Discontinuous Lyapunov Functions for Discontinuous Nonlinear Systems

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Piecewise smooth systems

Polyhedral partition

\[ \mathbb{R}^n = \bigcup_{s \in \Sigma} X_s, \quad \Sigma = \{1, 2, \ldots, N\} \]

\[ \text{int} X_i \cap \text{int} X_j = \emptyset \quad \forall i \neq j \]

Piecewise-smooth dynamics

\[ \dot{x} = f_s(x) \quad x \in X_s, \; s \in \Sigma \]

Piecewise-affine (PWA) systems:

\[ f_s(x) = A_s x + b_s \]
Motivation and problem formulation

Occurrence of PWA systems:
› Linear systems coupled with saturation and friction
› Electrical circuits with diodes and transistors
› Feedback control with gain scheduling
› Linearization of nonlinear systems around different operating points

Goal: Stability proof

Construct global Lyapunov function $V$ for

$$\dot{x} = f_s(x), \quad x \in X_s, \quad s \in \Sigma$$

(PWS)

with the help of local Lyapunov functions $V_s$ on $X_s$
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Problem formulation

**Solution concepts for PWS systems**

Lyapunov function construction

Outlook
Behavior on boundaries

\[ \dot{x} = f_s(x), \quad x \in X_s, \ s \in \Sigma \]  
(PWS)

\( X_s \) are closed convex polyhedra (i.e. finite intersection of half spaces)

**Question**

What to do for \( x \in X_i \cap X_j \)

**Answer:** We don’t care

More formally:

\[ \dot{x} \in \{f_s(x) \mid s \in \Sigma^x\} \]

where

\[ \Sigma^x := \{s \in \Sigma \mid x \in X_s\} \]
Caratheodory solutions

\[ \dot{x} = f_s(x), \quad x \in X_s, \; s \in \Sigma \]  

(PWS)

**Definition**

\( x : [0, \omega) \rightarrow \mathbb{R}^n \) is called **Caratheodory solution** of (PWS): \( \iff \)

1. \( x \) is absolutely continuous (hence differentiable a.e.)
2. \( \dot{x}(t) \in \{ f_s(x) \mid s \in \Sigma^x \} \) for a.a. \( t \in [0, \omega) \)

\[ f_s(x) = A_s x \]

\[ A_1 = A_3 = \begin{bmatrix} 1 & -5 \\ 0.2 & 1 \end{bmatrix} \]

\[ A_2 = A_4 = \begin{bmatrix} 1 & -0.2 \\ 5 & 1 \end{bmatrix} \]
Well-posedness issues

Non-existence

Example

\[ \dot{x} = \begin{cases} -1, & x \geq 0 \\ 1, & x \leq 0 \end{cases} \]

No solution with \( x(0) = 0 \).

Non-uniqueness

Example

\[ \dot{x} = \begin{cases} 1, & x \geq 0 \\ -1, & x \leq 0 \end{cases} \]

Two solutions with \( x(0) = 0 \).

In the context of stability analysis

Non-existence: Not OK

Non-uniqueness: OK
Accumulation of switching times (Zeno)

Right-accumulation

$$\dot{x} = \begin{cases} 
(-1, 1)^T & x_1 \geq 0, \ x_2 \geq 0 \\
(-1, -1)^T & x_1 \leq 0, \ x_2 \geq 0 \\
(1, -1)^T & x_1 \leq 0, \ x_2 \leq 0 \\
(1/2, 1)^T & x_1 \geq 0, \ x_2 \leq 0 
\end{cases}$$

Left-accumulation

$$\dot{x} = \begin{cases} 
(1, -1)^T & x_1 \geq 0, \ x_2 \geq 0 \\
(1, 1)^T & x_1 \leq 0, \ x_2 \geq 0 \\
(-1, 1)^T & x_1 \leq 0, \ x_2 \leq 0 \\
(-1/2, -1)^T & x_1 \geq 0, \ x_2 \leq 0 
\end{cases}$$
Filippov solutions

\[ \dot{x} = f_s(x), \quad x \in X_s, \ s \in \Sigma \]  

(PWS)

Definition

\( x : [0, \omega) \rightarrow \mathbb{R}^n \) is called Filippov solution of (PWS): \( \iff \)

1. \( x \) is absolutely continuous (hence differentiable a.e.)
2. \( \dot{x}(t) \in \text{conv} \{ f_s(x) | s \in \Sigma^x \} \) for a.a. \( t \in [0, \omega) \)

Theorem (cf. Filippov 1988)

\( \forall x_0 \in \mathbb{R}^n \) exists Filippov solution \( x : [0, \omega) \rightarrow \mathbb{R}^n \) of (PWS) with \( x(0) = x_0 \).

Furthermore, if \( f_s(x) = A_s x + b_s \) and \( \Sigma \) finite then \( \omega = \infty \).

Problem: “Unnecessary” sliding

\[ 0 \quad x \]

has \( x(t) \equiv 0 \) as Filippov solution.
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From local to global LF-function

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<td>(L1) $V_s \in \mathcal{C}$ and $V_s</td>
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<td>(L2) $V_s$ is <strong>positive definite</strong> on $X_s$</td>
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<td>(L3) $V_s$ is <strong>radially unbounded</strong> in the following sense:</td>
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<td>$\forall \nu \in V_s(X_s) \subseteq \mathbb{R} : V_s^{-1}([0, \nu]) \cap X_s$ is compact</td>
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<td>(L4) $V_s$ is <strong>decreasing along solutions</strong> within $X_s$, i.e.</td>
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<td>$\nabla V_s(x)f_s(x) &lt; 0 \quad \forall x \in X_s \setminus {0}$,</td>
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<td><strong>let global Lyapunov function</strong> $V : \mathbb{R}^n \to \mathbb{R}$ be given by</td>
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<td>$V(x) = V_s(x) \quad \text{for } x \in X_s, \ s \in \Sigma$</td>
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$f_s(x) = A_s x + b_s$ and $V_s(x) = x^T P_s x + 2q_s^T x + r_s \quad \rightarrow \quad \text{LMIs}$
Continuous global Lyapunov function

Theorem (Cf. Johansson 2003)

Let $V_s : \mathbb{R}^n \to \mathbb{R}$ be local LF on $X_s$ such that

$$V(x) = V_s(x) \quad \text{for } x \in X_s, \ s \in \Sigma$$

is continuous.
Then $x(t) \to 0$ for all Caratheodory solutions $x : [0, \infty) \to \mathbb{R}^n$.

Extension to Filippov solution

Johansson extended the above result for PWA systems to Filippov solutions by additionally requiring

$$\nabla V_s(x)(A_s x + b_s) < 0, \ \forall x \in X_s \cap X_{\overline{s}}$$
Example to motivate discontinuous LF

\[ \dot{x} = A_s x + b_s \text{ on } X_s, \quad \text{(PWA)} \]

with

\[
A_0 = -I, \quad A_1 = A_2 = A_3 = A_4 = 0, \\
b_0 = 0, \quad b_1 = \begin{pmatrix} 1 \\ -0.1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \\
b_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b_4 = -b_2.
\]

PWQ-LF: \[ V_s(x) = x^T P_s x + 2 q_s^T x + r_s \]

Lemma

The above PWA system does not permit a continuous PWQ-LF.
Crossing and splitting boundaries

\[ \dot{x} = A_s x + b_s \text{ on } X_s, \]  
(PWA)

**Definition (For \((n-1)\)-dimensional boundaries)**

\( X_{ij} := X_i \cap X_j \) is called \((i, j)\)-crossing boundary \(\iff\) there are solutions of (PWA) going from region \(i\) to region \(j\).

\( X_{ij} := X_i \cap X_j \) is called \((i, j)\)-splitting boundary \(\iff\) there is no solutions reaching points in \(X_{ij}\).

**Lemma (Iervolino, T., Vasca; CDC 2017)**

Assume there exists local LF \(V_s : \mathbb{R}^n \rightarrow \mathbb{R}\), such that

\[ \forall x \in (i,j)\text{-crossing boundaries} \quad V_i(x) \geq V_j(x) \]

\[ \forall x \in \text{non-crossing and non-splitting boundaries} \quad V_i(x) = V_j(x) \]

Then \(x(t) \rightarrow 0\) for all Carathéodory solutions \(x : [0, \infty) \rightarrow \mathbb{R}^n\)
Forward, backward, sliding modes

For $x \in \mathbb{R}^n$ let

1. $\Sigma^x := \{ s \in \Sigma \mid x \in X_s \}$ the set of current modes
2. $\Sigma^x_+ := \bigcup_{\xi \in FS_+ (x)} \cap_{\epsilon > 0} \bigcup_{\tau \in (0, \epsilon)} \Sigma^{\xi (\tau)}$ the set of forward modes
3. $\Sigma^x_- := \bigcup_{\xi \in FS_- (x)} \cap_{\epsilon > 0} \bigcup_{\tau \in (0, \epsilon)} \Sigma^{\xi (-\tau)}$ the set of backward modes
4. $\Sigma^x_{\text{slide}} := \begin{cases} \Sigma^x_+, & CS(x) = \emptyset \\ \emptyset, & CS(x) \neq \emptyset \end{cases}$ the set of sliding modes

$\Sigma^0 = \{1, 2\}$, $\Sigma^0_+ = \{1\}$, $\Sigma^0_- = \{2\}$, $\Sigma^0_{\text{slide}} = \emptyset$

$\Sigma^0 = \Sigma^0_+ = \{1, 2\}$, $\Sigma^0_- = \Sigma^0_{\text{slide}} = \emptyset$

$\Sigma^0 = \Sigma^0_+ = \Sigma^- = \Sigma^0_{\text{slide}} = \{1, 2\}$
Forward, backward, sliding modes

For $x \in \mathbb{R}^n$ let

$\Sigma^x := \{ s \in \Sigma \mid x \in X_s \}$  the set of current modes

$\Sigma^x_+ := \bigcup_{x \in \mathcal{F}_+} \bigcap_{\varepsilon > 0} \bigcup_{\tau \in (0, \varepsilon)} \Sigma^x(\tau)$  the set of forward modes

$\Sigma^x_- := \bigcup_{x \in \mathcal{F}_-} \bigcap_{\varepsilon > 0} \bigcup_{\tau \in (0, \varepsilon)} \Sigma^x(-\tau)$  the set of backward modes

$\Sigma^x_{\text{slide}} := \begin{cases} \Sigma^x_+, & CS(x) = \emptyset \\ \emptyset, & CS(x) \neq \emptyset \end{cases}$  the set of sliding modes

\[ \Sigma^0 = \Sigma^0_+ = \Sigma^0_- = \Sigma^0_{\text{slide}} = \{1, 2, 3, 4\} \]
For $x \in \mathbb{R}^n$ let

$\Sigma^x := \{ s \in \Sigma \mid x \in X_s \}$ the set of current modes

$\Sigma^x_+ := \bigcup_{\xi \in \mathcal{F}S_+ (x)} \bigcap_{\tau > 0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(\tau)}$ the set of forward modes

$\Sigma^x_- := \bigcup_{\xi \in \mathcal{F}S_- (x)} \bigcap_{\tau > 0} \bigcup_{\tau \in (0,\varepsilon)} \Sigma^{\xi(-\tau)}$ the set of backward modes

$\Sigma^x_{\text{slide}} := \begin{cases} \Sigma^x_+ \mid CS(x) = \emptyset, & \text{the set of sliding modes} \\ \emptyset, & CS(x) \neq \emptyset \end{cases}$

$\Sigma^0 = \Sigma^0_+ = \{1, 2, 3, 4\}, \Sigma^- = \Sigma^0_{\text{slide}} = \emptyset$
Proposed generalized stability result

\[ \dot{x} = f_s(x), \quad x \in X_s, \quad s \in \Sigma \]  

(PWS)

Conjecture

Let $V_s : \mathbb{R}^n \to \mathbb{R}$ be local Lyapunov functions such that for all $x \in \mathbb{R}^n$

- Jump condition: $V_i(x) \geq V_j(x) \quad \forall (i, j) \in \Sigma_- \times \Sigma_+$
- Sliding condition: $\Sigma_{\text{slide}} \neq \emptyset \land x \neq 0 \quad \implies \exists i \in \Sigma_{\text{slide}} :$

\[ \nabla V_{ix}(x)(f_j(x)) < 0 \quad \forall j \in \Sigma_{\text{slide}} \]

Then (PWS) (with Filippov solutions) is globally asymptotically stable.

Proof idea: Consider global Lyapunov function $V(x) = \max_{s \in \Sigma_+} V_s(x)$

Problem: Is $t \mapsto V(x(t))$ differentiable almost everywhere?
Future work

TODO:
› Pass from point-wise jump/sliding-cond. to boundary-wise cond.
› Formulate sufficient LMIs to ensure validity of jump/sliding conditions

Long term goal: Automated stability proof for nonlinear systems
1. Chose polyhedral partition of $\mathbb{R}^n$
2. Linearize nonlinear dynamics around one interior point per region → PWA system with quantifiable approximation accuracy
3. Automatically set up LMIs
4. Try to find solution of LMIs with standard solvers
5. Solution found → global Lyapunov function for PWA
6. No solution found → refine partition and try again

Hope: LF for (PWA) also LF for original nonlinear system