Switch induced instabilities for stable power system DAE models ⋆

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Abstract: It is well known that for switched systems the overall dynamics can be unstable despite stability of all individual modes. We show that this phenoma can indeed occur for a linearized DAE model of power grids. By making certain topological assumptions on the power grid, we can ensure stability under arbitrary switching.

Keywords: Power Systems, DAE, switching, stability

1. INTRODUCTION

In the precursor (Gross et al., 2016) to this work we have discussed properties of a (linearized) differential-algebraic equation (DAE) model of power grids. We were able to show that the resulting DAE is regular, of index one and also stable (i.e. all solutions remain bounded). The presence of line failures or disconnection of generators can mathematically be modelled in the framework of switched DAEs (Trenn, 2012). It is well known, that switching between stable systems can lead to an overall unstable behavior (Liberzon, 2003). It is therefore of interest to study the stability properties of power DAE models in the presence of switching.

There is a large amount of literature concerning the stability of power systems, however, we are not aware of any references studying the destabilizing effects induced by structural changes in the modelling framework of switched DAEs.

This paper is structured as follows. We will first present a simple example of a power systems which exhibits an unstable behavior under a specific switching signal. Afterwards we present the general power system DAE model from Gross et al. (2016) and recall some basic facts from the theory of switched DAEs. In Section 5 we present sufficient conditions in terms of the power grid topology which guarantees stability under arbitrary switching.

2. UNSTABLE POWER GRID EXAMPLE

Consider a power grid with two generators as shown in Figure 1. We will consider the situation that the parameters of the line between busses one and four abruptly changes (the susceptance of the line is three orders of magnitude larger in mode two than in mode one).

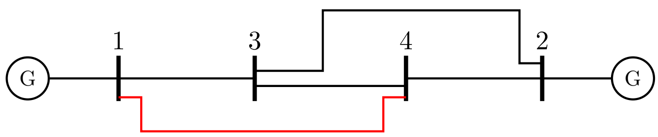
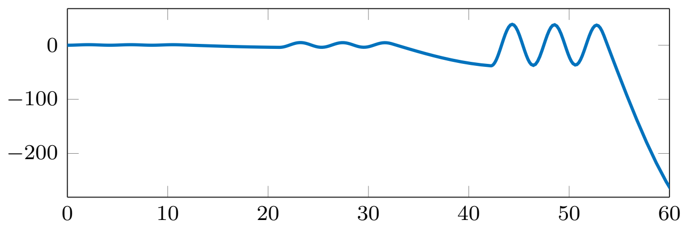
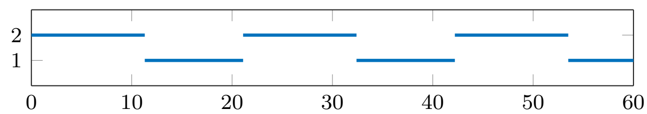


Fig. 1. A simple power grid with two generators. The red line will be subject to sudden changes in the line parameter.



(a) Evolution of x_1 in time.



(b) Destabilizing switching signal.

Fig. 2. Illustration of destabilizing effect of switching.

The simulation shows clearly an unstable behavior, see Figure 2(a) for a plot of the first component of the state vector.

For the DAE descriptions given by the matrix pairs $(E, A^1), (E, A^2) \in \mathbb{R}^{8\times 8} \times \mathbb{R}^{8\times 8}$ as recalled in the next section, the parameters are given by

$$\left(\begin{smallmatrix}\eta_1\\ \eta_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right), \quad M = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right], \; Z = \left[\begin{smallmatrix}0.1 & 0\\ 0 & 0.1\end{smallmatrix}\right], \; D = \left[\begin{smallmatrix}0.01 & 0\\ 0 & 0.01\end{smallmatrix}\right],$$

and, for mode 1,

$$\mathfrak{L}^1 = \begin{bmatrix} -0.01 & 0 & 0.005 & 0.005 \\ 0 & -5.005 & 0.005 & 5 \\ 0.005 & 0.005 & -0.02 & 0.01 \\ 0.005 & 5 & 0.01 & -5.015 \end{bmatrix} \qquad (1)$$

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and, for mode 2,

$$\mathfrak{L}^2 = \begin{bmatrix} \boxed{-2.005} & 0 & 0.005 & \boxed{2} \\ 0 & -5.005 & 0.005 & 5 \\ 0.005 & 0.005 & -0.02 & 0.01 \\ \boxed{2} & 5 & 0.01 & \boxed{-7.01} \end{bmatrix}. \quad (2)$$

The dashed boxes in (1) and (2) highlight the changes in the system matrices induced by the susceptance change in the line between bus one and four. As (consistent) initial value we choose

$$x_0 := [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^\top .$$

For the illustration of the destabilizing effect of switching it suffices (due to linearity) to consider the system with zero input.

We were able to achieve a destabilizing effect with a periodic switching signal, where the dwell time for each mode is chosen to be the inverse of the dominant natural frequency of the systems[1], i.e. with dwell times $\tau_1 = 11.3 \approx 1/(0.08838)$ and $\tau_2 = 9.8 \approx 1/(0.101633)$, see Figure 2(b).

3. DAE MODEL OF POWER SYSTEMS

As in (Gross et al., 2016), we consider a power grid consisting of $n_g \in \mathbb{N}$ generators (connected to n_g generator busses) and $n_b \in \mathbb{N}$ additional busses (which are not directly connected to a generator). The dynamical behaviour of the i-th generator is modelled as $\eta_i \in \mathbb{N}$ coupled rotating masses (the turbines) given by the linear differential-equation

$$\begin{aligned} \dot\alpha^i(t) &= \omega^i(t), \\ M^i\dot\omega^i(t) &= -D^i\omega^i(t) - K^i\alpha^i(t) + P_g^i(t) - P_e^i(t), \end{aligned}$$

where $\alpha^i = (\alpha_1^i, \dots, \alpha_{\eta_i}^i)$ and $\omega^i = (\omega_1^i, \dots, \omega_{\eta_i}^i)$ are the angles and the (relative) angular velocities of the η_i rotating masses, P_g^i is the vector of generator power acting on the turbines and $P_e^i = (0, \dots, 0, p_e^i)$ is the electrical power acting on the last rotating mass (the actual generator). The diagonal matrix M^i contains the (positive) moments of inertia of the rotating masses; the tridiagonal, symmetric, positive definite matrix D^i contains the friction coefficients and K^i is a tridiagonal, symmetric, positive semidefinite matrix containing the spring constants of the shafts connecting the rotating masses (and is zero if $\eta_i = 1$), for details see Gross et al. (2016).

The electrical interconnections of the generators with the power grid are represented by constant-voltage-behind-transient-reactance models (see e.g. Kimbark (1948); Kundur (1994); Machowski et al. (2008)); in particular, under the assumption that the difference $\alpha_{\eta_i}^i - \theta_i$ between generator angle and bus voltage angle is small, the electrical power p_e^i is approximately given by (cf. Pasqualetti et al. (2011); Gross et al. (2014))

$$p_e^i(t) = \frac{1}{z^i}(\alpha_{\eta_i}^i(t) - \theta_i(t)),$$

where $z^i > 0$ is the transient reactance of the generator.

[1] We have not investigated so far, whether this choice always leads to a worst case behavior; this is a possible topic for future research.

The transmission lines are described by the Π-model (see e.g. Elgerd (1982); Kundur (1994)); it is assumed that the conductance between the busses is negligible and that the difference of the bus voltage angles is small, then the power flow equations can be linearized as follows (Gross et al., 2014), $i = 1, \dots, n_g + n_b$,

$$p^i(t) = \sum_{j=1}^{n_g+n_b} b_{ij}(\theta^i(t) - \theta^j(t)) - p_e^i(t)$$

where $p^i(t)$ is the active power infeed (usually negative) at the i-bus, $b_{ij} = b_{ji} \geq 0$ is the susceptance between bus i and j and $p_e^i = 0$ for $i > n_g$. Note that $[b_{ij}]_{i,j=1,\dots,n_g+n_b}$ is the (weighted) adjacency matrix of the coupling graph of the power grid. Let $\mathfrak{L} \in \mathbb{R}^{(n_g+n_b)\times(n_g+n_b)}$ be the corresponding (weighted) Laplacian matrix of the graph, i.e. $\mathfrak{L} = [\ell_{ij}]$ with

$$\begin{aligned} \ell_{ii} &= \sum_{j=1}^{n_g+n_b} b_{ij}, \quad \forall i, \\ \ell_{ij} &= -b_{ij}, \quad \forall i \neq j. \end{aligned}$$

The overall DAE describing the power grid is now given by

$$E\dot x = Ax + Bu, \quad (3)$$

where, for $n_\eta = \sum_{i=1}^{n_g} \eta_i$, $x = (\alpha^\top, \omega^\top, \theta^\top)^\top \in \mathbb{R}^{n_\eta+n_\eta+(n_g+n_b)}$, $u = (P_g^\top, P^\top)^\top \in \mathbb{R}^{n_\eta+(n+m)}$ with α, ω, θ, P^g, P being each composed from α^i, ω^i, θ^i, P_g^i, p^i;

$$E = \begin{bmatrix} I_{n_\eta} & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ I_{n_\eta} & 0 \\ 0 & I_{n_g+n_b} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & I_{n_\eta} & 0 \\ -K - HZ^{-1}H^\top & -D & \left[\begin{smallmatrix} HZ^{-1} & 0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} Z^{-1}H^\top \\ 0 \end{smallmatrix}\right] & 0 & -\mathfrak{L} - \left[\begin{smallmatrix} Z^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] \end{bmatrix},$$

with M, D, K, Z being (block) diagonal matrices composed from M^i, D^i, K^i, z^i and

$$H = \begin{bmatrix} H^1 \\ \vdots \\ H^{n_g} \end{bmatrix}, \quad H^i = \begin{bmatrix} 0_{(\eta_i-1)\times n_g} \\ e_i^\top \end{bmatrix}$$

with $e_i \in \mathbb{R}^n$ being the i-th unit vector.

In the context of switching, each of the possible $\mathsf{q} \in \mathbb{N}$ operation modes is given by a DAE of the form (3) with matrices (E^1, A^1, B^1), ..., $(E^\mathsf{q}, A^\mathsf{q}, B^\mathsf{q})$. Here we restrict our attention to the case that the switches are induced by changes in the line parameters, i.e. the changes occur only in the Laplacian matrix $\mathfrak{L}$, i.e.

$$E^1 = \ldots = E^\mathsf{q} =: E, \quad B^1 = \ldots = B^\mathsf{q} =: B$$

and, for $q = 1, \dots, \mathsf{q}$,

$$A^q = \begin{bmatrix} 0 & I_{n_\eta} & 0 \\ -K - HZ^{-1}H^\top & -D & \left[\begin{smallmatrix} HZ^{-1} & 0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} Z^{-1}H^\top \\ 0 \end{smallmatrix}\right] & 0 & -\mathfrak{L}^q - \left[\begin{smallmatrix} Z^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] \end{bmatrix},$$

where $\mathfrak{L}^1, \dots, \mathfrak{L}^\mathsf{q}$ are the Laplacian matrices of the different couplings.

4. SWITCHED DAES

A switched differential-algebraic equation (DAE) is a time-varying, linear, implicit differential equation of the form

$$E^{\sigma(t)}\dot x = A^{\sigma(t)}x + B^{\sigma(t)}u \quad (4)$$

where $\sigma : \mathbb{R} \to \Sigma := \{1, 2, \ldots, \mathsf{q}\}$ is the switching signal choosing at each time which of the $\mathsf{q} \in \mathbb{N}$ modes is active and, for $q \in \Sigma$, $E^q, A^q \in \mathbb{R}^{n\times n}$, $B^q \in \mathbb{R}^{n\times m}$. We assume that σ is piecewise constant and right continuous and has only finitely many jumps in each finite interval (no Zeno-behavior); the matrix pairs (E^q, A^q) are each assumed to be regular, i.e. for each $q \in \Sigma$ the polynomial $\det(sE^q - A^q)$ is not identically zero. A very important characterization for regularity which goes back to Weierstraß (1868) is given by the following well known result:

Lemma 1. A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that (E, A) is equivalent to a *quasi-Weierstrass form* (QWF):

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{5}$$

where $N \in \mathbb{R}^{n_N \times n_N}$ is *nilpotent* and $J \in \mathbb{R}^{n_J \times n_J}$ with $n_N + n_J = n$.

Note that we do not consider the Weierstrass canonical form, because in the QWF the matrices N and J are not assumed to be in Jordan canonical form. An easy way to obtain the QWF is via the Wong sequences (Wong, 1974), for details see Berger et al. (2012). In particular, the limit $\mathscr{V}$ of the first Wong sequence is exactly the subspace of consistent initial values:

$$\mathscr{V} = \{\ x_0 \in \mathbb{R}^n \ \mid \exists \text{ solution of } E\dot{x} = Ax, x(0) = x_0\ \}.$$

The index of (E, A) (or the corresponding DAE) is defined to be the nilpotency index of N in the QWF. In case (E, A) has the special structure (semi-explicit form)

$$(E, A) = \left(\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \tag{6}$$

with E_1 being invertible, it is easily seen, that (E, A) is regular if, and only if, $[A_3, A_4]$ has full row rank; if this is the case, then (E, A) is of index one if, and only if, A_4 is invertible. In fact, if A_4 is invertible, one obtains the QWF (5) (with $N = 0$ and $J = E_1^{-1}(A_1 - A_2 A_4^{-1} A_3)$) via

$$S = \begin{bmatrix} E_1^{-1} & -E_1^{-1} A_2 A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix}, \quad T = \begin{bmatrix} I & 0 \\ -A_4^{-1} A_3 & I \end{bmatrix}. \tag{7}$$

In general, existence and uniqueness of solution of the switched DAE (4) is guaranteed provided all matrix pairs (E_p, A_p) are regular; however, solutions have to be considered in a certain distributional solution framework (Trenn, 2012). In particular, solutions of (4) will be discontinuous and may even contain derivatives of jumps (Dirac impulses). If the solutions do not contain Dirac impulses (impulse-freeness), then one can interpret the distributional solutions again as piecewise-smooth functions (right-continuous) and we will simply write $x(t)$ or $x(t^-)$ for the evaluation of x at time t (or t^-, i.e. the left limit) although, formally, the evaluation of a general distribution at some specific point in time is not well defined. Independently of the index, the unique jump in the solution of (4) with $u \equiv 0$ is given by

$$x(t) = \Pi^{\sigma(t)} x(t^-)$$

where $\Pi^q \in \mathbb{R}^{n\times n}$ is the *consistency projector* of mode q, given by

$$\Pi^q = T^q \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} (T^q)^{-1},$$

where the block sizes correspond to the block sizes in the QWF of (E^q, A^q) obtained by some invertible matrices S^q, T^q.

We will now introduce the following stability notion for (4):

Definition 2. The regular switched DAE (4) (for given switching signal) is called stable iff 1) all solutions are impulse-free [2] and 2) for all $\varepsilon > 0$ there exists $\delta > 0$ such that all solutions x_1, x_2 for the same input u satisfy the following implication:

$$\|x_1(0^-) - x_2(0^-)\| \le \delta \implies |x_1(t) - x_2(t)| \le \varepsilon.$$

Due to linearity, it suffices to consider $u = 0$ and $x_2 = 0$; furthermore, it is easily seen that stability is equivalent to boundedness of all solutions.

In contrast to Liberzon and Trenn (2012) we do not consider *asymptotic stability*, because, as was shown in Gross et al. (2016), the power grid DAE models considered here are only stable and not asymptotically stable. We will now give a sufficient condition for stability of the switched DAE (4) in terms of (multiple) Lyapunov functions:

Theorem 3. Consider the regular switched DAE (4) with corresponding consistency spaces $\mathscr{V}^q$ and consistency projectors Π^q, $q \in \Sigma$. If

I. all solutions are impulse-free;
II. for each $q \in \Sigma$, there exist a symmetric $P^q \in \mathbb{R}^{n\times n}$ such that $V^q(x) := x^\top (E^q)^\top P^q E^q x$ is positive definite on the consistency space $\mathscr{V}^q$ and $\dot{V}^q(x) := x^\top \left((A^q)^\top P^q E^q + (E^q)^\top P^q A^q \right) x$ is negative semidefinite on $\mathscr{V}^q$;
III. for all $p, q \in \Sigma$ the Lyapunov-functions are not increasing at switches, i.e.
$$V^q(\Pi^q x) \le V^p(x) \quad \forall x \in \mathscr{V}^p,$$

then (4) is stable for any switching signal.

Proof. The proof is a straightforward adaption of the proof in Liberzon and Trenn (2012), where the stronger case of asymptotic stability was considered.

Remark 4. Existence of a Lyapunov function as in assumption II. of Theorem 3 for a regular matrix pair (E, A) is equivalent with stability of the unswitched DAE $E\dot{x} = Ax$, in fact, stability of the latter is equivalent with solvability of the generalized Lyapunov equation

$$A^\top PE + E^\top PA = -Q \tag{8}$$

for some symmetric matrices $P, Q \in \mathbb{R}^{n\times n}$ being positive semidefinite on $\mathscr{V}$, c.f. Liberzon and Trenn (2012, Rem. 2.8). However, in contrast to ODEs, not for all Q the equation (8) has a solution P. If the regular matrix pair (E, A) has index one (or two) then stability actually is equivalent to solvability of

$$A^\top PE + E^\top PA = -E^\top QE, \tag{9}$$

for details see Groß (2016, Thm. 5.4.2) (which is a slight modification of Stykel (2002, Thm. 4.8) to the non-asymptotic case).

Remark 5. For a regular, index-one matrix pair (E, A) in semi-explicit form (6) the consistency projector is given by

$$\Pi = \begin{bmatrix} I & 0 \\ -A_4^{-1} A_3 & 0 \end{bmatrix}.$$

[2] but jumps are still allowed

Furthermore, for any function $V : \mathbb{R}^n \to \mathbb{R}$ given by $V(x) = x^\top E^\top P E x$ it is easily seen that

$$V(\Pi x) = x_1^\top E_1^\top P_1 E_1^\top x_1 = V(x),$$

where $P = \left[\begin{smallmatrix} P_1 & P_2 \\ P_3 & P_4 \end{smallmatrix}\right]$ and $x = (x_1^\top, x_2^\top)^\top$ with partitions corresponding to the block sizes in (6). Consequently, for index-one switched systems in semi-explicit form and parameter changes only in the A-matrix, Theorem 3 yields that the existence of a *common* Lyapunov function is sufficient to ensure stability under arbitrary switching (in general, a common Lyapunov Function is *not* sufficient to guarantee stability under arbitrary switching, see e.g. Liberzon and Trenn (2009, Ex. 1)).

5. STABILITY OF SWITCHED POWER SYSTEMS

We have seen that in general switching may result in an overall unstable behavior; however, under certain restrictions on the topology of the power grid as well as on the allowed topological changes stability may be preserved under switching. A key lemma to formulate such a topological restriction is the following.

Lemma 6. Consider a matrix pair (E, A) with the following structure:

$$(E, A) = \left(\begin{bmatrix} E_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 & 0 \\ A_3 & -\mathfrak{L}_1 + A_4 & -\mathfrak{L}_2 \\ 0 & -\mathfrak{L}_3 & -\mathfrak{L}_4 \end{bmatrix} \right)$$

with $E_1 \in \mathbb{R}^{n_1 \times n_1}$, $n_1 \in \mathbb{N}$, invertible, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2, A_3^\top \in \mathbb{R}^{n_1 \times n_{21}}$, $n_{21} \in \mathbb{N}$, $A_4 \in \mathbb{R}^{n_{21} \times n_{21}}$, and $\mathfrak{L} := \left[\begin{smallmatrix} \mathfrak{L}_1 & \mathfrak{L}_2 \\ \mathfrak{L}_3 & \mathfrak{L}_4 \end{smallmatrix}\right] \in \mathbb{R}^{n_2 \times n_2}$, $n_2 > n_{21}$, is a (weighted) Laplacian matrix of some (undirected) graph with n_2 nodes. Assume that

(1) (E, A) is regular, stable and index one;
(2) $\operatorname{rank} \mathfrak{L}_3 = 1$.

Then there is a Lyapunov function for $E\dot{x} = Ax$ which is also valid for any regularity preserving topological change in $\mathfrak{L}_4$. In particular, there is a *common* Lyapunov function for the corresponding switched systems where parameter changes only occur in $\mathfrak{L}_4$.

Proof. According to Remark 4, stability of (E, A) with index-one guarantees existence of a Lyapunov function $V(x) = x^\top E^\top P E x$ where P is a symmetric positive semidefinite solution of (9) for some positive semidefinite Q. We will now show that the possible choices for P are independent of the entries in $\mathfrak{L}_4$, which then proves the claim of the lemma. For that, we consider a partition of P according to the partition of E and A, i.e.

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}.$$

Evaluating (9) also blockwise we see that only the following two block equations depend on $\mathfrak{L}_4$:

$$-\mathfrak{L}_3 P_{21} E_1 - \mathfrak{L}_4 P_{31} E_1 = 0,$$
$$-E_1^\top P_{12} \mathfrak{L}_2 - E_1^\top P_{13} \mathfrak{L}_4 = 0.$$

Due to symmetry of $\mathfrak{L}$ and P, both are equivalent and can be rewritten as (invoking invertibility of E_1):

$$\operatorname{im} \begin{bmatrix} P_{21} \\ P_{31} \end{bmatrix} \subseteq \ker[\mathfrak{L}_3 \ \mathfrak{L}_4].$$

Due to regularity, $[\mathfrak{L}_3 \ \mathfrak{L}_3]$ has full row rank $n_{22} := n_2 - n_{21}$, hence $\dim \ker[\mathfrak{L}_3 \ \mathfrak{L}_4] = n_2 - n_{22} = n_{21}$. For any Laplacian matrix $\mathfrak{L}$ we have $(1, \ldots, 1)^\top \in \ker \mathfrak{L} \subseteq \ker[\mathfrak{L}_3 \ \mathfrak{L}_4]$ and since $\operatorname{rank} \mathfrak{L}_3 = 1$ by assumption we additionally have $\dim \ker \mathfrak{L}_3 = n_{21} - 1$. Altogether this yields

$$\ker[\mathfrak{L}_3 \ \mathfrak{L}_4] = \begin{pmatrix} \ker \mathfrak{L}_3 \\ \{0\} \end{pmatrix} \oplus \operatorname{im} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

which shows that $\ker[\mathfrak{L}_3 \ \mathfrak{L}_4]$ is independent of the specific entries in $\mathfrak{L}_4$ and, therefore, the solution of (9) is independent of $\mathfrak{L}_4$.

The result of Lemma 6 can now be utilized to give a topological condition on a power grid which ensures stability under arbitrary switching. Therefore, we will make the following topological assumptions on the power grid network.

Assumptions
Consider an electrical grid as in Section 3 with a corresponding coupling graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$. Assume that

$$\mathfrak{V} = \mathfrak{V}_g \cup \mathfrak{V}_c \cup \mathfrak{V}_l$$

such that

(i) $\mathfrak{V}_g$ are the nodes corresponding to the generator busses (in particular, $|\mathfrak{V}_g| = n_g$);
(ii) there are no edges betweens nodes in $\mathfrak{V}_g$ and nodes in $\mathfrak{V}_l$;
(iii) all nodes in $\mathfrak{V}_g$ are connected with all nodes in $\mathfrak{V}_c$;
(iv) the weights of the edges between $\mathfrak{V}_g$ and $\mathfrak{V}_c$ are such that the corresponding submatrix of the Laplacian has rank one [3];
(v) topological changes (addition/removal of edges or sudden change of the weight) are allowed in all edges between nodes in $\mathfrak{V}_c \cup \mathfrak{V}_l$ as long as the resulting graph remains connected.

Note that Assumption (iv) already "implies" Assumption (iii), because assuming that a node in $\mathfrak{V}_c$ is not connected to all generators implies (due to the rank-one assumption) that it cannot be connected to any generator, hence this node should be in the set $\mathfrak{V}_l$.

As an example consider a power grid with underlying graph as shown in Figure 3.

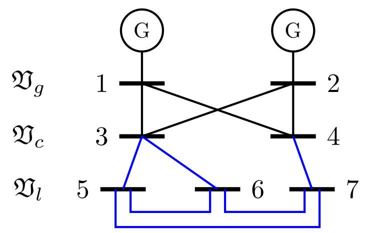

Fig. 3. A simple power grid satisfying Assumptions (i)-(v).

The corresponding Laplacian has the following structure

[3] In particular, this is the case if either A) for each generator bus all adjacent edges have the same weight or B) for each node in $\mathfrak{V}_c$ all adjacent edges have the same weight.

$$\mathfrak{L} = \begin{bmatrix} \mathfrak{L}_1 & \mathfrak{L}_2 \\ \mathfrak{L}_3 & \mathfrak{L}_4 \end{bmatrix} = \left[\begin{array}{cc|ccccc} * & 0 & \ell_{13} & \ell_{14} & 0 & 0 & 0 \\ 0 & * & \ell_{23} & \ell_{23} & 0 & 0 & 0 \\ \hline \ell_{13} & \ell_{23} & * & 0 & \ell_{35} & \ell_{36} & 0 \\ \ell_{14} & \ell_{24} & 0 & * & 0 & 0 & \ell_{47} \\ 0 & 0 & \ell_{35} & 0 & * & \ell_{56} & \ell_{57} \\ 0 & 0 & \ell_{36} & 0 & \ell_{56} & * & \ell_{67} \\ 0 & 0 & 0 & \ell_{47} & \ell_{57} & \ell_{67} & * \end{array}\right]$$

and Assumptions (i)-(v) are satisfied if, and only if, only the entries in $\mathfrak{L}_4$ (highlighted in blue) are subject to changes and the matrix $\left[\begin{smallmatrix} \ell_{13} & \ell_{23} \\ \ell_{14} & \ell_{24} \end{smallmatrix}\right]$ only contains positive entries and has rank one.

Theorem 7. Consider a switched power grid model satisfying Assumptions (i)-(v). Then it remains stable for arbitrary switching signals.

Proof. Since each mode by assumption has a connected coupling graph, Gross et al. (2016, Thms. 3.2,4.3,5.3) have shown that each mode is regular, index-one and stable. The topological assumptions ensure that all parameter changes only occur in $\mathfrak{L}_4$ and also that $\mathfrak{L}_3$ has rank one, hence all requirements of Lemma 6 are satisfied and there exist a common Lyapunov-Function V. Now Remark 5 concludes the proof.

Consider again the example from Section 2. The conditions from Theorem 7 are not satisfied, because the switches occur for a power line directly connected to a generator bus and, furthermore, the susceptances for the lines connected to the generator buses are not identical (i.e. the rank-one-assumption from Lemma 6 is not satisfied); therefore, stability for arbitrary switching cannot be guaranteed and indeed instability occurs as shown with the simulations in Section 2.
However setting the susceptance between bus 1 and 4 to the value 5 and switching the line between bus 3 and 4, the assumptions of Theorems 7 are satisfied with $\mathfrak{V}_g = \{1, 2\}$, $\mathfrak{V}_c = \{3, 4\}$, $\mathfrak{V}_l = \emptyset$ and for Case B in the footnote of Assumption (iv). Therefore, stability is guaranteed for arbitrary switching.

Already for this 8×8 example it is not possible to obtain a common Lyapunov function via the standard LMI-Toolbox (Gahinet et al., 1994). However, based on the QWF (5) obtained via (7) one can easily find $Y = Y^\top > 0$, such that $YJ + J^\top Y \leq 0$, e.g.,

$$Y \approx \begin{bmatrix} 14225.38 & -14225.20 & 10.67275 & 4.24986 \\ -14225.20 & 14225.38 & 7.75326 & 14.17615 \\ 10.67275 & 7.75326 & 17006.95 & 8477.417 \\ 4.24986 & 14.17615 & 8477.417 & 17006.93 \end{bmatrix}.$$

It is now possible to construct a Lyapunov function for the original system with the help of Y via

$$V(x) := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top E^\top S^\top \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} SE \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0 \quad \forall x_1 \neq 0, \tag{10}$$

Then the symmetric matrices, $i = 1, 2$,

$$\Pi^\top \left(E^\top S^\top \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} SA^i + (A^i)^\top S^\top \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} SE \right) \Pi \tag{11}$$

corresponding to the derivates along solutions have negative or (numerically) zero eigenvalues. In view of Remark 5 we can therefore conclude directly with the help of a common Lyapunov function that the switched system is stable under arbitrary switching.

6. CONCLUSION

We have studied the stability property of a simple, linearized model of a power grid which is subject to sudden structural changes. Surprisingly, the switching itself may result in an unstable behavior although each configuration exhibits stable dynamics. At the moment this is just a theoretical observation and it remains a topic for future research whether this phenomena really plays an important role in real world power grids. In particular, unbounded trajectories in response to switching indicates that our model introduces energy into the system when a topological change occurs; whether this is physically justified needs to be clarified in the future.

We provide topological assumptions on the power grid which prevents instability due to switching. These assumptions in particular require that certain line parameters satisfy a rank-one assumption; an intuitive interpretation of this rank-one assumption in terms of the physical properties of the power grid is still an open question.

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