The one-step-map for switched singular systems in discrete-time

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***Abstract*— We study switched singular systems in discrete time and first highlight that in contrast to continuous time regularity of the corresponding matrix pairs is not sufficient to ensure a solution behavior which is causal with respect to the switching signal. With a suitable index-1 assumption for the whole switched system, we are able to define a one-step-map which can be used to provide explicit solution formulas for general switching signals.**

I. Introduction

We consider switched singular systems (SwSS) in discrete time of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad k \in \mathbb{N}, \quad (1)$$

where $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathtt{n}\}$ is the switching signal determining at each time $k \in \mathbb{N}$ which of the $\mathtt{n} \in \mathbb{N}$ system modes is active, $x(k) \in \mathbb{R}^n$, $n \in \mathbb{N}$, is the state and $u(k) \in \mathbb{R}^m$, $m \in \mathbb{N}$, is the input at time k. The different modes of the switched systems are described by the matrices $E_1, E_2, \ldots, E_{\mathtt{n}}, A_1, A_2, \ldots, A_{\mathtt{n}} \in \mathbb{R}^{n\times n}$ and $B_1, B_2, \ldots, B_{\mathtt{n}} \in \mathbb{R}^{n\times m}$. SwSS of the form (1) are a special case of time-varying singular systems where only finitely many different matrix triples (E_k, A_k, B_k) describe the dynamics. The introduction of a switching signal is motivated by the situation that mode changes are induced by relatively rare events, e.g. faults or event triggered control actions. In case all matrices E_k are invertible, then premultiplying (1) from the left with $E^{-1}_{\sigma(k)}$ leads to an equivalent switched system of the form

$$x(k+1) = \overline{A}_{\sigma(k)}x(k) + \overline{B}_{\sigma(k)}u(k)$$

for which existence and uniqueness of solutions is well established. Singular coefficients E_k naturally occur when modeling dynamical process subject to algebraic constraints, see e.g. [14].

For continuous time systems the solution theory of switched singular systems is well established [22], in particular, if considered in an appropriate distributional solution space, existence and uniqueness of solutions for all switching signals is guaranteed if, and only if, all matrix pairs (E_k, A_k) are regular (see the forthcoming Definition 2.1). An important property in the continuous time case is a causality of the solutions with respect to the switching signal, i.e. a future change in the switching signal will not change the solution at the current time. The following example shows that this is not the case anymore in the discrete time, which indicates that a straightforward generalization from the continuous time case to the discrete time case is not possible.

Example 1.1: Consider the SwSS (1) with $\mathtt{n} = 2$ modes and

$$E_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad E_2 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix},$$
$$A_1 = A_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad B_1 = B_2 = 0.$$

Note that both matrix pairs (E_1, A_1) and (E_2, A_2) are regular and of index-1 (see Section II). For the constant switching signals $\sigma_1 \equiv 1$ and $\sigma_2 \equiv 2$ the solutions are then given by $x(k) = \begin{pmatrix}x_0^1\\0\end{pmatrix}$ and $x(k) = \begin{pmatrix}0\\x_0^2\end{pmatrix}$, respectively, for all $k \in \mathbb{N}$. However, when consider a switching signal with one switch at time $k_s > 0$, e.g.

$$\sigma_{12}(k) = \begin{cases}1, & k < k_s\\2, & k \geq k_s,\end{cases}$$

it follows that the only solution of the corresponding SwSS is the zero solution! Indeed, for $k \leq k_s$ the solution has to satisfy $x(k) = (c, 0)^\top$ for some $c \in \mathbb{R}$. At $k = k_s$ it must additionally hold that $E_2x(k_s+1) = A_2x(k_s)$, in particular:

$$0 = x_1(k_s) = c,$$
$$x_2(k_s+1) = x_2(k_s) = 0,$$

from which $c = 0$ and $x(k) = (0, 0)^\top$ for all $k \in \mathbb{N}$ follows.

Although some authors have already studied discrete-time singular switched (or time-varying) systems (e.g. [14], [2], [15], [3], [25], [28], [30], [27], [5], [6], [26], [1]) it seems that the existence of a one-step-map was not investigated so far and we want to close this gap with this contribution.

This note is structured as follows. After some preliminaries on non-switched singular systems, we investigate first the homogeneous case in Section III where Theorem 3.5 is our main result about the existence of a one-step-map for general switched singular systems. In Section IV we present a constructive way to calculate the one-step-map, some technical results of the section will also play an important role in Section V, where we generalize the notion of the one-step-map to the inhomogeneous case.

II. Preliminaries

We first recall some important notation and properties of non-switched homogeneous singular systems of the form

$$Ex(k+1) = Ax(k), \quad (2)$$

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where $E, A \in \mathbb{R}^{n\times n}$ are given.

Definition 2.1: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called *regular* if, and only if, the polynomial $\det(sE - A)$ is not identically zero.

Lemma 2.2 ([33], [11]): A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{3}$$

where $N \in \mathbb{R}^{n_N\times n_N}$ is nilpotent and $J \in \mathbb{R}^{n_J\times n_J}$ with $n_N + n_J = n$.

In view of [4] we call (3) a *quasi Weierstrass form (QWF)* of (E, A). The QWF is unique up to similarity of the matrices J and N; in particular, the nilpotency index of N (the smallest number $\nu \in \mathbb{N}$ such that $N^\nu = 0$) is independent of the choices for S and T and we will define the *index* of a regular matrix pair (E, A) as the nilpotency index of N in the QWF. In the index-1 case it is actually easy to see that $T = [T_1, T_2]$ and $S = [ET_1, AT_2]^{-1}$ transform (E, A) into QWF if, and only if, the full column rank matrices T_1, T_2 matrices are chosen so that

$$\begin{aligned} \operatorname{im} T_1 &= \boxed{\mathcal{S} := A^{-1}(\operatorname{im} E)} := \{\xi \in \mathbb{R}^n \;:\; A\xi \in \operatorname{im} E\}, \\ \operatorname{im} T_2 &= \ker E; \end{aligned}$$

Note that then $\mathcal{S} \cap \ker E = \{0\}$. In fact, the following stronger results holds.

Lemma 2.3 ([8, Appendix A, Thm. 13]): The matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular and of index-1 if, and only if,

$$\mathcal{S} \cap \ker E = \{0\}. \tag{4}$$

Furthermore, if (4) holds then

$$\mathcal{S} \oplus \ker E = \mathbb{R}^n. \tag{5}$$

The main relevance of regularity and index-1 is the following statement about existence and uniqueness of solutions of the non-switched singular system (2).

Lemma 2.4: Assume (E, A) is regular and of index-1, i.e. it satisfies (4), then (2) with initial condition $x(0) = x_0 \in \mathbb{R}^n$ has a unique solution if, and only if $x_0 \in \mathcal{S}$ and the solution is then given by

$$x(k) = \Phi^k_{(E,A)} x_0, \quad \text{with } \Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

where T and J are given by the QWF (3) and $\Phi_{(E,A)}$ is independent from the specific choice of T and J.

Proof: Let $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = T^{-1}x$ then from the QWF (3) we can conclude that x solves (2) if, and only if,

$$\begin{aligned} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(k+1) \\ w(k+1) \end{pmatrix} &= \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} v(k) \\ w(k) \end{pmatrix}, \\ \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} &= T^{-1}x_0. \end{aligned}$$

In particular, for $k = 0$ we can conclude that $0 = w(0)$, hence a solution only exists if $w_0 = 0$. Since $T = [T_1, T_2]$ and $\operatorname{im} T_1 = \mathcal{S}$ we can conclude

$$w_0 = 0 \quad \Longleftrightarrow \quad x_0 \in \operatorname{im} T_1 = \mathcal{S},$$

which shows the existence statement. If $w_0 = 0$ then the unique solution for v and w is given by $v(k) = J^k v_0$ and $w(k) = 0$, hence the unique solution of (2) is given by

$$\begin{aligned} x(k) &= T \left(\begin{smallmatrix} v(k) \\ w(k) \end{smallmatrix}\right) = T \left[\begin{smallmatrix} J^k & 0 \\ 0 & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} v_0 \\ w_0 \end{smallmatrix}\right) = T \left[\begin{smallmatrix} J^k & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1} x_0 \\ &= \Phi^k_{(E,A)} x_0. \end{aligned}$$

The independence from the choice of the matrix follows from the observation that for any other transformation matrices $\overline{S}$ and $\overline{T}$ which transform (E, A) into QWF there are invertible matrices $M_J \in \mathbb{R}^{n_J\times n_J}$ and $M_N \in \mathbb{R}^{n_N\times n_N}$ such that

$$T = \overline{T} \left[\begin{smallmatrix} M_J & 0 \\ 0 & M_N \end{smallmatrix}\right], \quad S = \left[\begin{smallmatrix} M_J^{-1} & 0 \\ 0 & M_N^{-1} \end{smallmatrix}\right] \overline{S}.$$

Using the fact that $S^{-1} \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] T^{-1} = A = \overline{S}^{-1} \left[\begin{smallmatrix} \overline{J} & 0 \\ 0 & I \end{smallmatrix}\right] \overline{T}^{-1}$, we have

$$\begin{aligned} \Phi_{(E,A)} &= \overline{T} \left[\begin{smallmatrix} M_J & 0 \\ 0 & M_N \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S S^{-1} \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] T^{-1} \\ &= \overline{T} \left[\begin{smallmatrix} M_J & 0 \\ 0 & M_N \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} M_J^{-1} & 0 \\ 0 & M_N^{-1} \end{smallmatrix}\right] \overline{S}\, \overline{S}^{-1} \left[\begin{smallmatrix} \overline{J} & 0 \\ 0 & I \end{smallmatrix}\right] \overline{T}^{-1} \\ &= \overline{T} \left[\begin{smallmatrix} \overline{J} & 0 \\ 0 & 0 \end{smallmatrix}\right] \overline{T}^{-1} \end{aligned}$$

This concludes the proof. ■

Remark 2.5: The matrix $\Phi_{(E,A)}$ corresponds to the matrix A^{diff} in continuous time, see e.g. [24] and can be interpreted as the *one-step map* for (2), i.e. every solution of (2) satisfies

$$x(k+1) = \Phi_{(E,A)} x(k), \quad k \in \mathbb{N}. \tag{6}$$

However, it is important to note that this interpretation is only valid if we assume that (2) holds for *at least two time steps*. In fact, from

$$Ex(1) = Ax(0)$$

we can only conclude that

$$x(1) \in \{\Phi_{(E,A)} x(0)\} + \ker E.$$

In order to conclude that $x(1) = \Phi_{(E,A)} x(0)$ we additionally have to take into account

$$Ex(2) = Ax(1) \text{ (which implies } x(1) \in \mathcal{S})$$

together with the index-1 assumption (4). If the matrix pair (E, A) is not index-1, i.e. (4) is not valid, then one has to consider also the equation

$$Ex(3) = Ax(2)$$

which implies that $x(2) \in \mathcal{S}$ and therefore $x(1) \in A^{-1}(E\mathcal{S})$. For index-2 systems $A^{-1}(E\mathcal{S}) \cap \ker E = \{0\}$ which now can be used to conclude uniqueness of $x(1)$. In general, for index ν, the one-step-map (6) from $x(k)$ to $x(k+1)$ is only valid if the difference equation (2) is assumed to also hold for the *future* times $k+2, k+3, \ldots, k+\nu$.

Our goal will be to define a suitable one-step-map also for the switched case and to be able to define a state-transition map. Therefore, we conclude this section by recalling the definition of the state-transition map for non-singular (switched) systems.

Definition 2.6: Consider a switched (non-singular) linear system

$$x(k+1) = A_{\sigma(k)}x(k), \quad k \in \mathbb{N}, \tag{7}$$

where $\sigma : \mathbb{N} \to \{1, 2, .., \mathtt{n}\}$, $\mathtt{n} \in \mathbb{N}$, $A_1, \ldots, A_{\mathtt{n}} \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$. The *state transition matrix* $\Phi_\sigma(k, h)$ for system (7) is defined as $\Phi_\sigma(k, h) = I$ for $k = h$ and, for $k > h$,

$$\Phi_\sigma(k, h) = A_{\sigma(k-1)}A_{\sigma(k-2)} \ldots A_{\sigma(h)}.$$

It is easily seen that all solutions of (7) satisfy

$$x(k) = \Phi_\sigma(k, h)x(h), \quad \forall k, h \in \mathbb{N} \text{ with } k \geq h,$$

in particular, the initial value problem (7), $x(0) = x_0 \in \mathbb{R}^n$, has the unique solution

$$x(k) = \Phi_\sigma(k, 0)x_0, \quad k \in \mathbb{N}.$$

Note that, in contrast to the continuous time, the transition matrix can in general not be defined backwards in time, i.e. for $k < h$, because the matrices A_i can be singular.

III. Homogeneous switched singular systems

In this section we consider the homogeneous case of (1), i.e. the following SwSS:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{8}$$

and first define a desired solvability property:

Definition 3.1: SwSS (8) is called *causal (with respect to the switching signal)* iff for all switching signals σ and all corresponding solutions x the following implication holds for any switching signal $\widetilde{\sigma}$ and any $\widetilde{k} \in \mathbb{N}$

$$\begin{aligned} &\sigma(k) = \widetilde{\sigma}(k)\ \forall k \leq \widetilde{k} \\ &\quad \implies \exists \text{ sol. } \widetilde{x} \text{ of (8) with } \widetilde{\sigma} :\ \widetilde{x}(k) = x(k)\ \forall k \leq \widetilde{k}. \end{aligned}$$

In other words, (8) is called causal if changing the switching signal in the future, does not make it necessary to change the solution in the past.

Example 1.1 already showed that regularity and index-1 of the individual matrix pairs will not be enough to guarantee causality (in contrast to the continuous time case).

We now propose a generalization of the index-1 property from individual matrix pairs to the a whole family of matrix pairs as follows:

Definition 3.2 (cf. [2], [3], [13]): A family of matrix pairs $\{(E_1, A_1), \ldots, (E_{\mathtt{n}}, A_{\mathtt{n}})\}$ or the corresponding SwSS (8) is called *index-1* iff it satisfies the following conditions

(i) $\operatorname{rank} E_i = r,\ \forall i = 1, 2, .., \mathtt{n}$,
(ii) $\mathcal{S}_i \cap \ker E_j = \{0\}, \quad \forall i, j \in \{1, 2, .., \mathtt{n}\}$, where $\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$.

In view of Lemma 2.3 Condition (ii) is indeed a generalization of the regularity and index-1 property (4) for a single matrix pair; in particular, it implies that each pair of the family is regular and index-1.

Note that in (i) the case $r = n$ is not excluded; however, then (ii) is trivially satisfied and the switched system is equivalent to a switched nonsingular system (7) for which a solution theory is already established, hence in the following we will only consider the case $r < n$.

For Example 1.1, it is easily seen that

$$\mathcal{S}_1 = \operatorname{im}\begin{bmatrix}1\\0\end{bmatrix}, \ker E_1 = \operatorname{im}\begin{bmatrix}0\\1\end{bmatrix},$$
$$\mathcal{S}_2 = \operatorname{im}\begin{bmatrix}0\\1\end{bmatrix}, \ker E_2 = \operatorname{im}\begin{bmatrix}1\\0\end{bmatrix},$$

and Condition (ii) is clearly not satisfied for $i = 1$, $j = 2$ as well as for $i = 2$, $j = 1$.

Before providing the main result concerning the solvability of (8), we first highlight an important consequence from the index-1 property.

Lemma 3.3: Suppose $\operatorname{rank} E_i = r,\ i = 1, \ldots, \mathtt{n}$. Then Condition (ii) in Definition 3.2 is equivalent to the relation

$$\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n \tag{9}$$

for all $i, j \in \{1, 2, .., \mathtt{n}\}$.

Proof: Obviously, condition (9) implies (ii). Conversely, suppose that condition (ii) holds. Due to Lemma 2.3 each matrix pair (E_i, A_i) is therefore regular and index-1, in particular (9) holds for $i = j$. This implies that $\dim \mathcal{S}_i = r$ for all i because by assumption we have that $\dim \ker E_i = n - r$ for all $i \in \{1, \ldots, \mathtt{n}\}$. From this we can conclude $\dim \mathcal{S}_i + \dim \ker E_j = n$ for all $i, j \in \{1, \ldots, \mathtt{n}\}$ which together with (ii) implies (9). ∎

The following is a simple geometric property of subspaces and will be crucial for deriving the upcoming explicit solution formula for (8).

Lemma 3.4: Consider two subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ such that

$$\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n$$

and let $\Pi_{\mathcal{V}}^{\mathcal{W}}$ be the unique projector onto $\mathcal{V}$ along $\mathcal{W}$ (i.e. $\operatorname{im} \Pi_{\mathcal{V}}^{\mathcal{W}} = \mathcal{V}$ and $\ker \Pi_{\mathcal{V}}^{\mathcal{W}} = \mathcal{W}$). Then for any $x \in \mathbb{R}^n$ the following holds:

$$\mathcal{V} \cap (\{x\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}} x\},$$

in other words, for any $x \in \mathbb{R}$ there exists a unique vector $y \in \mathcal{V}$ for which there exists $w \in \mathcal{W}$ with $y = x + w$ and this vector is given by $y = \Pi_{\mathcal{V}}^{\mathcal{W}} x$.

Proof: We decompose x uniquely as $x = x_{\mathcal{V}} + x_{\mathcal{W}}$ with $x_{\mathcal{V}} \in \mathcal{V}$ and $x_{\mathcal{W}} \in \mathcal{W}$. Then $\{x\} + \mathcal{W} = \{x_{\mathcal{V}}\} + \mathcal{W}$ and for any $w \in \mathcal{W} \setminus \{0\}$ we have $x_{\mathcal{V}} + w \notin \mathcal{V}$. Hence $\mathcal{V} \cap (\{x\} + \mathcal{W}) = \{x_{\mathcal{V}}\}$. Since $\Pi_{\mathcal{V}}^{\mathcal{W}} x = x_{\mathcal{V}}$ the claim is shown. ∎

We are now ready to present our main result.

Theorem 3.5: The SwSS (8) of index-1 in the sense of Definition 3.2 has for every switching signal σ a solution with $x(0) = x_0 \in \mathbb{R}$ if, and only if, $x_0 \in \mathcal{S}_{\sigma(0)}$, where $\mathcal{S}_\ell := A_\ell^{-1}(\operatorname{im} E_\ell)$, $\ell \in \{1, 2, \dots, \mathtt{n}\}$. This solution is unique and satisfies

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k) \quad \forall k \in \mathbb{N} \tag{10}$$

where Φ_{ij} is the *one-step map* from mode j to mode i given by

$$\Phi_{ij} := \Pi_{\mathcal{S}_i}^{\ker E_j} \Phi_{(E_j, A_j)}$$

with $\Pi_{\mathcal{S}_i}^{\ker E_j}$ being the unique projector onto $\mathcal{S}_i$ along $\ker E_j$ and $\Phi_{(E_j,A_j)}$ being the one-step map corresponding to mode j as in Lemma 2.4.

Proof: With the same argument as in the proof of Lemma 2.4 we can conclude that $x_0 \in \mathcal{S}_{\sigma(0)}$ is necessary for existence of a solution. We show sufficiency by induction: Assume that $x(\ell)$ already satisfies (8) for $\ell = 0, 1, \dots, k$ and that $x(k) \in \mathcal{S}_{\sigma(k)}$. In order to extend this solution to $k+1$ it suffices to find $x(k+1)$ such that

$$E_{\sigma(k)} x(k+1) = A_{\sigma(k)} x(k)$$

and

$$E_{\sigma(k+1)} \xi = A_{\sigma(k+1)} x(k+1) \text{ for some } \xi \in \mathbb{R}^n.$$

In view of Remark 2.5 the first condition is equivalent to

$$x(k+1) \in \{\Phi_{(E_{\sigma(k)}, A_{\sigma(k)})} x(k)\} + \ker E_{\sigma(k)}$$

and the second condition is equivalent to

$$x(k+1) \in A_{\sigma(k+1)}^{-1}(\operatorname{im} E_{\sigma(k+1)}) = \mathcal{S}_{\sigma(k+1)}.$$

By assumption $\ker E_{\sigma(k)} \cap \mathcal{S}_{\sigma(k+1)} = \{0\}$, hence Lemma 3.3 together with Lemma 3.4 yields that $x(k+1)$ is uniquely given by (10). ∎

Remark 3.6: In contrast to the nonsingular case (7), the one-step-map (10) from $x(k)$ to $x(k+1)$ depends not only on the mode at time k but also on the mode at time $k+1$.

The existence of a one-step-map now allows us to define a state-transition map in a similar way as for the non-singular case (cf. Definition 2.6).

Definition 3.7: Consider a family of matrix pairs $\{(E_i, A_i) \mid i = 1, 2, \dots, \mathtt{n}\}$ of index-1 and the corresponding SwSS (8). The transition matrix for (8) is given by

$$\Phi_\sigma(k, h) = \Phi_{\sigma(k),\sigma(k\text{-}1)} \Phi_{\sigma(k\text{-}1),\sigma(k\text{-}2)} \cdots \Phi_{\sigma(h+1),\sigma(h)}$$

for $k > h$ and

$$\Phi_\sigma(h, h) = \Pi_{\mathcal{S}_{\sigma(h)}}^{\ker E_{\sigma(h)}}.$$

With this definition we arrive at the following corollary of Theorem 3.5.

Corollary 3.8: Consider an index-1 SwSS (1) with corresponding transition matrix Φ_σ as in Definition 3.7. Then all solutions are given by

$$x(k) = \Phi_\sigma(k, 0) x_0, \quad x_0 \in \mathbb{R}^n. \tag{11}$$

In particular,

$$x(0) = \Pi_{\mathcal{S}_{\sigma(0)}}^{\ker E_{\sigma(0)}} x_0$$

and $x(0) = x_0$ if, and only if, $x_0 \in \mathcal{S}_{\sigma(0)}$.

Remark 3.9: One may wonder how *necessary* the index-1 assumption from Definition 3.2 really is for existence and uniqueness of solutions of the SwSS (1). It is not difficult to see that in the non-switched case only regularity is necessary to ensure existence and uniqueness of solutions. However, in view of Remark 2.5 for higher index systems it is not possible to conclude existence of the one-step-map by just looking at the current and the next mode. In particular, the switched system would then not be causal w.r.t. the switching signal (Definition 3.1). Furthermore, assuming a one-step-map Φ_{ij} exists, then $x(0) = 0$ should imply $x(1) = 0$ for any switching signal σ with $\sigma(0) = j$ and $\sigma(1) = i$ (in particular, independently from the values $\sigma(k)$ for $k > 1$). However $E_j x(1) = A_j x(0) = 0$ and $E_i \xi = A_i x(1)$ for some $\xi \in \mathbb{R}^n$ is satisfied if, and only if, $x(1) \in \ker E_j \cap \mathcal{S}_1 \supsetneq \{0\}$, hence $x(1) = 0$ is not the only possible solution of (1) considered for $k = 0, 1$, therefore, a one step map cannot exist. Altogether, the index-1 assumption for (1) is necessary for causality of the switched system as well as for the existence of a one-step-map (which only depends on the current and past mode).

IV. A CONSTRUCTIVE FORUMULA FOR THE ONE-STEP-MAP

In what follows we give a constructive formula for the matrix Φ_{ij} as well as for the unique solution of SDSL (8).

Although the following results are partially obtained by similar arguments as in [2], [3], [13], we will give their proofs here to make our presentation self-contained. Furthermore, these properties also play a crucial rule for the treatment of the inhomogeneous case later.

Lemma 4.1: Consider the SwSS (8) and assume that it is index-1. For $i = 1, \dots, \mathtt{n}$, let $V_i := [s_i^1, \dots, s_i^r, h_i^{r+1}, \dots, h_i^n]$ be such that it columns form bases of $\mathcal{S}_i$ and $\ker E_i$, respectively. Let $P := \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{R}^{n \times n}$, where I_r is an $r \times r$, identity matrix, $Q := I_n - P$. Finally, let $P_i := V_i P V_i^{-1} = \Pi_{\mathcal{S}_i}^{\ker E_i}$, $Q_i := I - P_i = \Pi_{\ker E_i}^{\mathcal{S}_i}$ and $Q_{ij} := V_j Q V_i^{-1}$ for $i, j = 1, \dots, \mathtt{n}$. Then the following properties hold

(i) $G_{ij} := E_i + A_i Q_{ij}$ is nonsingular for all $i, j \in \{1, 2, \dots, \mathtt{n}\}$,
(ii) $\Pi_{\mathcal{S}_i}^{\ker E_j} = I - Q_{ij} G_{ij}^{-1} A_i$,
(iii) $\Phi_{(E_i, A_i)} = P_i G_{ii}^{-1} A_i$,
(iv) $\Phi_{ij} = (I - Q_{ij} G_{ij}^{-1} A_i) P_j G_{jj}^{-1} A_j$.

Proof:

(i) Assume that $x \in \ker G_{ij}$, then $A_i Q_{ij} x = -E_i x \in \operatorname{im} E_i$, hence $Q_{ij} x \in \mathcal{S}_i$. Further, $Q_{ij} x = V_j Q V_i^{-1} x \in \operatorname{im} V_j Q = \ker E_j$. Since $\mathcal{S}_i \cap \ker E_j = \{0\}$, we get $Q_{ij} x = 0$, hence, $E_i x = -A_i Q_{ij} x = 0$, therefore $x \in \ker E_i = \operatorname{im} Q_i$. Since Q_i is a projector, we have $x = Q_i x$. On the other hand $Q_i x = V_i V_j^{-1} Q_{ij} x = 0$, thus $x = Q_i x = 0$. This shows that $\ker G_{ij} = \{0\}$, i.e., the square matrix G_{ij} is nonsingular.
(ii) We will show that $Q_{ij} G_{ij}^{-1} A_i$ is the projection along $\mathcal{S}_i$ onto $\ker E_j$, it then follows that $I - Q_{ij} G_{ij}^{-1} A_i = \Pi_{\mathcal{S}_i}^{\ker E_j}$. First observe that

$G_{ij}V_iQ = (E_i + A_iQ_{ij})V_iQ = A_iV_jQ$ because $E_iV_iQ = 0$ by definition, hence $(Q_{ij}G_{ij}^{-1}A_i)^2 = Q_{ij}G_{ij}^{-1}\underline{A_iV_jQ}V_i^{-1}G_{ij}^{-1}A_i = Q_{ij}G_{ij}^{-1}\underline{G_{ij}V_iQ}V_i^{-1}G_{ij}^{-1}A_i = Q_{ij}G_{ij}^{-1}A_i$, i.e. $Q_{ij}G_{ij}^{-1}A_i$ is idempotent and therefore a projector.

It remains to be shown that $\operatorname{im} Q_{ij}G_{ij}^{-1}A_i = \ker E_j$ and $\ker Q_{ij}G_{ij}A_i = \mathcal{S}_i$. From $E_jV_jQ = 0$ it follows that immediately that $\operatorname{im} Q_{ij}G_{ij}^{-1}A_i \subseteq \ker E_j$. For $x \in \ker E_j \subseteq \operatorname{im} Q_j$ we have $x = Q_jx$ and therefore $\operatorname{im} Q_{ij}G_{ij}^{-1}A_i \ni Q_{ij}G_{ij}^{-1}A_ix = Q_{ij}G_{ij}^{-1}\underline{A_iV_jQ}V_j^{-1}x = Q_{ij}G_{ij}^{-1}A_ix = Q_{ij}G_{ij}^{-1}\underline{G_{ij}V_iQ}V_j^{-1}x = V_jQV_j^{-1}x = x$, which shows that also $\ker E_j \subseteq \operatorname{im} Q_{ij}G_{ij}^{-1}A_i$. Finally, the following equivalences hold:

$$\begin{aligned} x \in \mathcal{S}_i &\iff A_ix = E_i\xi \text{ for some } \xi \\ &\iff G_{ij}^{-1}A_ix = G_{ij}^{-1}E_i\xi = P_i\xi \\ &\iff V_i^{-1}G_{ij}^{-1}A_ix = PV_i^{-1}\xi \\ &\iff QV_i^{-1}G_{ij}^{-1}A_ix = 0 \\ &\iff Q_{ij}G_{ij}^{-1}A_ix = 0. \end{aligned}$$

This shows $\operatorname{im} \mathcal{S}_i = \ker Q_{ij}G_{ij}^{-1}A_i$.

(iii) In view of Lemma 2.2 and the discussing thereafter, it holds that

$$(E_iV_iP + A_iV_iQ)^{-1}A_iV_i = \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix}$$

for some $J_i \in \mathbb{R}^{r\times r}$. Hence,

$$\Phi_{(E_i,A_i)} = V_iP(E_iV_iP + A_iV_iQ)^{-1}A_i.$$

On the other hand

$$P_iG_{ii}^{-1}A_i = V_iP(E_iV_i + A_iV_iQ)^{-1}A_i$$

and since $E_iV_iP = E_iV_i$ the claim is shown.

(iv) This is a direct consequence from (ii),(iii) and Theorem 3.5.

■

We illustrate the usefulness of the derived formulas via the following example.

Example 4.2: Let

$$(E_1, A_1) = \left(\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 5 \\ 3 & 3 & -1 \\ -5 & -1 & 7 \end{bmatrix} \right),$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & -1 \\ -5 & -3 & 1 \end{bmatrix} \right),$$

$$(E_3, A_3) = \left(\begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 5 & 1 \end{bmatrix} \right)$$

A simple computation shows that

$$\begin{aligned} \ker E_1 &= \operatorname{span}\{(-1, 1, 0)^\top\}, \\ \ker E_2 &= \operatorname{span}\{(0, -1, 1)^\top\}, \\ \ker E_3 &= \operatorname{span}\{(1, 0, 1)^\top\}, \\ \mathcal{S}_1 &= \operatorname{span}\{(-2, 1, -1)^\top, (-1, -1, 0)^\top\}, \\ \mathcal{S}_2 &= \operatorname{span}\{(0, 1, 1)^\top, (1, -1, 2)^\top\}, \\ \mathcal{S}_3 &= \operatorname{span}\{(0, 1, 1)^\top, (-1, -1, 0)^\top\}, \end{aligned}$$

hence $\mathcal{S}_i \cap \ker E_j = \{0\},\ i, j = 1, 2,$. It means that system (8) with the above data is of index-1. Furthermore, we can chose V_1, V_2, V_3 such that it columns form bases of $\mathcal{S}_i$ and $\ker E_i$, respectively, then we can calculate G_{ij} and obtain

$$\Phi_{11} = \begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} & \frac{1}{2} \\ 2 & 2 & -1 \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \quad \Phi_{12} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix},$$

$$\Phi_{13} = \begin{bmatrix} 1 & -4 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}, \quad \Phi_{21} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix},$$

$$\Phi_{22} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -1 & -1 \end{bmatrix}, \quad \Phi_{23} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} \end{bmatrix},$$

$$\Phi_{31} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & 0 \\ -\frac{3}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}, \quad \Phi_{32} = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix},$$

$$\Phi_{33} = \begin{bmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Choosing $\sigma(k) = (k \mod 3) + 1$, we can compute the corresponding solution as follows

$$\begin{aligned} x(3k) &= (\Phi_{1,3}\Phi_{3,2}\Phi_{2,1})^k x_0 \\ x(3k+1) &= \Phi_{2,1}(\Phi_{1,3}\Phi_{3,2}\Phi_{2,1})^k x_0 \\ x(3k+2) &= \Phi_{3,2}\Phi_{2,1}(\Phi_{1,3}\Phi_{3,2}\Phi_{2,1})^k x_0. \end{aligned}$$

In particular, we can conclude that this switched systems is unstable because the repeating three-step transition matrix $\Phi_{1,3}\Phi_{3,2}\Phi_{2,1}$ has an eigenvalue with magnitude larger than one.

V. Inhomogeneous SwSS

We return our attention to the inhomogeneous SwSS (1).

Theorem 5.1: Suppose that the family of matrix pairs $\{(E_i, A_i)\}_{i=1}^N$ of SwSS (1) is of index-1. Then there exists matrices $\overline{\Phi}_{ij\ell} \in \mathbb{R}^{n\times n}$ and $\overline{\Psi}_{ij\ell}, \widetilde{\Psi}_{ij} \in \mathbb{R}^{n\times m}$ such that all solutions of (1) satisfy for all $k \in \mathbb{N}$

$$\begin{aligned} x(k+1) = &\,\overline{\Phi}_{\sigma(k+1)\sigma(k)\sigma(k-1)}x(k) \\ &\overline{\Psi}_{\sigma(k+1)\sigma(k)\sigma(k-1)}u(k) + \widetilde{\Psi}_{\sigma(k+1)\sigma(k)}u(k+1), \end{aligned} \tag{12}$$

where $\sigma(-1) := \sigma(0)$.

In fact, with V_i and G_{ij} as defined in Lemma 4.1 let

$$V_i^{-1}G_{ij}^{-1}A_iV_j = \begin{bmatrix} \bar{A}_{ij}^1 & 0 \\ \bar{A}_{ij}^2 & I_{n-r} \end{bmatrix}, \tag{13}$$

where $\bar{A}_{ij}^1 \in \mathbb{R}^{r\times r}$, $\bar{A}_{ij}^2 \in \mathbb{R}^{(n-r)\times r}$ and $\bar{B}_{ij} = V_i^{-1}G_{ij}^{-1}B_i = \begin{bmatrix} \bar{B}_{ij}^1 \\ \bar{B}_{ij}^2 \end{bmatrix}$, where $\bar{B}_{ij}^1 \in \mathbb{R}^{r\times m}$ and $\bar{B}_{ij}^2 \in \mathbb{R}^{(n-r)\times m}$. Then

$$\begin{aligned} \overline{\Phi}_{ij\ell} &= V_j \begin{bmatrix} \bar{A}_{j\ell}^1 & 0 \\ -\bar{A}_{ij}^2\bar{A}_{j\ell}^1 & 0 \end{bmatrix} V_\ell^{-1}, \\ \overline{\Psi}_{ij\ell} &= V_j \begin{bmatrix} \bar{B}_{j\ell}^1 \\ -\bar{A}_{ij}^2\bar{B}_{j\ell}^1 \end{bmatrix}, \\ \widetilde{\Psi}_{ij} &= V_j \begin{bmatrix} 0 \\ -\bar{B}_{ij}^2 \end{bmatrix}. \end{aligned}$$

Furthermore, there exists a solution of (1) with $x(0) = x_0$ if, and only if,

$$x_0 \in \operatorname{im} V_{\sigma(-1)} \begin{bmatrix} I & 0 \\ -\bar{A}_{\sigma(0)\sigma(-1)}^2 & \bar{B}_{\sigma(0)\sigma(-1)}^2 \end{bmatrix}.$$

Proof: Observe that $G_{ij}P_i = (E_i + A_iV_jQV_i^{-1})V_iPV_i^{-1} = E_iP_i + A_iV_jQPV_i^{-1} = E_iP_i$. Further, since Q_i is the projection onto $\ker E_i$ along S_i, it follows $E_iQ_i = 0$, therefore, $E_iP_i = E_i(P_i + Q_i) = E_i$. Thus, $G_{ij}P_i = E_i$, hence $P_i = G_{ij}^{-1}E_i$.
According to the proof of item (ii) of Lemma 4.1, $G_{ij}V_iQ = A_iV_jQ$, hence, $V_i^{-1}G_{ij}^{-1}A_iV_jQ = Q$.
Therefore, we obtain

$$\bar{A}_{ij} := V_i^{-1}G_{ij}^{-1}A_iV_j = \begin{bmatrix} \bar{A}_{ij}^1 & O \\ \bar{A}_{ij}^2 & I_{n-r} \end{bmatrix},$$
$$\bar{E}_{ij} := V_i^{-1}G_{ij}^{-1}E_iV_i = \begin{bmatrix} I_r & O \\ O & O_{n-r} \end{bmatrix}.$$

Multiplying both sides of system (1) by $V_{\sigma(k)}^{-1}G_{\sigma(k)\sigma(k-1)}^{-1}$, and using the transformation $\bar{x}(k) = V_{\sigma(k-1)}^{-1}x(k)$, we get

$$\bar{E}_{\sigma(k)\sigma(k-1)}\bar{x}(k+1) = \bar{A}_{\sigma(k)\sigma(k-1)}\bar{x}(k) + \bar{B}_{\sigma(k),\sigma(k-1)}u(k). \quad (14)$$

Putting $\bar{x}(k) := (v(k)^\top, w(k)^\top)^\top$, where $v(k) \in \mathbb{R}^r$, $w(k) \in \mathbb{R}^{n-r}$, we can reduce system (14) to system

$$v(k+1) = \bar{A}_{\sigma(k)\sigma(k-1)}^1 v(k) + \bar{B}_{\sigma(k)\sigma(k-1)}^1 u(k)$$

together with the constraint

$$w(k) = -\bar{A}_{\sigma(k)\sigma(k-1)}^2 v(k) - \bar{B}_{\sigma(k),\sigma(k-1)}^2 u(k),$$

which is equivalent to

$$\begin{aligned} w(k+1) &= -\bar{A}_{\sigma(k+1)\sigma(k)}^2 v(k+1) - \bar{B}_{\sigma(k+1),\sigma(k)}^2 u(k+1) \\ &= -\bar{A}_{\sigma(k+1)\sigma(k)}^2 \bar{A}_{\sigma(k)\sigma(k-1)}^1 v(k) \\ &\quad - \bar{A}_{\sigma(k+1)\sigma(k)}^2 \bar{B}_{\sigma(k)\sigma(k-1)}^1 u(k) \\ &\quad - \bar{B}_{\sigma(k+1),\sigma(k)}^2 u(k+1). \end{aligned}$$

By transforming back to the original coordinates via $x(k+1) = V_{\sigma(k)} \begin{bmatrix} v(k+1) \\ w(k+1) \end{bmatrix}$ and $\begin{bmatrix} v(k) \\ w(k) \end{bmatrix} = V_{\sigma(k-1)}^{-1}x(k)$ we arrive at (12). Finally, existence of a solution is guaranteed, if and only if, $x(0)$ is consistent with (1), or in the (v, w)-coordinates, if and only if there exists $u(0) \in \mathbb{R}^m$ such that

$$w(0) = -\bar{A}_{\sigma(0)\sigma(-1)}^2 v(0) - \bar{B}_{\sigma(0),\sigma(-1)}^2 u(0)$$

where $v(0) \in \mathbb{R}^r$ is arbitrary. In other words,

$$\begin{bmatrix} v(0) \\ w(0) \end{bmatrix} \in \begin{bmatrix} I & 0 \\ -\bar{A}_{\sigma(0)\sigma(-1)}^2 & \bar{B}_{\sigma(0)\sigma(-1)}^2 \end{bmatrix},$$

which yields the claimed condition by applying the coordinate transformation $V_{\sigma(-1)}$. ■

Remark 5.2: In contrast to the homogeneous case the one-step-map from $x(k)$ to $x(k+1)$ in the inhomogeneous case not only depends on the modes at time $k+1$ and k but also on the mode $k-1$. Furthermore, the allowed space of consistent initial values seems to depend on the choice of $\sigma(-1)$, and it is ongoing research to investigate the significance of this fact.

VI. Conclusion

We have shown that for switched singular systems in discrete time with a certain index-1 property a unique one-step-map exists which can fully characterize all possible solutions. An application of this result could be the stability analysis for switched singular systems and is ongoing research.

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