V_{\sigma(-1)}$. ■

Remark 5.2: In contrast to the homogeneous case the one-step-map from $x(k)$ to $x(k+1)$ in the inhomogeneous case not only depends on the modes at time $k+1$ and k but also on the mode $k-1$. Furthermore, the allowed space of consistent initial values seems to depend on the choice of $\sigma(-1)$, and it is ongoing research to investigate the significance of this fact. //

VI. CONCLUSION

We have shown that for switched singular systems in discrete time with a certain index-1 property a unique one-step-map exists which can fully characterize all possible solutions. An application of this result could be the stability analysis for switched singular systems and is ongoing research.

REFERENCES

- [1] P.K. Anh, P.T. Linh, Stability of periodically switched discrete-time linear singular systems, *J. Difference Equ. Appl.*, 23(10) (2017) 1680-1693.
- [2] P.K. Anh, H.T.N. Yen, Floquet theorem for linear implicit nonautonomous difference systems, *J. Math. Anal. Appl.*, 321 (2006) 921-929.
- [3] P.K. Anh, N.H. Du, L.C. Loi, Singular difference equations: an overview, *Vietnam J. Math.*, 35 (2007) 339-372.
- [4] Berger, T. and Ilchmann, A. and Trenn, S., The quasi-Weierstraß form for regular matrix pencils, *Linear Algebra Appl.*, 436(10) (2012) 4052-4069.
- [5] Bonilla, M., Malabre, M., On the control of linear systems having internal variations. *Automatica* 39(11), 1989-1996 (2003)
- [6] M. Darouach and M. Chadli, Admissibility and control of switched discrete-time singular systems, *Syst. Sci. Control Engrg.: An Open Access J.*, 1:1 (2013) 43-51.
- [7] N.H. Du, V.H. Linh, and N.T. Nga, On stability and Bohl exponent of linear singular systems of difference equations with variable coefficients, *J. Difference Equ. Appl.* 22(9) (2016) 1350-1377.
- [8] D. Debeljković, N. Višnjić and M. Pješčić, The stability of linear discrete descriptor systems in the sense of Lyapunov: an overview, *Scient. Techn. Rev.*, Vol.LVII No.3-4 (2007).
- [9] E. Griepentrog, R. März, *Differential-Algebraic Equations and Their Numerical Treatment*, Teubner-Texte Math., vol. 88, Teubner, Leipzig, (1986).
- [10] D. Koenig, B. Marx, H_∞ filtering and state feedback control for discrete-time switched descriptor systems, *IET Control Theory*, 3(6) (2009) 661-670.
- [11] D. Liberzon, S. Trenn, On stability of linear switched differential algebraic equations, *Proc. IEEE 48th Conf. Decision Control*, December (2009) 2156-2161.
- [12] Gantmacher, Felix R., *The Theory of Matrices (Vol. I & II)*, Chelsea, New York, (1959)
- [13] H. Lin, P.J. Antsaklis, Stability and stabilizability of switched Linear Systems: a survey of recent results, *IEEE Trans. Aut. Contr.* 54 (2009) 308-322.
- [14] L.C. Loi, N.H. Du, and P.K. Anh, On linear implicit non-autonomous systems of difference equations, *J. Difference Equ. Appl.* 8 (2002) 1085-1105.
- [15] Luenberger, David G., Dynamic equations in descriptor form, *IEEE Trans. Autom. Control*, 22 (1977) 312-321.
- [16] B. Meng, J. Zhang, Output feedback based admissible control of switched linear singular systems, *Automatica Sinica*, 32(2) (2006) 179-185.
- [17] G. Polya and G. Szego, *Problems and Theorems in Analysis I*, Springer, Berlin, Heidelberg, (1998).
- [18] G. C. Rota and W. G. Strang, A note on the joint spectral radius, *Indag. Math.*, 22 (1960) 379-381.
- [19] W.J. Rugh, *Linear System Theory*, 2nd ed., Prentice Hall., New Jersey, (1996).
- [20] M. Shih, J. Wu, and C. T. Pang, Asymptotic stability and generalized Gelfand spectral radius formula, *Linear Algebra Appl.*, 251 (1997) 61-70.
- [21] Z. Sun, S.S. Ge, *Stability Theory of Switched Dynamical Systems*, Springer, London, (2011).
- [22] S. Trenn, *Distributional Differential Algebraic Equations*, PhD thesis, Technische Universität Ilmenau, (2009).
- [23] S. Trenn, Switched differential algebraic equations, in Vasca, Francesco; Iannelli, Luigi (Ed.): *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, London (2012), 189-216.
- [24] S. Trenn and F. Wirth, Linear switched DAEs: Lyapunov exponents, a converse Lyapunov theorem, and Barabanov norms. In *Decision and Control (CDC), IEEE 51st Annual Conference*, (2012) 2666-2671.
- [25] Tanwani, Aneel and Trenn, Stephan, On observability of switched differential-algebraic equations, *prcdc10* (2017) 5656-5661.
- [26] Y. Xia, J. Zhang, E. Boukas, Control for discrete singular hybrid systems, *Automatica*, 44 (2008) 2635-2641.
- [27] X. Su, W. Wu, The study of exponential stability of singular switched systems, *Intern. Conf. on Intell. Sys. Contr. (ISCO)* (10)(2016).
- [28] G. Zhai, X. Xu, D. W.C. Ho, Stability of switched linear discrete-time descriptor systems: a new commutation condition, *Internat. J. Contr.*, 85 (2012) 1779-1788.
- [29] G. Zhai, X. Xu, A unified approach to stability analysis of switched linear descriptor systems under arbitrary switching, *Internat. J. Appl. Math. Comput. Sci.*, 20(2) (2010) 249-259.
- [30] G. Zhai, H. Lin, A.N. Michel, and K. Yasuda, Stability analysis for switched systems with continuous-time and discrete-time subsystems, *Proc. Amer. Control Conf.*, (2004).
- [31] G. Zhai, R. Kou, J. Imae, and T. Kobayashi, Stability analysis and design for switched descriptor systems, *Internat. J. Control, Autom., Syst.*, 7(3)(2009) 349-355.
- [32] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, Qualitative analysis of discrete-time switched systems, *Proc. Amer. Control Conf.*, 3 (2002) 1880-1885.
- [33] Y. Zhang, G. Duan, Guaranteed cost control with constructing switching law of uncertain discrete-time switched systems, *J. Syst. Eng. Electron.*, 18 (2007) 846-851.
- [34] Weierstraß, Karl, *Zur Theorie der bilinearen und quadratischen Formen*, *Berl. Monatsb.*, (1868)310-338