If you have any questions concerning this material (in particular, specific pointers to literature), please don't hesitate to contact me via email: trenn@mathematik.uni-kl.de

7 Consistency projector

Definition (Initial trajectory problem (ITP))**.** Given past trajectory $x^0 : (-\infty,0) \to \mathbb{R}^n$ find $x : \mathbb{R} \to \mathbb{R}^n$ such that

$$\left.\begin{aligned} x\big|_{(-\infty,0)} &= x^0 \\ (E\dot{x})\big|_{[0,\infty)} &= (Ax+f)\big|_{[0,\infty)} \end{aligned}\right\} \tag{ITP}$$

"Theorem": Consider (ITP) with regular (E,A) and $f=0$. Choose S,T invertible such that

$$(SET,SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right).$$

Then any solution x of (ITP) satisfies

$$x(0+) = \Pi_{(E,A)} x(0-)$$

where

$$\boxed{\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}}$$

is the *consistency projector*.

Proof: Let $\begin{pmatrix} v \\ w \end{pmatrix} = T^{-1}x$ and $\begin{pmatrix} v^0 \\ w^0 \end{pmatrix} = T^{-1}x^0$, then x solves (ITP) with $f=0$ $\Leftrightarrow$ $\begin{pmatrix} v \\ w \end{pmatrix}$ solves

$$\left.\begin{aligned} v\big|_{(-\infty,0)} &= v^0 \\ \dot{v}_{[0,\infty)} &= (Jv)_{[0,\infty)} \end{aligned}\right\} \tag{$*$}$$

and

$$\left.\begin{aligned} w\big|_{(-\infty,0)} &= w^0 \\ (N\dot{w})_{[0,\infty)} &= w_{[0,\infty)} \end{aligned}\right\} \tag{$**$}$$

Since $(*)$ is an ODE on $[0,\infty)$ we have

$$v(t) = e^{Jt}v(0-) \quad \forall t \geq 0$$

In particular, $\boxed{v(0+) = v(0-)}$ From Lecture 1 we know that $(**)$ considered on $(0,\infty)$ implies

$$w(t) = 0 \quad \forall t > 0$$

In particular, $\boxed{w(0+) = 0}$ (independently of $w(0-)$)

Altogether we have

$$\begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = \begin{pmatrix} v(0+) \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix}$$

hence

$$x(0+) = T\begin{pmatrix} v(0+) \\ w(0+) \end{pmatrix} = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} v(0-) \\ w(0-) \end{pmatrix} = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}x(0-) = \Pi_{(E,A)}x(0-)$$

Remarks.

a) $\Pi_{(E,A)}$ does not depend on the specific choice of S and T.

b) At this point we haven't actually shown that (ITP) has a solution!

Theorem. *Let (E,A) be regular. In the correct* distributional solution space *the ITP has a unique solution for all f.*

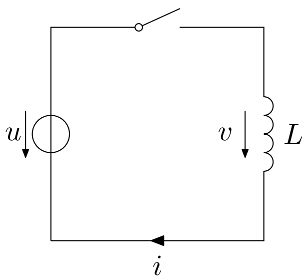
In particular, jumps and Dirac impulses at initial time are uniquely determined.

Attention: Choosing the right solution space is crucial and not immediately clear!

Here: Solution space = Space of *piecewise-smooth distributions* $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$

8 Switched DAEs: Definition

Recall example from Lecture 2:



Switch → Different DAE models (=modes) depending on time-varying position of switch

Switching signal $\sigma : \mathbb{R} \to \{1,\dots,N\}$ picks mode number $\sigma(t)$ at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma\cancel{(t)}}\dot{x}\cancel{(t)} &= A_{\sigma\cancel{(t)}}x\cancel{(t)} + B_{\sigma\cancel{(t)}}u\cancel{(t)} \\ y\cancel{(t)} &= C_{\sigma\cancel{(t)}}x\cancel{(t)} + D_{\sigma\cancel{(t)}}u\cancel{(t)} \end{aligned} \tag{swDAE}$$

Each mode might have different consistency spaces
⇒ inconsistent initial values at each switch
⇒ distributional solutions, i.e. $x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$

Corollary. *Let*

$$\Sigma_0 := \left\{ \sigma : \mathbb{R} \to \{1,\dots,N\} \;\middle|\; \sigma \textit{ is piecewise constant and } \sigma\big|_{(-\infty,0)} \textit{ is constant} \right\}.$$

Consider (swDAE) with regular (E_p,A_p) $\forall p \in \{1,\dots,N\}$. Then

$$\forall u \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}\ \forall \sigma \in \Sigma_0\ \exists \textit{ solution } x \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$$

and $x(0-)$ uniquely determines x.

9 Impulse-freeness of switched DAEs

Question: When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ *impulse free*?
(jumps are OK)

Lemma (Sufficient conditions)**.**

- (E_p,A_p) all *have index one (i.e. $N_p = 0$ in QWF)*
  ⇒ *(swDAE) impulse free*

- *all consistency spaces of (E_p,A_p) coincide (i.e. Wong limits $\mathcal{V}_p^*$ are identical)*
  $\Rightarrow$ *(swDAE) impulse free*

Proof:

- Index-1-case: Consider nilpotent DAE-ITP:

$$\begin{aligned}(N\dot{w})_{[0,\infty)} &= w_{[0,\infty)}\\ \Rightarrow 0 &= w_{[0,\infty)}\\ \Rightarrow w[0] := w_{[0,0]} &= 0\end{aligned}$$

  Hence an inconsistent initial value does not induce Dirac-impulse

- Same consistency space for all modes
  $\Rightarrow$ no inconsistent initial values at switch
  $\Rightarrow$ no jumps and no Dirac-impulse

Theorem. *The switched DAE $E_\sigma \dot{x} = A_\sigma x$ is impulse free $\forall \sigma \in \Sigma_0$*

$$\Leftrightarrow \quad E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p,q \in \{1,\dots,N\}$$

where $\Pi_p := \Pi_{(E_p,A_p)}$, $p \in \{1,\dots,N\}$ is the consistency projector.

Remarks.

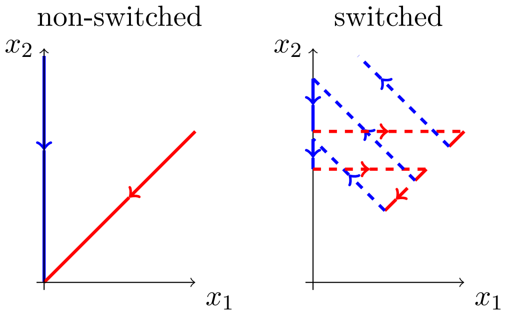
a) Index 1 $\Leftrightarrow E_p(I - \Pi_p) = 0\ \forall p$

b) Consistency spaces equal $\Leftrightarrow (I - \Pi_q)\Pi_p = 0\ \forall p,q$

10 Stability of switched DAEs

Examples.

a) $E_1 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $A_1 = \begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}$ $E_2 = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$, $A_1 = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$



$\rightarrow$ jumps destabilize

b) (E_1,A_1) as above, $E_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$, $A_1 = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$
non-switched behavior exactly the same as above, but switched behavior now stable: