Discontinuous Lyapunov Functions for Discontinuous Nonlinear Systems

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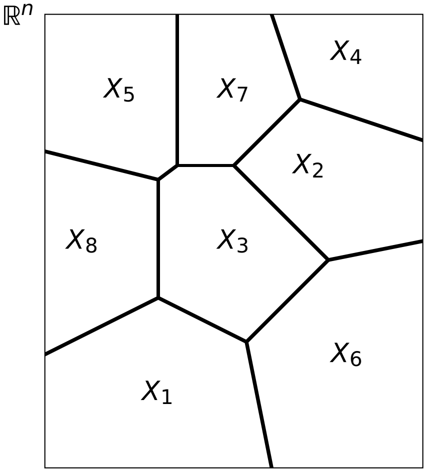
Piecewise smooth systems



Polyhedral partition

$$\mathbb{R}^n = \bigcup_{s\in\Sigma} X_s, \quad \Sigma = \{1, 2, \ldots, N\}$$

$$\operatorname{int} X_i \cap \operatorname{int} X_j = \emptyset \quad \forall i \neq j$$

Piecewise-smooth dynamics

$$\dot{x} = f_s(x) \quad x \in X_s,\ s \in \Sigma$$

Piecewise-affine (PWA) systems:

$$f_s(x) = A_s x + b_s$$

Motivation and problem formulation

Occurrence of PWA systems:

- Linear systems coupled with saturation and friction
- Electrical circuits with diodes and transistors
- Feedback control with gain scheduling
- Linearization of nonlinear systems around different operating points

Goal: Stability proof

Construct global Lyapunov function V for

$$\dot{x} = f_s(x), \quad x \in X_s,\ s \in \Sigma \tag{PWS}$$

with the help of local Lyapunov functions V_s on X_s

Contents

Behavior on boundaries

$$\dot{x} = f_s(x), \quad x \in X_s, \; s \in \Sigma \tag{PWS}$$

X_s are closed convex polyhedra (i.e. finite intersection of half spaces)

Question

What to do for $x \in X_i \cap X_j$

Answer: We don't care

More formally:

$$\dot{x} \in \left\{ f_s(x) \,\middle|\, s \in \Sigma^x \right\}$$

where

$$\Sigma^x := \{ s \in \Sigma \mid x \in X_s \}$$

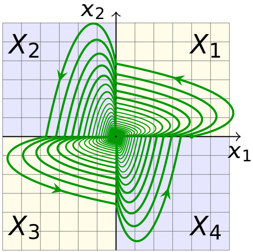
Caratheodory solutions

$$\dot{x} = f_s(x), \quad x \in X_s, \ s \in \Sigma \tag{PWS}$$

Definition

$x : [0, \omega) \to \mathbb{R}^n$ is called Caratheodory solution of (PWS): $\iff$

1. x is absolutely continuous (hence differentiable a.e.)
2. $\dot{x}(t) \in \{ f_s(x) \mid s \in \Sigma^x \}$ for a.a. $t \in [0, \omega)$



$$f_s(x) = A_s x$$

$$A_1 = A_3 = \begin{bmatrix} 1 & -5 \\ 0.2 & 1 \end{bmatrix}$$

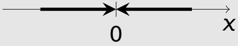
$$A_2 = A_4 = \begin{bmatrix} 1 & -0.2 \\ 5 & 1 \end{bmatrix}$$

Well-posedness issues

Non-existence

Example

$$\dot{x} = \begin{cases} -1, & x \geq 0 \\ 1, & x \leq 0 \end{cases}$$

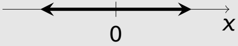


No solution with $x(0) = 0$.

Non-uniqueness

Example

$$\dot{x} = \begin{cases} 1, & x \geq 0 \\ -1, & x \leq 0 \end{cases}$$



Two solutions with $x(0) = 0$

In the context of stability analysis

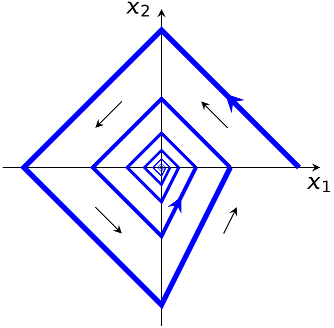
Non-existence: Not OK

Non-uniqueness: OK

Accumulation of switching times (Zeno)

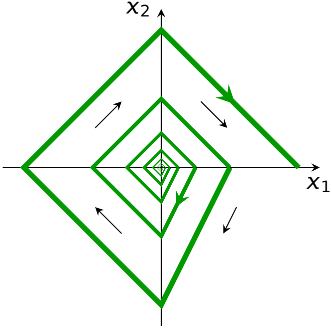
Right-accumulation

$$\dot{x} = \begin{cases} (-1,\ 1)^\top & x_1 \geq 0,\ x_2 \geq 0 \\ (-1,\ -1)^\top & x_1 \leq 0,\ x_2 \geq 0 \\ (1,\ -1)^\top & x_1 \leq 0,\ x_2 \leq 0 \\ (1/2,\ 1)^\top & x_1 \geq 0,\ x_2 \leq 0 \end{cases}$$



Left-accumulation

$$\dot{x} = \begin{cases} (1,\ -1)^\top & x_1 \geq 0,\ x_2 \geq 0 \\ (1,\ 1)^\top & x_1 \leq 0,\ x_2 \geq 0 \\ (-1,\ 1)^\top & x_1 \leq 0,\ x_2 \leq 0 \\ (-1/2,\ -1)^\top & x_1 \geq 0,\ x_2 \leq 0 \end{cases}$$



Filippov solutions

$$\dot{x} = f_s(x), \quad x \in X_s, \ s \in \Sigma \qquad \text{(PWS)}$$

Definition

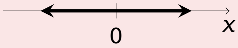
$x : [0, \omega) \to \mathbb{R}^n$ is called Filippov solution of (PWS): $\Longleftrightarrow$

1. x is absolutely continuous (hence differentiable a.e.)
2. $\dot{x}(t) \in \operatorname{conv} \{ f_s(x) \mid s \in \Sigma^x \}$ for a.a. $t \in [0, \omega)$

Theorem (cf. Filippov 1988)

$\forall x_0 \in \mathbb{R}^n$ exists Filippov solution $x : [0, \omega) \to \mathbb{R}^n$ of (PWS) with $x(0) = x_0$. Furthermore, if $f_s(x) = A_s x + b_s$ and Σ finite then $\omega = \infty$.

Problem: "Unnecessary" sliding



has $x(t) \equiv 0$ as Filippov solution.

Contents

From local to global LF-function

General idea

Given local Lyapunov function $V_s : \mathbb{R}^n \to \mathbb{R}$, i.e.

(L1) $V_s \in \mathcal{C}$ and $V_s\big|_{X_s} \in \mathcal{C}^1$.

(L2) V_s is positive definite on X_s

(L3) V_s is radially unbounded in the following sense:

$$\forall \overline{v} \in V_s(X_s) \subseteq \mathbb{R} : V_s^{-1}([0, \overline{v}]) \cap X_s \text{ is compact}$$

(L4) V_s is decreasing along solutions within X_s, i.e.

$$\nabla V_s(x) f_s(x) < 0 \quad \forall x \in X_s \setminus \{0\},$$

let global Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ be given by

$$V(x) = V_s(x) \quad \text{for } x \in X_s,\ s \in \Sigma$$

$f_s(x) = A_s x + b_s$ and $V_s(x) = x^\top P_s x + 2q_s^\top x + r_s \quad \longrightarrow \quad$ LMIs

Continuous global Lyapunov function

Theorem (Cf. Johansson 2003)

Let $V_s : \mathbb{R}^n \to \mathbb{R}$ be local LF on X_s such that

$$V(x) = V_s(x) \quad \text{for } x \in X_s,\ s \in \Sigma$$

is continuous.
Then $x(t) \to 0$ for all Caratheodory solutions $x : [0, \infty) \to \mathbb{R}^n$.

Extension to Filippov solution

Johansson extended the above result for PWA systems to Filippov solutions by additionally requiring

$$\nabla V_s(x)(A_{\bar{s}}x + b_{\bar{s}}) < 0,\ \forall x \in X_s \cap X_{\bar{s}}$$

Example to motivate discontinuous LF

$$\dot{x} = A_s x + b_s \text{ on } X_s, \qquad \text{(PWA)}$$

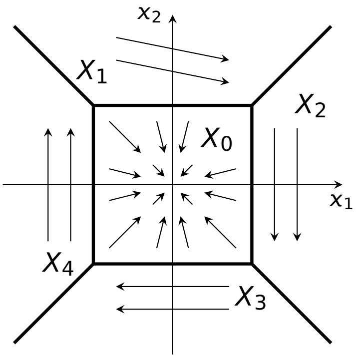
with

$A_0 = -I, \quad A_1 = A_2 = A_3 = A_4 = 0,$

$b_0 = 0, \quad b_1 = \begin{pmatrix} 1 \\ -0.1 \end{pmatrix}, \; b_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$

$b_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \; b_4 = -b_2.$

PWQ-LF: $V_s(x) = x^\top P_s x + 2q_s^\top x + r_s$



Lemma

The above PWA system does not permit a continuous PWQ-LF.

Crossing and splitting boundaries

$$\dot{x} = A_s x + b_s \text{ on } X_s, \qquad \text{(PWA)}$$

Definition (For $(n-1)$-dimensional boundaries)

$X_{ij} := X_i \cap X_j$ is called (i,j)-crossing boundary $:\iff$ there are solutions of (PWA) going from region i to region j.
$X_{ij} := X_i \cap X_j$ is called (i,j)-splitting boundary $:\iff$ there is no solutions reaching points in X_{ij}

Lemma (Iervolino, T., Vasca; CDC 2017)

Assume there exists local LF $V_s : \mathbb{R}^n \to \mathbb{R}$, such that

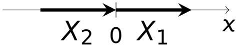
- $V_i(x) \geq V_j(x)$ *for all x in (i,j)-crossing boundaries*
- $V_i(x) = V_j(x)$ *for all x in non-crossing and non-splitting boundaries*

Then $x(t) \to 0$ for all Caratheodory solutions $x : [0,\infty) \to \mathbb{R}^n$

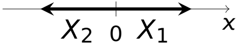
Forward, backward, sliding modes

For $x \in \mathbb{R}^n$ let

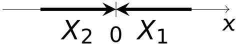
- $\Sigma^x := \{s \in \Sigma \mid x \in X_s\}$ the set of current modes
- $\Sigma^x_+ := \bigcup_{\xi \in \mathcal{FS}_+(x)} \bigcap_{\varepsilon>0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(\tau)}$ the set of forward modes
- $\Sigma^x_- := \bigcup_{\xi \in \mathcal{FS}_-(x)} \bigcap_{\varepsilon>0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(-\tau)}$ the set of backward modes
- $\Sigma^x_{\text{slide}} := \begin{cases} \Sigma^x_+, & \mathcal{CS}(x) = \emptyset \\ \emptyset, & \mathcal{CS}(x) \neq \emptyset \end{cases}$ the set of sliding modes



$$\Sigma^0 = \{1,2\},\ \Sigma^0_+ = \{1\},\ \Sigma^0_- = \{2\},\ \Sigma^0_{\text{slide}} = \emptyset$$



$$\Sigma^0 = \Sigma^0_+ = \{1,2\},\ \Sigma^0_- = \Sigma^0_{\text{slide}} = \emptyset$$



$$\Sigma^0 = \Sigma^0_+ = \Sigma^0_- = \Sigma^0_{\text{slide}} = \{1,2\}$$

Forward, backward, sliding modes

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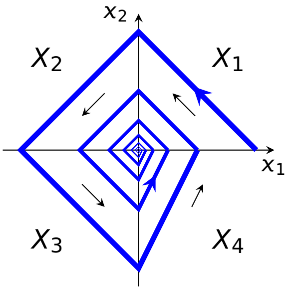

$$\Sigma^0 = \Sigma^0_+ = \Sigma^0_- = \Sigma^0_{\text{slide}} = \{1, 2, 3, 4\}$$

Forward, backward, sliding modes

For $x \in \mathbb{R}^n$ let

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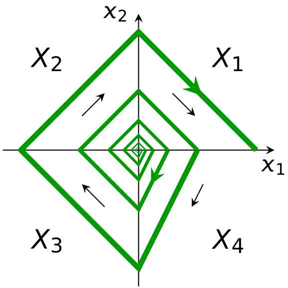

$$\Sigma^0 = \Sigma^0_+ = \{1, 2, 3, 4\}, \ \Sigma^0_- = \Sigma^0_{\text{slide}} = \emptyset$$

Proposed generalized stability result

$$\dot{x} = f_s(x), \quad x \in X_s, \; s \in \Sigma \qquad \text{(PWS)}$$

Conjecture

Let $V_s : \mathbb{R}^n \to \mathbb{R}$ be local Lyapunov functions such that for all $x \in \mathbb{R}^n$

- Jump condition: $V_i(x) \geq V_j(x) \quad \forall (i,j) \in \Sigma_-^x \times \Sigma_+^x$
- Sliding condition: $\Sigma_{\text{slide}}^x \neq \emptyset \wedge x \neq 0 \implies \exists i_x \in \Sigma_{\text{slide}}^x$:

$$\nabla V_{i_x}(x)(f_j(x)) < 0 \quad \forall j \in \Sigma_{\text{slide}}^x$$

Then (PWS) (with Filippov solutions) is globally asymptotically stable.

Proof idea: Consider global Lyapunov function $V(x) = \max_{s \in \Sigma_+^x} V_s(x)$

Problem: Is $t \mapsto V(x(t))$ differentiable almost everywhere?

Future work

TODO:

- Pass from point-wise jump/sliding-cond. to boundary-wise cond.
- Formulate sufficient LMIs to ensure validity of jump/sliding conditions

Long term goal: Automated stability proof for nonlinear systems

1. Chose polyhedral partition of $\mathbb{R}^n$
2. Linearize nonlinear dynamics around one interior point per region
   $\rightarrow$ PWA system with quantifiable approximation accuracy
3. Automatically set up LMIs
4. Try to find solution of LMIs with standard solvers
5. Solution found $\rightarrow$ global Lyapunov function for PWA
6. No solution found $\rightarrow$ refine partition and try again

Hope: LF for (PWA) also LF for original nonlinear system