Zeno-like behavior in coupled systems of switched DAEs and PDEs

Stephan Trenn

Jan C. Willems Center for Systems and Control
University of Groningen, Netherlands

Joint work with **Damla Kocuglu** and **Raul Borsche**
AG Technomathematik, TU Kaiserslautern

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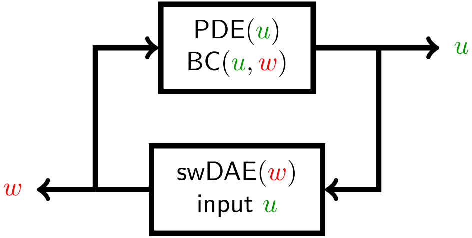
Setup: Coupling of PDE and swDAE



Motivation: **Human blood flow**

PDE: Blood flow in vessels

swDAE: Heart (valve = switch)

Extremely simplified linear model

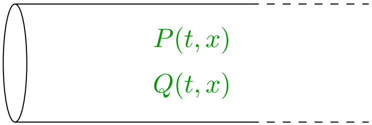
swDAE heart model

Valve open ($p(t) > P(t,0)$)

$$\dot{p} = -q$$
$$\dot{q} = p$$
$$Q(t,0) = q(t)$$

Valve closed ($p(t) \leq P(t,0)$)

$$\dot{p} = -q$$
$$q = 0$$
$$Q(t,0) = q(t)$$



PDE vessel model

$$P_t + Q_x = 0$$
$$Q_t + c^2 P_x = 0$$

Domain: $t \geq 0$, $x \geq 0$
Initial conditions:

$$P(0,x) = P_0(x), \quad Q(0,x) = Q_0(x)$$

Decoupling of PDE

Applying coordinate transformation $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix}^{-1} \begin{pmatrix} P \\ Q \end{pmatrix}$ yields **decoupled** PDEs

$$\boxed{\begin{aligned} v_{1t} - c v_{1x} &= 0 \\ v_{2t} + c v_{2x} &= 0 \end{aligned}}$$

Unique solution

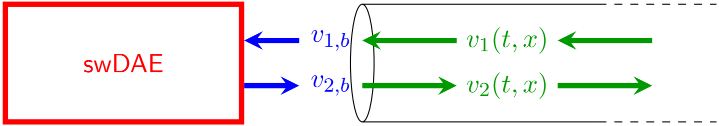
$$v_1(t,x) = v_1^0(ct + x)$$

$$v_2(t,x) = \begin{cases} v_2^0(x - ct), & t \leq x/c \\ v_{2,b}(t - x/c), & t > x/c \end{cases}$$

Output: $v_{1,b}(t) := v_1(t,0)$

Input: $v_{2,b}(t) := v_2(t,0)$

Combining swDAE with decoupled PDE



swDAE model

Valve open $(p > v_{1,b} + v_{2,b})$

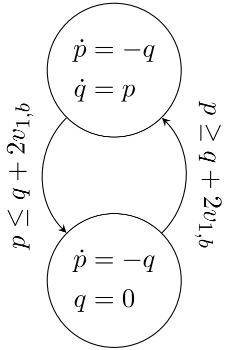
$$\dot{p} = -q$$
$$\dot{q} = p$$
$$v_{2,b} = q + v_{1,b}$$

Valve closed $(p \leq v_{1,b} + v_{2,b})$

$$\dot{p} = -q$$
$$q = 0$$
$$v_{2,b} = q + v_{1,b}$$

$\hookrightarrow$ **open loop analysis** possible

Numerical simulation of swDAE: Naive approach



Euler steps with fixed step-size

Given: $w_k^- = \begin{pmatrix} p_k^- \\ q_k^- \end{pmatrix}$ state value at t_k^-

u_k input value at t_k

u_{k+1} input value at t_{k+1}

m_k^- active mode at t_k^-

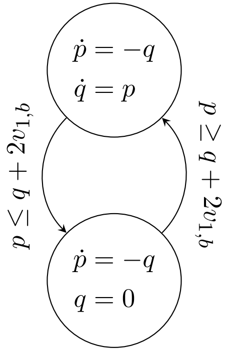
Step 1: Switching? $\exists m : g_{m_k^-,m}(w_k^-, u_k) \leq 0$?

YES: $m_k^+ := m$, $w_k^+ := \Pi_{m_k^+} w_k^-$

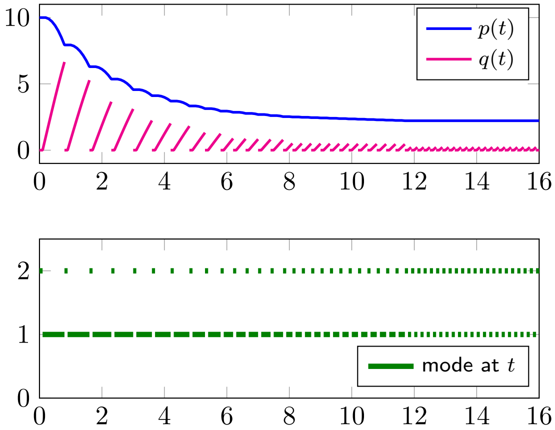
NO: $m_k^+ := m_k^-$, $w_k^+ := w_k^-$

Step 2: Euler Step $w_{k+1}^- = ES_{m_k^+}(w_k^+, u_k, u_{k+1})$

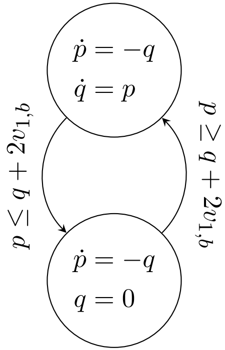
Numerical simulation of swDAE: Naive approach



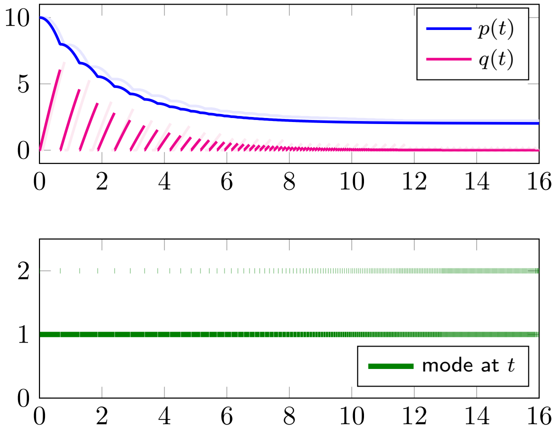
Results for $p(0^-) = 10$, $q(0^-) = 10$, $v_{1,b} = 1$
time step size $= 0.1$



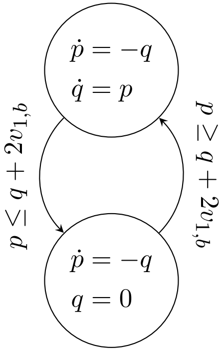
Numerical simulation of swDAE: Naive approach



Results for $p(0^-) = 10$, $q(0^-) = 10$, $v_{1,b} = 1$
time step size $= 0.01$



Numerical simulation of swDAE: Naive approach

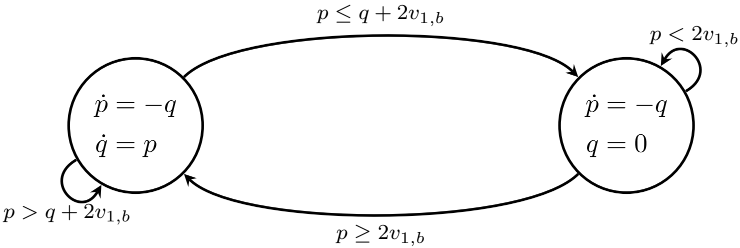


Major numerical problems

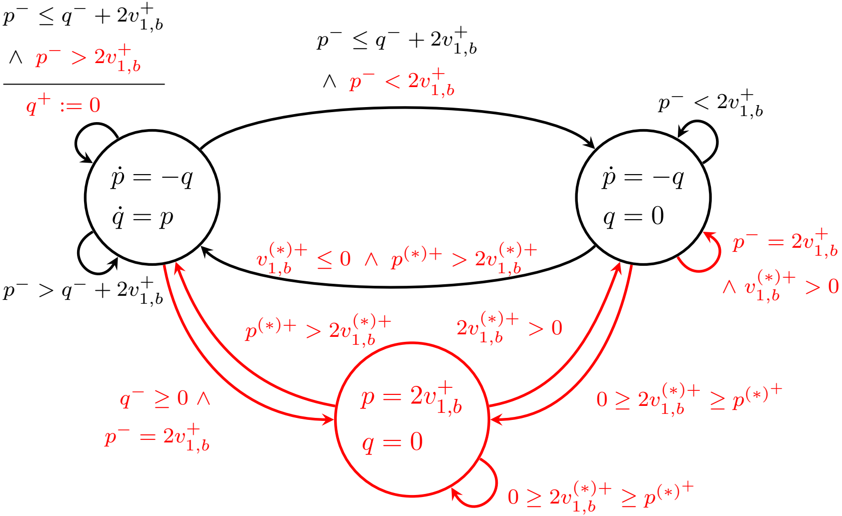
For exact solution:

1. Leaving Mode 2 immediately after entering
2. Zeno-like behavior
   $\rightarrow$ no upper bound on switching frequency
   $\rightarrow$ any adaptive step size method breaks down
3. What happens after Zeno?

Refining swDAE model - going beyond Zeno



Refining swDAE model - going beyond Zeno



Inspired by: Zhen et al.: **Beyond Zeno: Get on with it!**, HSCC 2006

Guards relaxation for numerical integration

Key features of refinement

- Multiple mode changes at one time instant avoided
- Sliding mode (after zeno) explicitly visible

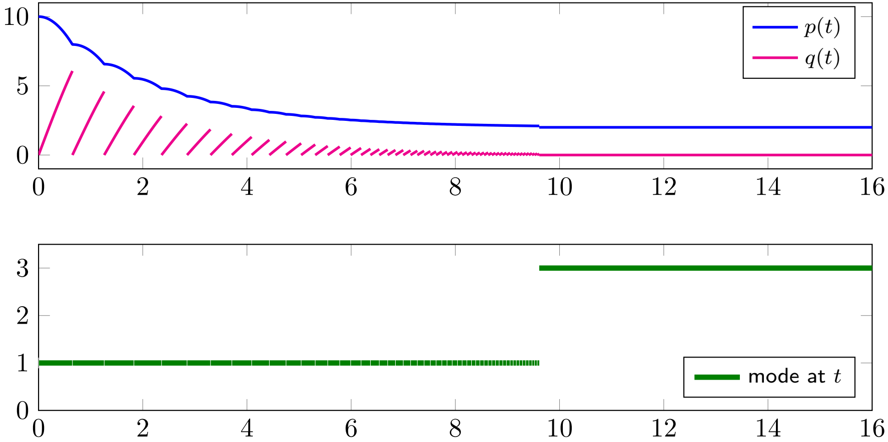
Unresolved problem

Zeno behavior sill present and leads to numerical problems

Numerical treatmant

Relax all ingoing transition towards a sliding mode

Improved numerical simulation



Conclusion

Numerical challenges in coupling swDAEs and PDEs

- Identify inputs/outputs in the coupling $\rightarrow$ Coordinate transformations
- Refine swDAE-model by introducing additional jump maps and Zeno states
- Relax guards to avoid accumulation of switching times
- Adaptive time stepping near Zeno
- Guards depending on derivatives of input

Theoretical challenges

- Solution framework which allows for Zeno-solution
- Dirac impulses (derivatives of jumps)
- Dynamics in Zeno-state