Asymptotic Tracking with Funnel Control

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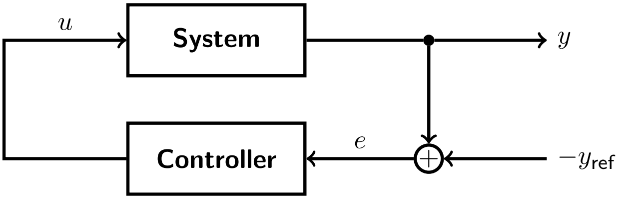
Control Task

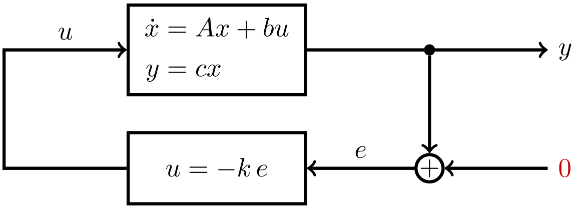


Goal: Output tracking $y(t) \approx y_{\text{ref}}(t)$

without

- exact knowledge of system model
- model for reference signal

High-gain-feedback: linear case



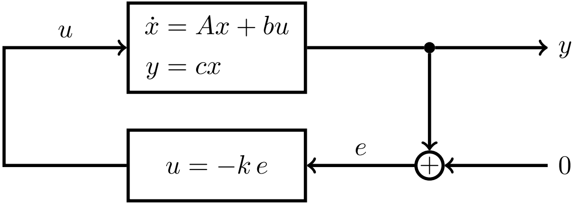
Assumptions:

- Relative degree $r = 1$ $\Leftrightarrow$ $\gamma := cb \neq 0$, in particular:

$$\text{System} \quad \Leftrightarrow \quad \begin{aligned} \dot{y} &= a_{11}y + a_{12}z + \gamma u \\ \dot{z} &= a_{21}y + A_{22}z \end{aligned}$$

- positive high frequency gain $\Leftrightarrow$ $\gamma > 0$
- stable zero-dynamics (minimum phase) $\Leftrightarrow$ A_{22} Hurwitz

High-gain-feedback: linear case



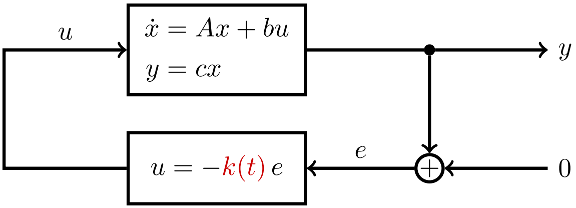
Theorem (High-gain feedback)

$cb > 0$ and stable zero-dynamics

$$\Rightarrow \quad \exists k_0 > 0\ \forall\, k \geq k_0 : \ \textit{Closed loop is asymptotically stable}$$

Problem: How to find k_0?

High-gain-feedback: linear case



Idea: Make gain k time varying

Theorem (Adaptive high-gain feedback, BYRNES & WILLEMS 1984)

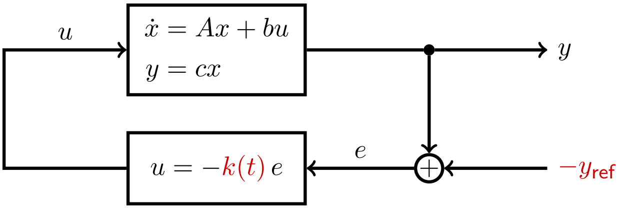
$cb > 0$ *and stable zero-dynamics* $\Rightarrow$

$$\dot{k}(t) = e(t)^2 \text{ makes closed loop asymptotically stable}$$

and $k(\cdot)$ *remains bounded*

Problem: Disturbances or $y_{\text{ref}} \neq 0$ lead to unbounded $k(\cdot)$!

High-gain-feedback: linear case



Solution: Aim for practical stability, i.e. $|e(t)| \leq \lambda$ for $t >> 0$ and some small $\lambda > 0$

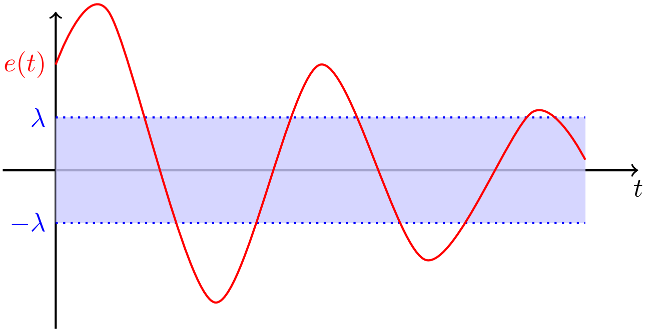
Theorem (λ-tracking, Ilchmann & Ryan 1994)

Assume $cb > 0$, stable zero-dynamics and $y_{\text{ref}}, \dot{y}_{\text{ref}}$ bounded. For $\lambda > 0$ consider

$$\dot{k}(t) = \begin{cases} |e(t)|\big(|e(t)| - \lambda\big), & |e(t)| > \lambda, \\ 0, & |e(t)| \leq \lambda. \end{cases}$$

Then the closed loop is practically stable.

Remaining problems of λ-tracker

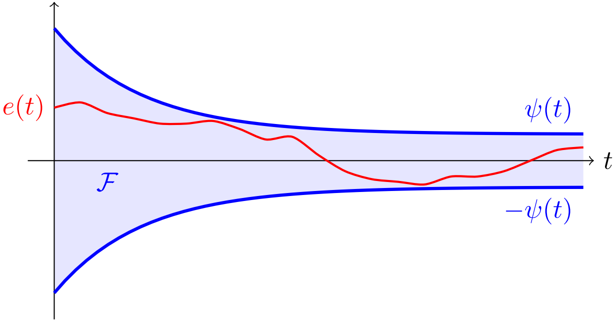


Problems:

- No guarantees when $|e(t)| \leq \lambda$
- No bounds on transient behaviour
- Monotonically growing $k(\cdot)$ $\Rightarrow$ Measurement noise unnecessarily amplified

The funnel as time-varying error bound

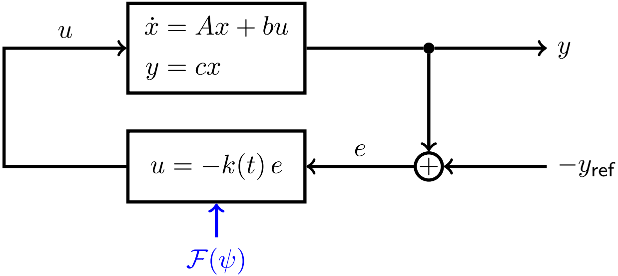
$$\mathcal{F} = \mathcal{F}(\psi) := \Big\{(t,e) \;\Big|\; |e| < \psi(t)\Big\}$$



Idea: $k(t)$ large $\Leftrightarrow$ Distance of $e(t)$ to funnel boundary small

$$u(t) = -k(t)e(t) \quad \text{with} \quad \boxed{k(t) = \frac{1}{\psi(t) - |e(t)|}}$$

Funnel control for linear SISO systems



Theorem (Funnel Control, Ilchmann, Ryan, Sangwin 2002)

Let $cb > 0$, A_{22} *Hurwitz,* $y_{\text{ref}}, \dot{y}_{\text{ref}}, \psi, \dot{\psi}$ *bounded,* $\liminf_{t\to\infty} \psi(t) =: \lambda > 0$ *and* $|e(0)| < \psi(0)$. *Then*

$$k(t) = \frac{1}{\psi(t) - |e(t)|}$$

remains bounded in the closed loop, i.e. $e(t)$ *remains within funnel.*

Asymptotic tracking impossible by design

$$u(t) = -k(t)e(t) \quad \text{with} \quad \boxed{k(t) = \frac{1}{\psi(t) - |e(t)|}}$$

Asymptotic tracking only with unbounded gain

Asymptotic tracking $\Leftrightarrow \quad \psi(t) \underset{t\to\infty}{\to} 0 \quad \Rightarrow \quad \psi(t) - |e(t)| \underset{t\to\infty}{\to} 0$

$\Leftrightarrow \quad k(t) \underset{t\to\infty}{\to} \infty$

Conclusion: Funnel control and asymptotic tracking not compatible **?**

Important observation

$\psi(t) \underset{t\to\infty}{\to} 0 \quad \Longrightarrow \quad |e(t)| \underset{t\to\infty}{\to} 0$ hence

$$u(t) = -k(t)e(t) \to \infty \cdot 0$$

not necessarily unbounded!

Rewrite rule for funnel control

$$u(t) = -\frac{1}{\psi(t) - |e(t)|} e(t) = -\frac{1}{1 - \left|\frac{e(t)}{\psi(t)}\right|} \frac{e(t)}{\psi(t)} =: -\alpha(\eta(t)) \cdot \beta(\eta(t))$$

where $\eta(t) := \frac{e(t)}{\psi(t)}$ and

- $1 - \eta(t)$ is the relative difference between error and funnel boundary
- $\alpha : (-1, 1) \to \mathbb{R}$ with $\alpha(\eta) \to \infty$ as $|\eta| \to 1$
- $\beta : (-1, 1) \to \mathbb{R}$ bounded with $\beta(\eta) \not\to 0$ as $|\eta| \to 1$ and $\operatorname{sgn} \beta(\eta) = \operatorname{sgn} \eta$

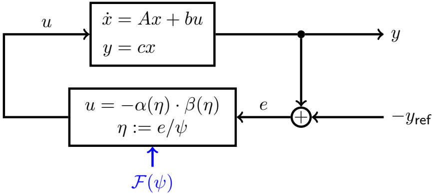
Remark

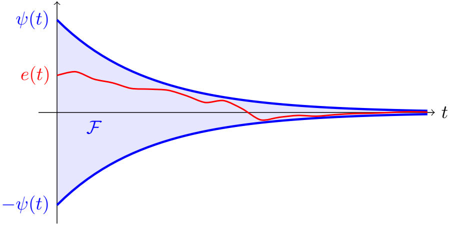
The original funnel controller already had a very similar structure:

$$u(t) = -\alpha(\varphi(t)e(t)) \cdot e(t)$$

with $\varphi := 1/\psi$, but $e(t) \to 0$ still requires unbounded gain.

Main result SISO case





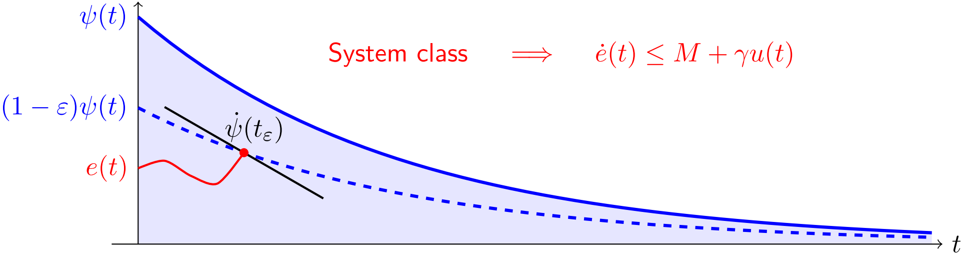
Theorem

Let $cb > 0$, A_{22} *Hurwitz,* $y_{\text{ref}}, \dot{y}_{\text{ref}}, \psi, \dot{\psi}$ *bounded,* $\psi > 0$, $|e(0)| < \psi(0)$,
$\alpha : (-1,1) \to \mathbb{R}$ *with* $\alpha(\eta) \to \infty$ *as* $|\eta| \to 1$, $\beta : (-1,1) \to \mathbb{R}$ *bounded with* $\beta(\eta) \not\to 0$ *as* $|\eta| \to 1$ *and* $\operatorname{sgn}\beta(\eta) = \operatorname{sgn}\eta$ *then* $\exists > 0$

$$|e(t)| < (1-\varepsilon)\psi(t) \quad \forall t \geq 0$$

In particular, for $\psi(t) \to 0$ *asymptotic tracking is achieved.*

Proof idea: Show positive invariance of funnel



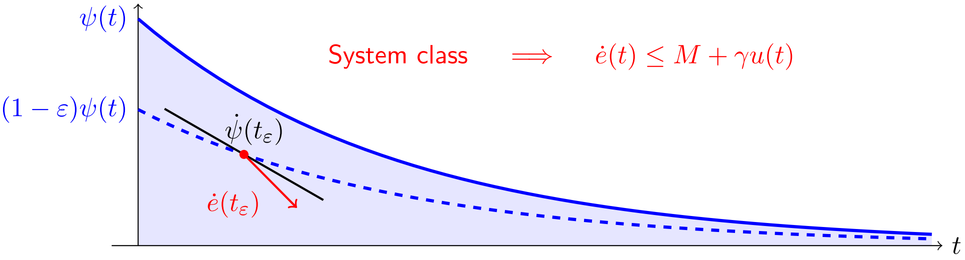
System class $\implies$ $\dot e(t) \leq M + \gamma u(t)$

Assume $e(t_\varepsilon) = (1-\varepsilon)\psi(t_\varepsilon)$ then

$$u(t_\varepsilon) = -\alpha(1-\varepsilon)\beta(1-\varepsilon) < -C_\varepsilon \text{ with } C_\varepsilon \to \infty \text{ as } \varepsilon \to 0$$

Hence $\dot e(t_\varepsilon) < M - \gamma C_\varepsilon < \dot\psi(t_\varepsilon)$ for ε small

Proof idea: Show positive invariance of funnel



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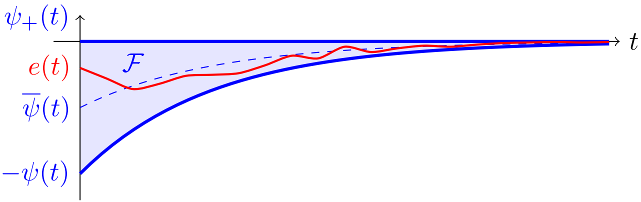
Hence $\dot e(t_\varepsilon) < M - \gamma C_\varepsilon < \dot\psi(t_\varepsilon)$ for ε small

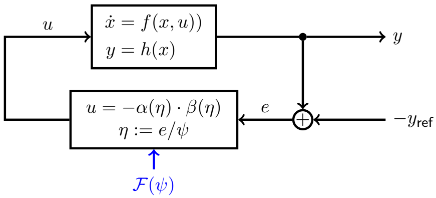
Possible generalizations

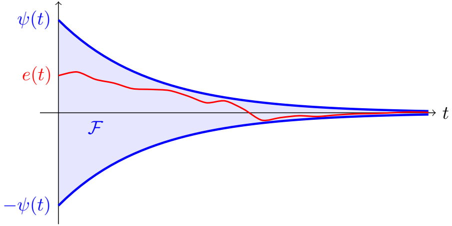
- Much more general system class (nonlinear, hysteresis, ...)
- Unknown sign of $\gamma = cb$
  chose α such that $\limsup_{t\to\pm 1} \alpha(t) = \infty$ and $\liminf_{t\to\pm 1} \alpha(t) = -\infty$
- unsymmetric funnel with ψ_+, ψ_-, allowing also e.g. $\psi_+ \equiv 0$
  $u(t) = -\alpha(\eta)\beta(\eta)$ with $\eta := \frac{e(t)-\overline{\psi}(t)}{\psi_+(t)-\overline{\psi}(t)}$, where $\overline{\psi} := \frac{1}{2}(\psi_+ + \psi_-)$



- finite time convergence via $\psi(T) = 0$ for some $T > 0$
- MIMO: need to generalize $cb > 0$ and $\operatorname{sgn}\beta(\eta) = \operatorname{sgn}\eta$
- higher relative degree

Summary





New insight

Practical tracking is NOT a theoretical limitation of funnel control

Key observation

Boundedness of gain in $u(t) = -k(t)\cdot e(t)$ is NOT required for boundedness of input!