Optimal Control of Switched DAEs

Stephan Trenn

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

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From LTIs to switched DAEs

Classical linear systems:

$$\dot{x} = Ax + Bu$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

Include algebraic constraints (e.g. Kirchhoff laws)

$$E\dot{x} = Ax + Bu$$

$E \in \mathbb{R}^{n\times n}$ singular

Consider sudden structural changes (switches)

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u$$

$\sigma : \mathbb{R} \to \{1, 2, ..., N\}$ piecewise-constant switching signal

Challenges for switched DAEs

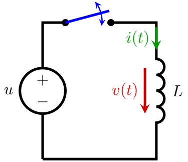
Inconsistent state values when switching

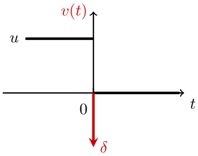
At a switching time $t_s \in \mathbb{R}$ the state value $x(t_s^-)$ may be inconsistent with algebraic constraints of mode $\sigma(t_s^+)$.
$\Rightarrow$ state jump necessary: $x(t_s^-) \to x(t_s^+)$

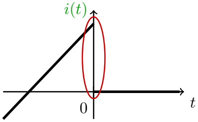
Dirac impulses in response to switches

In addition to jumps, switches may also induce Dirac impulses in the state.

Example:





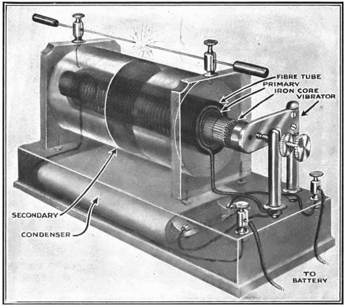


Dirac impulse is “real”

Dirac impulse

Not just a mathematical artefact!

Induction coil



Drawing: Harry Winfield Secor, public domain

Spark plug

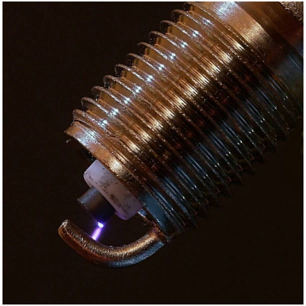


Photo: Ralf Schumacher, CC-BY-SA 3.0

Different (optimal) control setups

$u(t) \rightarrow$ $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ $\Rightarrow$ time-varying, linear

$\sigma(t) \rightarrow$ $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ $\Rightarrow$ optimal switching sequence
$+$ optimal switching times

$u(t), \sigma(t) \rightarrow$ $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ $\Rightarrow$ combined continuous
and discrete optimization

Available results for time-varying case

$$E(t)\dot{x} = A(t)x + B(t)u$$

Treated in e.g. Kunkel & Mehrmann 1997, Kurina & März 2004, ...

BUT: Continuity assumption

Existing results restricted to (at least) continuous coefficient matrices $E(\cdot)$, $A(\cdot)$.

In particular:

- no jumps considered
- avoiding of Dirac impulses not addressed
- role of switches for guaranteeing controllability not relevant

Illustrative Example

Minimize $\int_0^{t_f} (x_1^2 + u^2)$ subject to

$$\text{on } [0,1): \quad \begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= 0 \end{aligned} \qquad\qquad \text{on } [1,t_f): \quad \begin{aligned} \dot{x}_1 + \dot{x}_2 &= 0 \\ 0 &= x_2 \end{aligned}$$

Trivial optimal control $u^* \equiv 0$ on $[1, t_f)$ with corresponding solution:

$$x_1(t) \overset{\dot{x}_1+0=0}{=} x_1(1^+) \overset{x_2=0}{=} x_1(1^+) + x_2(1^+) \overset{\frac{\mathrm{d}}{\mathrm{d}t}(x_1+x_2)|_{t=1}=0}{=} x_1(1^-) + x_2(1^-)$$

Two competing control objectives

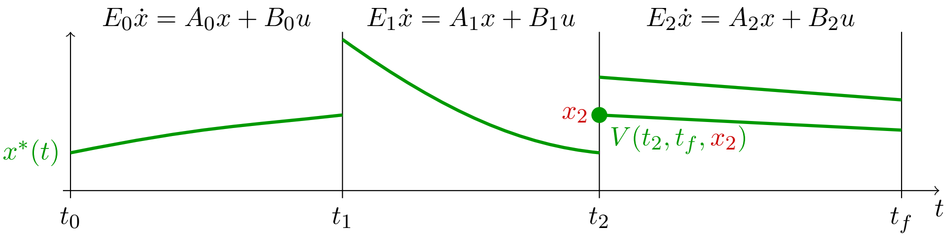
1. Make x_1 small with minimal control effort on $[0,1)$
2. Steer $x_1(1^-)$ close to $-x_2(1^-) = -x_2(0)$

Proposed optimization method

(OPT) Minimize $J(t_0, t_f, x_0, u) := \int_{t_0}^{t_f} (x^\top Q x + u^\top R u)$ subject to

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \quad x(t_0^-) = x_0 \in \mathbb{R}^n, \quad u \in \mathcal{U}$$

Value function: $\quad V(\tau_1, \tau_2, x_1) = \inf_{u \in \mathcal{U}} J(\tau_1, \tau_2, x_1, u)$



New opt. problem on $[t_0, t_2]$: Minimize $J(t_0, t_2, x_0, u) + V(t_2, t_2, x_2)$
$\hookrightarrow$ **Dynamic programming**

Recursive solution approach

(OPT) Minimize $\int_0^{t_f}(x^\top Qx + u^\top Ru)$ subject to

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \quad x(t_0^-) = x_0 \in \mathbb{R}^n, \quad u \in \mathcal{U}$$

with switching times $t_0, t_1, t_2, \ldots, t_N = t_f$.

Optimization "Algorithm"

Step N:
Solve (OPT) on $[t_{N-1}, t_N]$ with variable initial value x_{N-1}
$\hookrightarrow$ cost function $V_{N-1}(x_{N-1}) := V(t_{N-1}, t_N, x_{N-1})$
$\hookrightarrow$ optimal u on $[t_{N-1}, t_N]$ parametrized by x_{N-1}

Step $k = N-1, N-2, \ldots, 1$:
Solve (OPT) on $[t_{k-1}, t_k]$ with additional terminal costs $V_k(x(t_k^-))$ and variable initial value x_{k-1}
$\hookrightarrow$ cost function $V_{k-1}(x_{k-1})$
$\hookrightarrow$ optimal u $[t_{k-1}, t_k]$parametrized by x_{N-1}

Open issues with recursive algorithm

Recursive algorithm summary: For any $x_{k-1} \in \mathbb{R}^n$ solve

$$\min_{u \in \mathcal{U}} \int_{t_{k-1}}^{t_k} (x^\top Q x + u^\top R u) + V_k(x(t_k^-)), \text{ subj. to } \begin{array}{r}E_{k-1}\dot{x} = A_{k-1}x + B_{k-1}u\\ x(t_{k-1}^-) = x_{k-1}\end{array}$$

where $V_k(x_k) := \min_{u \in \mathcal{U}} \int_{t_k}^{t_f} x^\top Q x + u^\top R u, \quad x(t_k^-) = x_k$

Questions and challenges

- Is it true that $V_k(x_k) = x_k^\top M_k x_k$ for some pos. semi-def. M_k?
- Can M_k be calculated efficiently and numerically?
- How to take into account that x_k is restricted to some subspace?
- Optimal control u^* in state-feedback form?
- Avoiding Dirac-impulses?
- More global result, e.g. in terms of adjoint system?

Different optimization setups

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$\sigma(t) \rightarrow$ $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ $\Rightarrow$ **optimal switching sequence**
$+$ **optimal switching times**

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and discrete optimization

Optimal switching

$$E_\sigma \dot{x} = A_\sigma x$$

Possible setup

$$\boxed{E\dot{x} = Ax + Bu} \quad + \quad \boxed{u = F_\sigma x} \quad = \quad \boxed{E\dot{x} = (A + BF_\sigma)x}$$

$\hookrightarrow$ In general: **Mixed integer programming problem** $\rightarrow$ NP-hard
  $\hookrightarrow$ Relaxations, Heuristics, Branch & Bound methods, ...

Fix switching sequence: Optimize switching times only
$\hookrightarrow$ Available results for ODEs: Egerstedt et al. 2006, Xu et al. 2004, ...

DAE-specific

Costs for induced jumps / Dirac impulses

Different optimization setups

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$u(t)$, $\sigma(t)$ → $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$ $\Rightarrow$ **combined continuous and discrete optimization**

Most general problem

$\min_{u,\sigma} J(x,u,\sigma)$ subject to $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$

Suprisingly ...

Maximum principle available for ODE case (Sussmann 1999)

$\hookrightarrow$ DAE-generalization seems possible (jumps already included in ODE case)
$\hookrightarrow$ However: role of induced Dirac impulses unclear
$\hookrightarrow$ Implementability?

Summary

Optimal Control for switched DAEs

- **Given switching signal** ($\rightarrow$ time-varying case)
  - Seems most tractable (via dynamic programming)
  - Some DAE-specifics still unclear (in particular role of Diracs)
  - Role of state and input constraints?
- **Switching signal is also an input**
  - Costs for induced jumps and Dirac impulses
  - NP-hard
  - DAE-specific heuristics